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Bayesian Analysis of Generalized Gamma Distribution using R Software.

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Abstract: In this paper, some structural properties of Generalized Gamma Distribution (GGD) have been established. Bayesian method of estimation has been employed to estimate the parameters of GGD using Jeffrey's and extension of Jeffrey's priors under four different loss functions. The estimator thus obtained was compared with classical Maximum Likelihood Estimator using MSE through simulation studies with varying sample sizes using R codes. The expression for survival function has also been established under Jeffrey's and extension of Jeffrey's prior.

Keywords: GGD, Jeffrey's and extension of Jeffrey's prior, loss functions, Survival function.

1. Introduction:

The Generalized Gamma Distribution (GGD) introduced by Stacy (1962) presents a flexible family in the varieties of shapes and hazard functions for modeling duration. Distributions that are used in duration analysis in economics include exponential (Kiefer (1984)), gamma (Lancaster (1979)) and Weibull (Favero*et. al* (1994)) which are also the subfamilies of GGD and lognormal as a limiting distribution has been used in economics by Jasggia (1991). The study of life testing models begins with the estimation of the unknown parameters involved in the models. Prentice (1974) has considered maximum likelihood estimators for GGD by using the technique of reparameterization. Hager and Bain (1970) inhibited applications of the GGD model. However, despite its long history and growing use in various applications, the GGD family and its properties has been remarkably presented in different papers. Huang and Hwang (2006) introduced a new moment estimation of parameters of the GGD using its characterization.

The pdf of GGD is given by

$$f(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1}e^{-(\lambda x)^{\beta}} \quad , for\ x>0\ and\ \lambda,\beta,k>0 \tag{1.1}$$

Where k and β are shape parameters, and λ is the scale parameter.

The CDF of GGD is given by

$$F(x;\lambda,\beta,k)=\frac{\gamma\left(k,(kx)^{\beta}\right)}{\Gamma k}$$

The Structural properties of GGD are given as below:

$$E(X)=\frac{\Gamma\left(k+\frac{1}{\beta}\right)}{\lambda\,\Gamma k} \qquad E(X^2)=\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda^2\,\Gamma k}$$

$$V(X)=\frac{\Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k)-\Gamma^2\left(k+\frac{1}{\beta}\right)}{\lambda^2\Gamma^2(k)} \qquad C.V(X)=\frac{\left[\Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k)-\Gamma^2\left(k+\frac{1}{\beta}\right)\right]^{\frac{1}{2}}}{\Gamma\left(k+\frac{1}{\beta}\right)}$$

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2. Materials and Methods:

There are two main philosophical approaches to statistics. The first is called the classical approach which was founded by Prof. R. A. Fisher in a series of fundamental papers round about 1930. In this approach the unknown parameters are taken as fixed. The probability statement that can be made about the statistic based on its sampling distribution is converted to a confidence statement about the parameter. The confidence is based on the average behaviour of the procedure under all possible samples.

The alternative approach is the Bayesian approach which was first discovered by Reverend Thomas Bayes. In this approach, parameters are treated as random variables and data is treated fixed. Recently this approach has received great attention by most researchers. Rahul *et al.* (2009) have discussed the application of Bayesian methods. An important pre-requisite in this approach is the appropriate choice of priors for the parameters. However, Bayesian analysts have pointed out that there is no clear cut way from which one can conclude that one prior is better than the other. Very often, priors are chosen according to ones subjective knowledge and beliefs.

The other integral part of Bayesian inference is the choice of loss function. As there is no specific analytical procedure that allows us to identify the appropriate loss function to be used. The number of symmetric and asymmetric loss functions have been shown to be functional; see Varian (1975), Ahmed *et al* (2013) and Ahmad and Kaisar (2013) etc.

2.1. Loss functions:

(i) The squared error loss function (SELF) was proposed by Legendere (1805). Later, it was used in estimation problems when unbiased estimations of (sayθ) were evaluated in terms of the risk function $R(\theta, a)$ which becomes nothing but the variance of the estimator.

$$l_{Sl}(\hat{\lambda},\lambda) = a(\hat{\lambda}-\lambda)^2 \tag{2.1.1}$$

(ii) The Al-Bayyati's new loss function is of the form

$$l_{Al}\left(\hat{\lambda},\lambda\right) = \lambda^{c_1}\left(\hat{\lambda}-\lambda\right)^2; \quad c_1 \varepsilon R. \tag{2.1.2}$$

Which is an asymmetric loss function, λ and $\hat{\lambda}$ represent the true and estimated parameters.

(iii)The precautionary loss function is given by

$$l_{pr}\left(\hat{\lambda},\lambda\right) = \frac{\left(\hat{\lambda}-\lambda\right)^2}{\hat{\lambda}} \tag{2.1.3}$$

Which is an asymmetric loss function, for details, see Norstrom (1996).

(iv)The quadratic loss function is given by

$$l_{qd}\left(\hat{\lambda},\lambda\right) = \frac{\left(\hat{\lambda}-\lambda\right)^2}{\lambda} \tag{2.1.4}$$

Where λ and $\hat{\lambda}$ represent the true and estimated parameter.

2.2. Maximum likelihood estimation:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n having pdf (1.1), then the likelihood function is given by

$$L(X;\lambda,\beta,k) \;=\; \frac{\lambda^{n(k\beta)}\beta^n}{\Gamma^n(k)}\prod_{i=1}^{n} x_i^{k\beta-1} e^{-\lambda^\beta \sum_{i=1}^{n} x_i^\beta} \tag{2.2.1}$$

The log likelihood function is given by

$$\log L^{*}(X;\lambda,\beta,k)=n(k\beta)\log\lambda+n\log\beta-n\log\Gamma(k)+(k\beta-1)\sum_{i=1}^{n}\log x_i-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta}$$

Differentiate $\log L^{*}(X;\lambda,\beta,k)$ w.r.t. λ and equate to zero, we get

$$\hat{\lambda}=\frac{[nk]^{\frac{1}{\beta}}}{\sum_{i=1}^{n}x_i} \tag{2.2.2}$$

3. Bayesian Method of Estimation Using Jeffrey's Prior Under Different Loss Functions:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n having pdf (1.1) and the likelihood function (2.2.1).

Prior Distribution**:**

Quite often, the derivation of the prior distribution based on information other than the current data is impossible or rather difficult. Moreover, the statistician may be required to employ as little subjective inputs as possible, so that the conclusion may appear solely based on sampling model and the current data.

Jeffrey (1946) proposed a formal rule for obtaining a non-informative prior as

$$g(\theta)\ \alpha\ \sqrt{\det(I(\theta))}$$

Where θ is k-vector valued parameter and I (θ) is the Fisher's information matrix of order k × k.

We consider the prior distribution of λ to be

$$g(\lambda)\ \propto\ \sqrt{\det(I(\lambda))}$$

$$g(\lambda)=K\frac{1}{\lambda} \tag{3.1}$$

Where *K* is a constant.

By using Bayes theorem, the posterior distribution of λ is given by

$$\pi_1(\lambda\,|\,\underline{x})\propto L(x\,|\,\lambda)g(\lambda) \tag{3.2}$$

Using (2.2.1) and (3.1) in (3.2), we get

$$\pi_1(\lambda\,|\,\underline{x})\propto\frac{\lambda^{nk\beta-1}\beta^{n}}{\Gamma^{n}(k)}\,e^{-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta}}\prod_{i=1}^{n}(x_i)^{k\beta-1}$$

$$\pi_1(\lambda\,|\,\underline{x})=K\lambda^{nk\beta-1}e^{-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta}} \tag{3.3}$$

And
$$K=\frac{\left(\sum_{i=1}^{n}x_i^{\beta}\right)^{nk-\frac{1}{\beta}+1}}{\Gamma(nk-\frac{1}{\beta}+1)}$$

Using the value of K in (3.3), we get

$$\pi_1(\lambda \mid \underline{x}) = \left(\frac{e^{-\lambda^\beta \sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta-1} \left(\sum_{i=1}^{n} x_i^{\beta} \right)^{nk-\frac{1}{\beta}+1}}{\Gamma(nk-\frac{1}{\beta}+1)} \right) \tag{3.4}$$

3.1. Estimation under Squared error loss function:

By using the loss function (2.1.1), the risk function is given by

$$R(\hat{\lambda}) = \int_0^\infty a(\hat{\lambda}-\lambda)^2 \pi_1(\lambda \mid \underline{x}) d\lambda \tag{3.1.1}$$

Using (3.4) in (3.1.1), we have

$$R(\hat{\lambda}) = \int_0^\infty a(\hat{\lambda}-\lambda)^2 \left(\frac{e^{-\lambda^\beta \sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta-1} \left(\sum_{i=1}^{n} x_i^{\beta} \right)^{nk-\frac{1}{\beta}+1}}{\Gamma(nk-\frac{1}{\beta}+1)} \right) d\lambda \tag{3.1.2}$$

On solving (3.1.2), we have

$$R(\hat{\lambda}) = a\hat{\lambda}^2 + a\frac{\Gamma(nk+\frac{1}{\beta}+1)}{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{2}{\beta}} \Gamma(nk-\frac{1}{\beta}+1)} - \frac{2a\hat{\lambda}\Gamma(nk+1)}{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{1}{\beta}} \Gamma(nk-\frac{1}{\beta}+1)}$$

To obtain the Bayes estimator, we have $\frac{\partial R(\hat{\lambda})}{\partial \hat{\lambda}} = 0$

$$\hat{\lambda} = \frac{1}{\left(\sum_{i=1}^{n} x_i\right)} \left\{ \frac{\Gamma(nk+1)}{\Gamma(nk-\frac{1}{\beta}+1)} \right\} \tag{3.1.3}$$

3.2. Estimation under Al-Bayyati's new loss function:

By using the loss function (2.1.2), the risk function is given by

$$R(\hat{\lambda}) = \int_0^\infty \lambda^{c_1} (\hat{\lambda}-\lambda)^2 \pi_1(\lambda \mid \underline{x}) d\lambda \tag{3.2.1}$$

Using (3.4) in (3.2.1), we have

$$R\left(\hat{\lambda}\right)=\int_0^\infty \lambda^{c_1}\left(\hat{\lambda}-\lambda\right)^2\left(\frac{e^{-\lambda^\beta\sum_{i=1}^{n}x_i^\beta}\lambda^{nk\beta-1}\left(\sum_{i=1}^{n}x_i^\beta\right)^{nk-\frac{1}{\beta}+1}}{\Gamma(nk-\frac{1}{\beta}+1)}\right)d\lambda \quad (3.2.2)$$

On solving (3.2.2), we have

$$R\left(\hat{\lambda}\right)=\hat{\lambda}^2\frac{\Gamma(nk+\frac{c_1}{\beta}-\frac{1}{\beta}+1)}{\left(\sum_{i=1}^{n}x_i^\beta\right)^{\frac{c_1}{\beta}}\Gamma(nk-\frac{1}{\beta}+1)}+\frac{\Gamma(nk+\frac{c_1}{\beta}+\frac{1}{\beta}+1)}{\left(\sum_{i=1}^{n}x_i^\beta\right)^{\frac{c_{1+2}}{\beta}}\Gamma(nk-\frac{1}{\beta}+1)}-2\hat{\lambda}\frac{\Gamma(nk+\frac{c_1}{\beta}+1)}{\left(\sum_{i=1}^{n}x_i^\beta\right)^{\frac{c_{1+1}}{\beta}}\Gamma(nk-\frac{1}{\beta}+1)}$$

To obtain Bayes estimator, we have $\frac{\partial R\left(\hat{\lambda}\right)}{\partial\hat{\lambda}}=0$

$$\hat{\lambda}=\frac{1}{\left(\sum_{i=1}^{n}x_i\right)}\left\{\frac{\Gamma(nk+\frac{c_1}{\beta}+1)}{\Gamma(nk+\frac{c_1}{\beta}-\frac{1}{\beta}+1)}\right\} \quad (3.2.3)$$

Remark: - Replacing $C_1=0$ in (3.2.3), the same Bayes estimator is obtained as in (3.1.3) corresponding to Jeffrey's prior.

3.3. Estimation under precautionary loss function:

By using the loss function (2.1.3), the risk function is given by

$$R\left(\hat{\lambda}\right)=\int_0^\infty\frac{\left(\hat{\lambda}-\lambda\right)^2}{\hat{\lambda}}\pi_1(\lambda\,|\,\underline{x})d\lambda \quad (3.3.1)$$

Using (3.4) in (3.3.1), we have

$$R\left(\hat{\lambda}\right)=\int_0^\infty\frac{\left(\hat{\lambda}-\lambda\right)^2}{\hat{\lambda}}\left(\frac{e^{-\lambda^\beta\sum_{i=1}^{n}x_i^\beta}\lambda^{nk\beta-1}\left(\sum_{i=1}^{n}x_i^\beta\right)^{nk-\frac{1}{\beta}+1}}{\Gamma(nk-\frac{1}{\beta}+1)}\right)d\lambda \quad (3.3.2)$$

On solving (3.3.2), we have

$$R\left(\hat{\lambda}\right)=\frac{\Gamma(nk+\frac{1}{\beta}+1)}{\hat{\lambda}\Gamma(nk-\frac{1}{\beta}+1)\left(\sum_{i=1}^{n}x_i^\beta\right)^{\frac{2}{\beta}}}+\hat{\lambda}-\frac{2\Gamma(nk+1)}{\left(\sum_{i=1}^{n}x_i^\beta\right)^{\frac{1}{\beta}}\Gamma(nk-\frac{1}{\beta}+1)}$$

To obtain Bayes estimator, we have $\frac{\partial R\left(\hat{\lambda}\right)}{\partial\hat{\lambda}}=0$

$$\hat{\lambda} = \frac{1}{\left(\sum_{i=1}^{n} x_i\right)} \left\{ \sqrt{\frac{\Gamma(nk + \frac{1}{\beta} + 1)}{\Gamma(nk - \frac{1}{\beta} + 1)}} \right\} \tag{3.3.3}$$

3.4. Estimation under quadratic loss function:

By using the loss function (2.1.4), the risk function is given by

$$R\left(\hat{\lambda}\right) = \int_0^\infty \frac{\left(\hat{\lambda} - \lambda\right)^2}{\lambda} \pi_1\left(\lambda \mid \underline{x}\right) d\lambda \tag{3.4.1}$$

By using (3.4) in (3.4.1), we have

$$R\left(\hat{\lambda}\right) = \int_0^\infty \frac{\left(\hat{\lambda} - \lambda\right)^2}{\lambda} \left(\frac{e^{-\lambda^\beta \sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta - 1} \left(\sum_{i=1}^{n} x_i^{\beta}\right)^{nk - \frac{1}{\beta} + 1}}{\Gamma(nk - \frac{1}{\beta} + 1)} \right) d\lambda \tag{3.4.2}$$

On solving (3.4.2), we have

$$R\left(\hat{\lambda}\right) = \frac{\Gamma(nk+1)}{\Gamma(nk - \frac{1}{\beta} + 1)\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{1}{\beta}}} + \frac{\hat{\lambda}^2 \Gamma(nk - \frac{2}{\beta} + 1)}{\Gamma(nk - \frac{1}{\beta} + 1)\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{-\frac{1}{\beta}}} - 2\hat{\lambda}$$

To obtain Bayes estimator, we have $\frac{\partial R\left(\hat{\lambda}\right)}{\partial \hat{\lambda}} = 0$

$$\hat{\lambda} = \frac{1}{\left(\sum_{i=1}^{n} x_i\right)} \left\{ \frac{\Gamma(nk - \frac{1}{\beta} + 1)}{\Gamma(nk - \frac{2}{\beta} + 1)} \right\} \tag{3.4.3}$$

3.5. Estimation of Survival function:

By using posterior pdf we can find the Survival function, such that

$$\hat{S}_1(\underline{x}) = \int_0^\infty e^{-(\lambda x)^\beta} \pi_1\left(\lambda / \underline{x}\right) d\lambda \tag{3.5.1}$$

Using (3.4) in (3.5.1), we have

$$\hat{S}_1(\underline{x}) = \left(\frac{\sum_{i=1}^{n} x_i^{\beta}}{x_i^{\beta} + \sum_{i-1}^{n} x_i^{\beta}} \right)^{nk - \frac{1}{\beta} + 1} \tag{3.5.2}$$

4. Bayesian Method of Estimation Using Extension of Jeffrey's Prior Under Different Loss Functions:

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n having pdf (1.1) and the likelihood function (2.2.1).

We consider the prior distribution of λ to be

$$g(\lambda) \propto \left[\det | I(\lambda) |\right]^{c} \ , c \in R^{+}$$

$$g(\lambda) = \frac{K}{\lambda^{2c}} \tag{4.1}$$

By using Bayes theorem, the posterior distribution of λ is given by

$$\pi_2(\lambda \,|\, \underline{x}) \propto L(x \,|\, \lambda) g(\lambda) \tag{4.2}$$

Using (2.2.1) and (4.1) in (4.2), we get

$$\pi_2(\lambda \,|\, \underline{x}) \propto \frac{\lambda^{nk\beta - 2c} \beta^{n}}{\Gamma^{n}(k)} e^{-\lambda^{\beta} \sum_{i=1}^{n} x_i^{\beta}} \prod_{i=1}^{n} (x_i)^{k\beta - 1}$$

$$\pi_2(\lambda \,|\, \underline{x}) = K \lambda^{nk\beta - 2c} e^{-\lambda^{\beta} \sum_{i=1}^{n} x_i^{\beta}} \tag{4.3}$$

And

$$K = \frac{\left(\sum_{i=1}^{n} x_i^{\beta} \right)^{nk - \frac{2c}{\beta} + 1}}{\Gamma(nk - \frac{2c}{\beta} + 1)}$$

Using the value of K in (4.3), we get

$$\pi_2(\lambda \,|\, \underline{x}) = \left(\frac{e^{-\lambda^{\beta} \sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta - 2c} \left(\sum_{i=1}^{n} x_i^{\beta} \right)^{nk - \frac{2c}{\beta} + 1}}{\Gamma(nk - \frac{2c}{\beta} + 1)} \right) \tag{4.4}$$

4.1. Estimation under Squared error loss function:

By using the loss function (2.1.1), the risk function is of the form

$$R\left(\hat{\lambda}\right)=\int_{0}^{\infty} a\left(\hat{\lambda}-\lambda\right)^{2}\pi_{2}\left(\lambda\,|\,\underline{x}\right)d\lambda \tag{4.1.1}$$

By using (4.4) in (4.1.1), we have

$$R\left(\hat{\lambda}\right)=\int_{0}^{\infty} a\left(\hat{\lambda}-\lambda\right)^{2}\left(\frac{e^{-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta}}\lambda^{nk\beta-2c}\left(\sum_{i=1}^{n}x_i^{\beta}\right)^{nk-\frac{2c}{\beta}+1}}{\Gamma(nk-\frac{2c}{\beta}+1)}\right)d\lambda \tag{4.1.2}$$

On solving (4.1.2), we have

$$R\left(\hat{\lambda}\right)=a\hat{\lambda}^{2}+a\frac{\Gamma(nk-\frac{2c}{\beta}+\frac{2}{\beta}+1)}{\left(\sum_{i=1}^{n}x_i^{\beta}\right)^{\frac{2}{\beta}}\Gamma(nk-\frac{2c}{\beta}+1)}-\frac{2a\hat{\lambda}\Gamma(nk-\frac{2c}{\beta}+\frac{1}{\beta}+1)}{\left(\sum_{i=1}^{n}x_i^{\beta}\right)^{\frac{1}{\beta}}\Gamma(nk-\frac{2c}{\beta}+1)}$$

To obtain Bayesian estimator, we have $\dfrac{\partial R\left(\hat{\lambda}\right)}{\partial\hat{\lambda}}=0$

$$\hat{\lambda}=\frac{1}{\left(\sum_{i=1}^{n}x_i\right)}\left\{\frac{\Gamma(nk-\frac{2c}{\beta}+\frac{1}{\beta}+1)}{\Gamma(nk-\frac{2c}{\beta}+1)}\right\} \tag{4.1.3}$$

Remark: Replacing C=1/2 in (4.1.3), the same Bayes estimator is obtained as in (3.1.3) corresponding Jeffrey's prior.

4.2. Estimation under Al-Bayyati's loss function:

By using the loss function (2.1.2), the risk function is given by

$$R\left(\hat{\lambda}\right)=\int_{0}^{\infty} \lambda^{c_1}\left(\hat{\lambda}-\lambda\right)^{2}\pi_{2}\left(\lambda\,|\,\underline{x}\right)d\lambda \tag{4.2.1}$$

Using (4.4) in (4.2.1), we have

$$R\left(\hat{\lambda}\right)=\int_{0}^{\infty} \lambda^{c_1}\left(\hat{\lambda}-\lambda\right)^{2}\left(\frac{e^{-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta}}\lambda^{nk\beta-1}\left(\sum_{i=1}^{n}x_i^{\beta}\right)^{nk-\frac{1}{\beta}+1}}{\Gamma(nk-\frac{1}{\beta}+1)}\right)d\lambda \tag{4.2.2}$$

On solving (4.2.2), we have

$$R(\hat{\lambda}) = \hat{\lambda}^2 \frac{\Gamma(nk + \frac{c_1}{\beta} - \frac{2c}{\beta} + 1)}{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{c_1}{\beta}} \Gamma(nk - \frac{2c}{\beta} + 1)} + \frac{\Gamma(nk - \frac{2c}{\beta} + \frac{c_1}{\beta} + \frac{2}{\beta} + 1)}{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{c_{1+2}}{\beta}} \Gamma(nk - \frac{2c}{\beta} + 1)} - 2\hat{\lambda} \frac{\Gamma(nk - \frac{2c}{\beta} + \frac{c_1}{\beta} + \frac{1}{\beta} + 1)}{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{c_{1+1}}{\beta}} \Gamma(nk - \frac{1}{\beta} + 1)}$$

To obtain Bayesian estimator, we have $\frac{\partial R(\hat{\lambda})}{\partial \hat{\lambda}} = 0$

$$\hat{\lambda} = \frac{1}{\left(\sum_{i=1}^{n} x_i\right)} \left\{ \frac{\Gamma(nk - \frac{2c}{\beta} + \frac{c_1}{\beta} + \frac{1}{\beta} + 1)}{\Gamma(nk - \frac{2c}{\beta} + \frac{c_1}{\beta} + 1)} \right\} \tag{4.2.3}$$

Remark: Replacing C=1/2 in (4.2.3), the same Bayes estimator is obtained as in (3.2.3) corresponding Jeffrey's prior.

Also if we replace C_1=0 in (4.2.3), the same Bayes estimator is obtained as in (4.1.3) corresponding to Extension of Jeffrey's prior.

4.3. Estimation under precautionary loss function:

By using the loss function (2.1.3), the risk function is given by

$$R(\hat{\lambda}) = \int_0^{\infty} \frac{(\hat{\lambda} - \lambda)^2}{\hat{\lambda}} \pi_2(\lambda \mid \underline{x}) d\lambda \tag{4.3.1}$$

By using (4.4) in (4.3.1), we have

$$R(\hat{\lambda}) = \int_0^{\infty} \frac{(\hat{\lambda} - \lambda)^2}{\hat{\lambda}} \frac{e^{-\lambda^{\beta} \sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta - 2c} \left(\sum_{i=1}^{n} x_i^{\beta}\right)^{nk - \frac{2c}{\beta} + 1}}{\Gamma(nk - \frac{2c}{\beta} + 1)} d\lambda \tag{4.3.2}$$

On solving (4.3.2), we have

$$R(\hat{\lambda}) = \frac{\Gamma(nk - \frac{2c}{\beta} + \frac{2}{\beta} + 1)}{\hat{\lambda}\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{2}{\beta}} \Gamma(nk - \frac{2c}{\beta} + 1)} + \hat{\lambda} - \frac{2\Gamma(nk - \frac{2c}{\beta} + \frac{1}{\beta} + 1)}{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{1}{\beta}} \Gamma(nk - \frac{2c}{\beta} + 1)}$$

To determine the Bayes estimator, we have $\frac{\partial R(\hat{\lambda})}{\partial \hat{\lambda}} = 0$

$$\hat{\lambda} = \frac{1}{\left(\sum_{i=1}^{n} x_i\right)} \left[\sqrt{\frac{\Gamma\left\{(nk - \frac{2c}{\beta} + \frac{2}{\beta} + 1)\right\}}{\Gamma\left\{(nk - \frac{2c}{\beta} + 1)\right\}}} \right] \tag{4.3.3}$$

Remark: Replacing C=1/2 in (4.3.3), the same Bayes estimator is obtained as in (3.3.3) corresponding Jeffrey's prior.

4.4. Estimation under quadratic loss function:

By using the loss function (2.1.4), the risk function is given by

$$R(\hat{\lambda})=\int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\lambda}\pi_2(\lambda\,|\,\underline{x})d\lambda \tag{4.4.1}$$

Using (4.4) in (4.4.1), we have

$$R(\hat{\lambda})=\int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\lambda}\,\frac{e^{-\lambda^\beta\sum_{i=1}^n x_i^{\beta}}\,\lambda^{nk\beta-2c}\left(\sum_{i=1}^n x_i^{\beta}\right)^{nk-\frac{2c}{\beta}+1}}{\Gamma(nk-\frac{2c}{\beta}+1)}\,d\,\lambda \tag{4.4.2}$$

On solving (4.4.2), we have

$$R(\hat{\lambda})=\frac{\Gamma(nk-\frac{2c}{\beta}+\frac{1}{\beta}+1)}{\left(\sum_{i=1}^n x_i^{\beta}\right)^{\frac{1}{\beta}}\Gamma(nk-\frac{2c}{\beta}+1)}+\hat{\lambda}^2\frac{\Gamma(nk-\frac{2c}{\beta}-\frac{1}{\beta}+1)}{\left(\sum_{i=1}^n x_i^{\beta}\right)^{-\frac{1}{\beta}}\Gamma(nk-\frac{2c}{\beta}+1)}-2\hat{\lambda}$$

To obtain the Bayes estimator, we have $\frac{\partial R(\hat{\lambda})}{\partial\hat{\lambda}}=0$

$$\hat{\lambda}=\frac{1}{\left(\sum_{i=1}^n x_i\right)}\left[\frac{\left\{\Gamma(nk-\frac{2c}{\beta}+1)\right\}}{\left\{\Gamma(nk-\frac{2c}{\beta}-\frac{1}{\beta}+1)\right\}}\right] \tag{4.4.3}$$

Remark: Replacing C=1/2 in (4.4.3), the same Bayes estimator is obtained as in (3.4.3) corresponding Jeffrey's prior.

4.5. Estimation of Survival function:

By using (4.4), the Survival function is given by

$$\hat{S}_2(\underline{x})=\int_0^\infty e^{-(\lambda x)^\beta}\pi_2(\lambda\,|\,\underline{x})d\lambda \tag{4.5.1}$$

Using (4.4) in (4.5.1), we have

$$\hat{S}_2(\underline{x})=\left(\frac{\sum_{i=1}^n x_i^{\beta}}{x_i^{\beta}+\sum_{i-1}^n x_i^{\beta}}\right)^{nk-\frac{2c}{\beta}+1} \tag{4.5.2}$$

5. Simulation Study:

A simulation study was conducted using R-software to examine and compare the performance of the estimates. We choose samples of size 25, 50 and 100 to generate the data set of GGD. The simulation study was carried out 3,000 times for each pairs of (λ, β, k) where $(\lambda = 1.0, 1.5, 2.0)$, $(\beta = 0.5, 1.0, 1.5)$ and $(k = 0.5, 1.0, 1.0)$. The scale parameter is estimated for GGD with MLE and Bayesian method of estimation using Jeffrey's & extension of Jeffrey's prior under different loss functions. The values for Jeffrey's extension are (C = 0.5, 1.0, 1.5) and the values for the loss parameter are (C_1=1,-1, 2, -2). The results are presented in the following tables.

Table 1: Mean Squared Error for $(\hat{\lambda})$ **u**nder Jeffrey's prior:

N	λ	β	k	λ_{ML}	λ_{sl}	λ_{pr}	λ_{qd}	λ_{Al}			
								C_1=1	C_1=-1	C_1=2	C_1=-2
25	1.0	0.5	0.5	1.0834	1.0244	1.2256	0.9843	1.1135	0.5734	1.1045	0.0174
	1.5	1.0	1.0	0.6857	0.6852	0.6632	0.7305	0.6413	0.7305	0.5989	0.4674
	2.0	1.5	1.0	0.6823	0.6810	0.6786	0.6863	0.6760	0.6863	0.6707	0.6591
50	1.0	0.5	0.5	1.1597	1.2941	1.0642	1.0851	1.1223	0.4792	1.0904	0.0848
	1.5	1.0	1.0	0.4263	0.4261	0.4152	0.4485	0.4044	0.4485	0.3832	0.3811
	2.0	1.5	1.0	0.7528	0.7523	0.7513	0.7544	0.7502	0.7544	0.7481	0.7392
100	1.0	0.5	0.5	1.7274	1.9086	1.9019	1.2507	0.9669	0.1199	0.1105	0.1104
	1.5	1.0	1.0	0.8330	0.8329	0.8275	0.8437	0.8222	0.8437	0.8116	0.7487
	2.0	1.5	1.0	0.7777	0.7775	0.9631	0.9647	0.7764	0.8081	0.7756	0.4665

ML=Maximum Likelihood, S_l=squared error loss function, Pr=precautionary loss function, qd=quadratic loss function, Al=Albayyti's new loss function

Table 2: Mean Squared Error for $(\hat{\lambda})$ under extension of Jeffrey's prior:

n	λ	β	k	C	λ_{ML}	λ_{sl}	λ_{pr}	λ_{qd}	λ_{Al}			
									C_1=1.0	C_1=-1.0	C_1=2.0	C_1=-2.0
25	1.0	0.5	0.5	0.5	0.5838	0.2311	0.2950	0.1948	0.2628	0.1948	0.3089	0.0571
				1.0	0.7738	0.2578	0.1734	0.3786	0.3251	0.1366	0.1491	0.0891
				1.5	0.1787	0.1868	0.5601	0.1028	0.3462	0.1028	0.2022	0.0753
	1.5	1.0	1.0	0.5	0.8879	0.8875	0.8666	0.9304	0.8456	0.9304	0.8047	0.6241
				1.0	0.8155	0.8586	0.8367	0.9037	0.8147	0.9037	0.7717	0.4614
				1.5	0.8769	0.9630	0.9411	1.0082	0.9191	0.9883	0.8761	0.2706
	2.0	1.5	1.0	0.5	1.1271	1.2578	1.3295	1.3889	1.5483	1.4604	1.2710	1.0525
				1.0	1.4098	1.1535	1.4532	1.4886	1.6567	1.2551	1.6148	1.075
				1.5	1.3308	1.4242	1.4725	1.2556	1.4919	1.2400	1.4484	1.0278
50	1.0	0.5	0.5	0.5	0.9244	0.4822	0.1015	0.4309	0.6661	0.4309	1.0517	0.1898
				1.0	0.9288	0.2332	1.0292	0.5212	0.2195	0.5212	0.3160	0.1965
				1.5	0.6407	0.8001	0.1095	0.2509	0.1676	0.7764	0.2780	0.1382
	1.5	1.0	1.0	0.5	0.8442	0.8441	0.8336	0.8657	0.8230	0.8657	0.8020	0.7097
				1.0	1.0558	1.0752	1.0655	1.0949	1.0557	1.0949	1.0361	0.9147
				1.5	0.9252	0.9670	0.9565	0.9883	0.9460	0.9883	0.9250	0.8198
	2.0	1.5	1.0	0.5	1.8551	1.8551	1.8434	1.8783	1.8320	1.8783	1.8088	1.6360
				1.0	1.6263	1.6511	1.6388	1.6761	1.6263	1.6761	1.6017	1.2201
				1.5	1.6536	1.7030	1.6907	1.7281	1.6783	1.7281	1.6535	1.2641
100	1.0	0.5	0.5	0.5	1.4077	0.5084	0.7676	0.3731	0.1842	0.2499	0.5785	0.1400
				1.0	1.1779	0.4541	0.2198	0.1993	0.5784	0.8096	0.1011	0.0982
				1.5	1.7517	0.3001	0.6765	0.7517	0.1431	0.5004	0.3364	0.1322
	1.5	1.0	1.0	0.5	0.8333	0.8332	0.8278	0.4557	0.8225	0.8440	0.8118	0.3914
				1.0	0.9041	0.9146	0.9092	0.9251	0.9041	0.9251	0.8937	0.7525
				1.5	0.8830	0.9043	0.8990	0.9150	0.8935	0.9150	0.8830	0.7436
	2.0	1.5	1.0	0.5	1.6398	1.6397	1.6336	1.6521	1.6275	1.6521	1.6153	1.1885
				1.0	1.6123	1.6248	1.6184	1.6370	1.6123	1.6370	1.5998	1.0497
				1.5	1.5403	1.5654	1.5592	1.5782	1.5527	1.5782	1.5403	1.1252

ML=Maximum Likelihood, S_l=squared error loss function, Pr=precautionary loss function, qd=quadratic loss function, Al=Albayyti's new loss function.

6. Result:

In table 1, Bayes estimation with Al-Bayyati's new loss function under Jeffrey's prior provides the smallest values in most cases especially when loss parameter (C_1= -2).

In table 2, Bayes estimation with Al-Bayyati's new loss function under extension of Jeffrey's prior provides the smallest values in most cases especially when loss parameter (C_1= -2) and (C=0.5, 1.0, 1.5).

7. Conclusion:

We observe that Bayesian method of estimation is better than classical method estimation. Also Bayesian estimator under Al-Bayyati's new loss function provides the smallest MSE values under Jeffrey's and extension of Jeffrey's prior as compared to other loss functions and the classical estimator when the loss parameter C_1 is -2. Thus we can say that Al-Bayyati's new loss is better than other loss functions.

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