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Comparison of Redundant Components Allocation Policies in Series Systems

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Abstract: Comparing different redundant components allocation policies in series systems is of great interest both in theory and in practice. This paper provides a simple criteria for comparing two policies in terms of hazard rate ordering. We conjecture that this simple criteria is applicable in terms of likelihood ratio ordering, excepting some extreme situations. Computer simulations strongly support this conjecture.

Keywords: Likelihood ratio order, hazard rate order, series system, stochastic comparison.

1 Introduction

To enhance the lifetime of a system, it is very common to allocate some redundant components. How to allocate redundant components in a system is an interesting and important issue in reliability engineering and system security. In the past three decades, many researchers devoted themselves to study the related theoretical problems. For instance, Shaked and Shanthikumar [5] considered the problem of allocation of K active components (or redundancies) to a series system when the lifetimes of the components and the redundancies are independent and identically distributed random variables. Specifically, let $\boldsymbol{k} = (k_1, \cdots, k_n)$ be an allocation vector, or an allocation policy, that is, for ith original component in the system, to allocate k_i redundant components. Denote $T(\boldsymbol{k})$ as the lifetime of the resulted series system. Shaked and Shanthikumar [5] showed that

$$T(\boldsymbol{k}) \leq_{st} T(\boldsymbol{k}') \quad \text{whenever } \boldsymbol{k} \succ \boldsymbol{k}'. \tag{1.1}$$

where $\leq_{st}$ and $\succ$ denote the usual stochastic order and the majorization order (the formal definitions of these order and others that will be used in this paper are given in Section 2). Singh and Singh [7] strengthened (1.1) from the usual stochastic order to the hazard rate order. For $n = 2$, Hu and Wang [1] posed an open problem that (1.1) can be strengthened to the likelihood ratio order. Zhao *et al*. [9] showed that (1.1) holds in terms of likelihood ratio order, thus providing a solution to this open problem.

The majorization condition $\boldsymbol{k} \succ \boldsymbol{k}'$ in (1.1) implies $k_1 + \cdots + k_n = k_1' + \cdots + k_n'$, that is, the total redundant components are the same. But how to compare two policies with different total numbers of redundant components is still a problem which deserves further study.

Intuitionally, by putting some redundant components can make the system more reliable. Therefore, the policy $(4,1,2,3)$ should be better than $(3,1,2,2)$, which in turns, better than $(2,0,2,1)$, and so on. Hence, we believe the majorization condition in (1.1) is too restrictive.

In this paper, we relax the majorization condition in (1.1) and generalize the result as

$$T(\boldsymbol{k}) \leq_{hr} T(\boldsymbol{k}') \quad \text{whenever } \boldsymbol{k} \overset{w}{\succ} \boldsymbol{k}', \tag{1.2}$$

where $\leq_{hr}$ denote the hazard rate order and $\overset{w}{\succ}$ denote the weak majorization order. We also investigate the policy comparison in terms of likelihood ratio ordering. We conjecture that, when $\min k = \min\{k_1, \cdots, k_n\} < \min k' =$

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$\min\{k'_1, \cdots, k'_n\}$, then

$$T(\boldsymbol{k}) \leq_{lr} T(\boldsymbol{k}') \quad \text{whenever } \boldsymbol{k} \overset{w}{\succ} \boldsymbol{k}'. \tag{1.3}$$

Some computer simulations are conducted to support this conjecture.

2 Notations and Lemmas

The following are some standard notations, which can be found in, say, Shaked and Shanthikumar [6], Müller and Stoyan [4], and, Hu and Wang [1].

Let X be a nonnegative continuous random variable with distribution function $F_X(t)$, survival function $\overline{F_X}(t) = 1 - F_X(t)$, density function $f_X(t)$, hazard function $\lambda_X = f_X/\overline{F_X}$, and reversed hazard function $r_X = f_X/F_X$, respectively. For two nonnegative continuous random variables X and Y, we say X is smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$), if $\overline{F_X}(t) \leq \overline{F_Y}(t)$; X is smaller than Y in hazard rate order (denoted by $X \leq_{hr} Y$), if $\lambda_X(t) \geq \lambda_Y(t)$; X is smaller than Y in reversed hazard rate order (denoted by $X \leq_{rh} Y$), if $r_X(t) \leq r_Y(t)$; X is smaller than Y in likelihood ratio order (denoted by $X \leq_{lr} Y$), if the ratio $f_Y(t)/f_X(t)$ is increasing in t. It is well known that likelihood ratio order implies both hazard rate order and reversed hazard rate order, and these two imply the usual stochastic order.

Given two vectors $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$, let $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ and $b_{(1)} \leq b_{(2)} \leq \cdots \leq b_{(n)}$ be the increasing arrangements of the elements of the two vectors. The vector $\boldsymbol{a}$ is said to majorize the vector $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \succ \boldsymbol{b}$) if and only if, $\sum_{i=1}^n a_{(i)} = \sum_{i=1}^n b_{(i)}$, and $\sum_{i=1}^k a_{(i)} \leq \sum_{i=1}^k b_{(i)}$, for $k = 1, \cdots, n-1$. If for all $k = 1, \cdots, n$, $\sum_{i=1}^k a_{(i)} \leq \sum_{i=1}^k b_{(i)}$, then the vector $\boldsymbol{a}$ is said to weakly majorize the vector $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \overset{w}{\succ} \boldsymbol{b}$). For more comprehensive details on the majorization order, see Marshall and Olkin [2], (Chapter 3). Followed Wang [8], we say a vector $\boldsymbol{a} = (a_1, \cdots, a_n)$ is *pseudo-positive*, if for all $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$, with $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_n \geq 0$, $\boldsymbol{a} \cdot \boldsymbol{\alpha} = \sum_{i=1}^n a_i \alpha_i \geq 0$. It is easy to confirm that, for $n = 2$, the vector $v = (1, a)$ is pseudo-positive if and only if $a \geq -1$. As proved in Wang (2015), $\boldsymbol{a} \overset{w}{\succ} \boldsymbol{b}$ is equivalent to $\boldsymbol{b} - \boldsymbol{a}$ is pseudo-positive.

To prove the main results, we need some lemmas.

Lemma 1. *Let* $f(x) = \dfrac{xa^x}{1-a^x}$, $0 \leq a < 1$. *Then, for* $x > 0$, $f'(x) \leq 0$, *and* $f''(x) \geq 0$.

Lemma 2. *For* $0 \leq a < 1$, *let*

$$\phi(x_1, \cdots, x_n) = \sum_{i=1}^n \frac{x_i a^{x_i}}{1 - a^{x_i}} \qquad 0 < x_1 \leq \cdots \leq x_n.$$

Then, the function $\phi(x_1, \cdots, x_n)$ *is decreasing in any pseudo-positive direction.*

Proof. By Lemma 1, for all $i = 1, \cdots, n$, $\frac{\partial \phi}{\partial x_i} \leq 0$, and for $1 \leq i < j \leq n$, $\frac{\partial \phi}{\partial x_i} \leq \frac{\partial \phi}{\partial x_j}$. Hence,

$$-\frac{\partial \phi}{\partial x_1} \geq \cdots \geq -\frac{\partial \phi}{\partial x_n} \geq 0.$$

Therefore, on a pseudo-positive direction $\boldsymbol{v}$, we have,

$$\begin{aligned}
\bigtriangledown_{\boldsymbol{v}} \phi &= \left(\frac{\partial \phi}{\partial x_1}, \cdots, \frac{\partial \phi}{\partial x_n}\right) \cdot \boldsymbol{v} \\
&= -\left(-\frac{\partial \phi}{\partial x_1}, \cdots, -\frac{\partial \phi}{\partial x_n}\right) \cdot \boldsymbol{v} \\
&\leq 0.
\end{aligned}$$

Thus, in the direction $\boldsymbol{v}$, the function $\phi(x_1, \cdots, x_n)$ is decreasing.

Lemma 3. *For any positive integers* k *and* a, *the function* $H(t) = \dfrac{1-t^{k+a}}{1-t^k}$ *is increasing in* $0 \leq t < 1$.

Proof. We have,

$$H'(t) = \frac{k(1-t^a) - at^a(1-t^k)}{(1-t^k)^2} t^{k-1}.$$

Let $g(t) = k(1-t^a) - at^a(1-t^k)$, then, $g'(t) = a(k+a)t^{a-1}(t^k - 1) \leq 0$, for $0 \leq t < 1$. Since $g(1) = 0$, thus, $g(t) \geq 0$, and hence, $H'(t) \geq 0$.

3 Series system with n components

Theorem 3.1. *For a series system with n components, put k_i redundant components in parallel with the ith original one. Let $\boldsymbol{k} = (k_1, \cdots, k_n)$ denote the allocation vector and $T(\boldsymbol{k})$ the lifetime of the resulted system. Then,*

$$T(\boldsymbol{k}) \leq_{hr} T(\boldsymbol{k}') \quad \textit{whenever} \;\; \boldsymbol{k} \overset{w}{\succ} \boldsymbol{k}'.$$

Proof. Denote by F and f the common distribution function and density function of the original components and active redundancies. As in Hu and Wang [1], the survival function of $T(\boldsymbol{k})$ is given by

$$\bar{F}_{T(\boldsymbol{k})}(t) = \prod_{i=1}^{n} \left(1 - F^{k_i+1}(t)\right),$$

and so, the hazard rate function of $T(\boldsymbol{k})$ is

$$\lambda_{T(\boldsymbol{k})}(t) = \sum_{i=1}^{n} \frac{(k_i+1)F^{k_i}(t)f(t)}{1-F^{k_i+1}(t)} = \sum_{i=1}^{n} \frac{(k_i+1)F^{k_i+1}(t)}{1-F^{k_i+1}(t)} \cdot \frac{f(t)}{F(t)}.$$

Without loss of generality, assume that $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$, and $0 \leq k_1' \leq k_2' \leq \cdots \leq k_n'$. From Proposition 1 in Wang [8], $\boldsymbol{k} \overset{w}{\succ} \boldsymbol{k}'$ implies the direction $\boldsymbol{k}' - \boldsymbol{k} = (\boldsymbol{k}' + \mathbf{1}) - (\boldsymbol{k} + \mathbf{1})$ is pseudo-positive, and therefore,

$$\lambda_{T(\boldsymbol{k}')}(t) = \sum_{i=1}^{n} \frac{(k_i'+1)F^{k_i'+1}(t)}{1-F^{k_i'+1}(t)} \cdot \frac{f(t)}{F(t)} \leq \sum_{i=1}^{n} \frac{(k_i+1)F^{k_i+1}(t)}{1-F^{k_i+1}(t)} \cdot \frac{f(t)}{F(t)} = \lambda_{T(\boldsymbol{k})}(t).$$

This complete the proof.

Theorem 3.1 indicates that, in terms of hazard rate order, put redundancies can make the system more reliable. A natural question now is: can such a quite reasonable conclusion hold in terms of likelihood ratio order? To investigate this question, we consider the simplest situation, the series systems with two components.

4 Likelihood ratio order for series systems with two nodes

For a series with two components, Hu and Wang [1] and Misra *et al.* [3] independently proved that,

$$T(k_1, k_2) \leq_{rh} T(k_1', k_2') \quad \text{whenever} \;\; (k_1, k_2) \succ (k_1', k_2'). \tag{4.1}$$

Hu and Wang [1] proposed an open problem whether the result (4.1) may be strengthened to the likelihood ratio order. Zhao *et al.* [9] provided a solution to this open problem. They showed that,

$$T(k_1, k_2) \leq_{lr} T(k_1', k_2') \quad \text{whenever} \;\; (k_1, k_2) \succ (k_1', k_2'). \tag{4.2}$$

This result shows, by moving components to make the system more balanced can make the system more reliable in terms of likelihood ratio order. By (4.2), we know, $T(2,8) \leq_{lr} T(3,7) \leq_{lr} T(4,6) \leq_{lr} T(5,5)$. The question now is, put some redundancies can increase the reliability of the system in terms of likelihood ratio order? The following figure shows the likelihood ratios of $T(2,3)$ over $T(2,2)$, and that of $T(2,4)$ over $T(2,3)$.

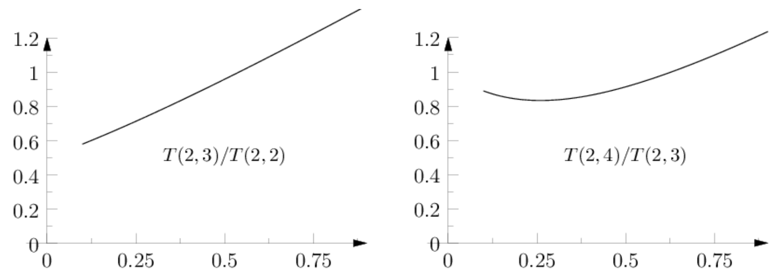

Fig. 1: Compare $T(2,2)$, $T(2,3)$, and $T(2,4)$

From the figure, we can see, if the original series is balanced, that is, the two nodes are the same, then, by adding some redundancies to a node can increase the reliability of the system in terms of likelihood ratio order. However, if the original series is not balanced, then by adding some redundancies to a node with more components will not increase the reliability in terms of likelihood ratio order. The following theorem confirms such a conclusion.

Theorem 4.1. *We have, $T(k,k+a) \geq_{lr} T(k,k)$. But there is no likelihood ratio order between $T(k,k_1)$ and $T(k,k_2)$ when $k < k_1 < k_2$, here k, a, k_1 and k_2 are all positive integers.*

Proof. Denote $U = T(k,k+a)$ and $V = T(k,k)$, and $G(t) = f_U(t)/f_V(t)$. We have,

$$\begin{aligned} G(t) &= \frac{(k+a)F^{k+a}(t)(1-F^k(t)) + kF^k(t)(1-F^{k+a}(t))}{2kF^k(t)(1-F^k(t))} \\ &= \frac{k+a}{2k}F^a(t) + \frac{k(1-F^{k+a}(t))}{2k(1-F^k(t))}. \end{aligned}$$

By Lemma 3, $G(t)$ is increasing, and thus, $T(k,k+a) \geq_{lr} T(k,k)$.

Without much notation confusion, we denote $U = T(k,k_2)$ and $V = T(k,k_1)$, and $G(t) = f_U(t)/f_V(t)$. This time,

$$G(t) = \frac{kF^k(t) + k_2F^{k_2}(t) - (k+k_2)F^{k+k_2}(t)}{kF^k(t) + k_1F^{k_1}(t) - (k+k_1)F^{k+k_1}(t)}.$$

Let $x = F(t)$ and consider the function

$$P(x) = \frac{k + k_2x^{k_2-k} - (k+k_2)x^{k_2}}{k + k_1x^{k_1-k} - (k+k_1)x^{k_1}}, \quad 0 \leq x < 1.$$

By some algebra,

$$\begin{aligned} P'(x) &\overset{\text{sgn}}{=} -kk_1(k_1-k)x^{k_1-k} + kk_2(k_2-k)x^{k_2-k} - k_1k_2(k_2-k_1)x^{k_1+k_2-2k} \\ &\quad - kk_2(k+k_2)x^{k_2} + kk_1(k+k_1)x^{k_1} \\ &\quad - (k_2-k_1)[k(k_1+k_2)+2k_1k_2-k^2]x^{k_1+k_2-k} + (k_2-k_1)(k+k_1)(k+k_2)x^{k_1+k_2} \\ &\overset{\text{sgn}}{=} -kk_1(k_1-k) + kk_2(k_2-k)x^{k_2-k_1} - k_1k_2(k_2-k_1)x^{k_2-k} \\ &\quad - kk_2(k+k_2)x^{k_2-k_1+k} + kk_1(k+k_1)x^k \\ &\quad - (k_2-k_1)[k(k_1+k_2)+2k_1k_2-k^2]x^{k_2} + (k_2-k_1)(k+k_1)(k+k_2)x^{k_2+k}. \end{aligned}$$

Since $-kk_1(k_1-k) < 0$, so, for some small enough x, we have $P'(x) < 0$. Hence, $T(k,k_2) \geq_{lr} T(k,k_1)$ will not hold.

We find the result $T(k,k+a) \geq_{lr} T(k,k)$ in Theorem 4.1 can be generalized to,

Theorem 4.2. *For any $k \leq k_1$, $k \leq k_2$, $T(k_1,k_2) \geq_{lr} T(k,k)$.*

Proof. The density ratio function of $T(k_1,k_2)$ over $T(k,k)$ is,

$$
\begin{aligned}
R(t) &= \frac{k_1F^{k_1}(1-F^{k_2})+k_2F^{k_2}(1-F^{k_1})}{2kF^k(1-F^k)} \\
&= \frac{k_1}{2k}F^{k_1-k}\frac{1-F^{k_2}}{1-F^k}+\frac{k_2}{2k}F^{k_2-k}\frac{1-F^{k_1}}{1-F^k},
\end{aligned}
$$

where $F=F(t)$ as before. By Lemma 3, this function is increasing in F, and then increasing in t. Therefore, $T(k_1,k_2)\geq_{lr} T(k,k)$.

Combine Theorem 4.1 and Theorem 4.2, we have,

Corollary 4.1. *For positive integers k_1 and k_2, denote $\bar{k}=(k_1+k_2)/2$, then, $T(k_1',k_2')\geq_{lr} T(k_1,k_2)$, if $k_1'\geq\bar{k}, k_2'\geq\bar{k}$.*

Proof. From Theorem 4.2, we know $T(k_1',k_2')\geq_{lr} T(\bar{k},\bar{k})$, and from (4.2), $T(\bar{k},\bar{k})\geq_{lr} T(k_1,k_2)$.

Based on Theorem 3.1, we know $T(k_1,k_2)\leq_{rh} T(k_1',k_2')$ whenever $(k_1,k_2)\overset{w}{\succ}(k_1',k_2')$. We strongly believe such a result can be extended to likelihood ratio order. However, the whole proof has not been available. So we put this as a conjecture.

Conjecture. *If $k_1=k_2$, then, $T(k_1,k_2)\leq_{lr} T(k_1',k_2')$ when $k_1'\geq k_1$ and $k_2'\geq k_2$. If $k_1<k_2$, then, $T(k_1',k_2')\geq_{lr} T(k_1,k_2)$ when $k_1'>k_1$ and $k_1'+k_2'\geq k_1+k_2$.*

A computer simulation has been conducted to check the conjecture. In this simulation, we set $(k_1,k_2)=(2,8)$. We locate all points (k_1',k_2') such that $T(k_1',k_2')\geq_{lr} T(k_1,k_2)$ holds. Since $T(k_1',k_2')=T(k_2',k_1')$, so, by symmetry, we assume $k_1'\leq k_2'$. The figure below is the simulation result.

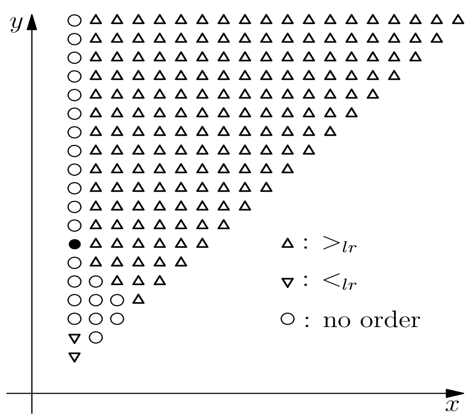

Fig. 2: Compare the policy $(2,8)$ with others

From the figure, we can see that the computer simulation agrees with the conjecture quite well. As we can see, between $T(2,8)$ and $T(2,9)$, or $T(2,7)$, there is no likelihood ratio order. But $T(2,8)\geq_{lr} T(2,2)$, as stated in Theorem 4.2. Interestingly, we see $T(2,8)\geq_{lr} T(2,3)$. The reason is the point $(2,3)$ is close to $(2,2)$.

5 Likelihood ratio order for series systems with *n* nodes

We believe most results in Section 4 for the series systems with two nodes can be generalized to the series system with n nodes. For instance, for two systems with the same smallest nodes, there is usually no likelihood ratio order between them. If we make the system more balanced by either putting some redundancies or by moving some components from other nodes, the system will become more reliable in terms of likelihood ratio order. So, we should have, $T(2,3,4)<_{lr} T(3,3,3)$, $T(2,4,6)<_{lr} T(3,4,5)$, $T(2,4,6)<_{lr} T(3,5,7)$, but there may have no likelihood ratio order between $T(2,3,5)$ and $T(2,4,4)$. The following figures confirm our speculations.

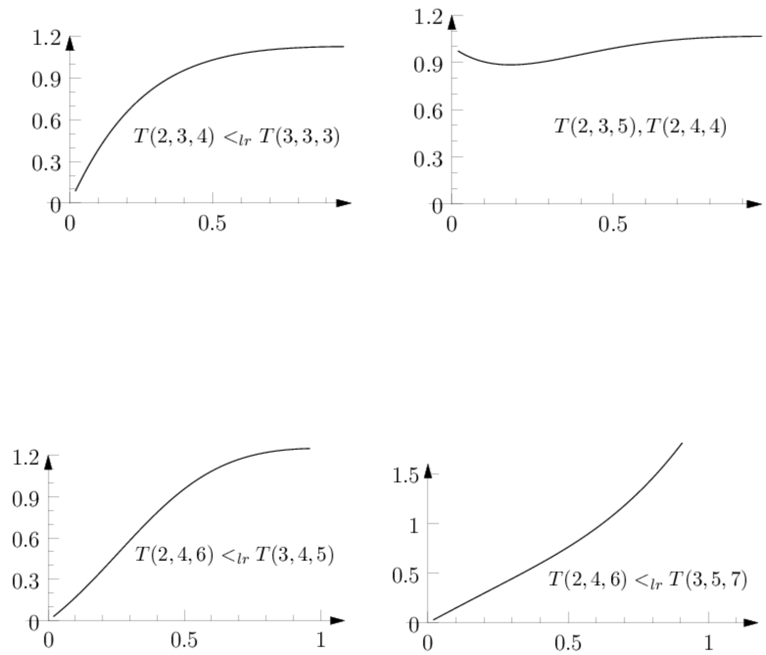

Fig. 3: Policy comparison with three nodes

Now we prove a generalization of Theorem 4.3.

Theorem 5.1 *For positive integers $k_1, \cdots, k_n$, denote $\bar{k} = (k_1 + \cdots + k_n)/n$, then,*

$$T(k_1', \cdots, k_n') \geq_{lr} T(k_1, \cdots, k_n),$$

if $k_i' \geq \bar{k}$, for $i = 1, \cdots, n$

Proof. By Theorem 4.1 in Zhao *et al.* [9], $T(k_1, \cdots, k_n) \leq_{lr} T(\bar{k}, \cdots, \bar{k})$. So, we just need to show $T(k_1', \cdots, k_n') \geq_{lr} T(\bar{k}, \cdots, \bar{k})$.

Consider the function

$$\begin{aligned} G(x) &= \frac{\sum_{i=1}^n k_i' x^{k_i'} \prod_{j \neq i} (1 - x^{k_j'})}{\bar{k} x^{\bar{k}} (1 - x^{\bar{k}})^{n-1}} \\ &= \sum_{i=1}^n \frac{k_i'}{\bar{k}} x^{k_i' - \bar{k}} \prod_{j \neq i} \left(\frac{1 - x^{k_j'}}{1 - x^{\bar{k}}} \right). \end{aligned}$$

By Lemma 4, the function $(1 - x^{k_j'})/(1 - x^{\bar{k}})$ is increasing, so is the function $G(x)$. Since $G(F(t))$ is the density ratio of $T(k_1', \cdots, k_n')$ over $T(\bar{k}, \cdots, \bar{k})$, the theorem is thus proved.

We conjecture the Theorem 3.1 can be strengthened to likelihood ratio order, by posing an extra requirement that $k_1 < k_1'$. So far, we have not been able to provide a proof for this result, and thus leave this as an open problem.

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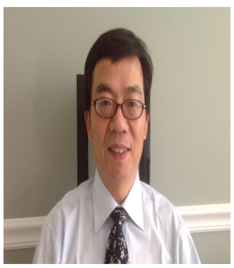

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