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On a Method of Finding Homoclinic and Heteroclinic Orbits in Multidimensional Dynamical Systems

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The presence of bounded orbits such as homoclinic and heteroclinic orbits are important in a dynamical system. Chaotical behavior or the presence of periodic orbits of a system is often preceded by destruction of a homoclinic or heteroclinic orbit. In this work we give insights on a method of detecting homoclinic or heteroclinic orbits in a three-dimensional dynamical system. The method can also be applied for multidimensional systems.

Keywords: Dynamical systems, heteroclinic orbits, chaos.

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1 Introduction and Necessary Theoretical Notions

In this paper we present insights on a method dealing with detecting homoclinic or heteroclinic orbits in a dynamical system generated by differential equations. The idea is to trace the separatrices (stable and unstable manifolds) until they hit a given surface and then look for conditions such that the separatrices meet one to another on this surface. Finding homoclinic and heteroclinic orbits in a given dynamical system is not an easy task but their presence tell us much about the behavior of the system. For example, destroying a homoclinic or heteroclinic orbit can bring a system in chaos in some cases while in others it leads only to the appearance of periodic orbits.

Consider a surface S in $\mathbb{R}^3$ given by $S:\ F(x,y,z)=0$, $F\in C^1(\mathbb{R}^3)$. A normal vector to S at a point $(x,y,z)\in S$ is the gradient vector $\text{grad}F(x,y,z)=(F'_x,F'_y,F'_z)$. The surface S splits the space in two regions according to $F>0$ (the positive region) or $F<0$ (the negative region) and assume the normal vector $\text{grad}F(x,y,z)$ lies in the positive region.

If a trajectory (for t increasing) of a differential system

$$\dot{u} = f(u), \tag{1.1}$$

where $\dot{u} = (\dot{x}, \dot{y}, \dot{z})$, $f(u) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$, hits the surface S at a point $A(x_0, y_0, z_0)$ coming from the positive region $F > 0$, then the vector field f at A (which is tangent to the trajectory) makes with the gradient vector grad$F$ at A an angle larger than $90°$, see Fig.1.1. Since

$$\cos(\text{grad}F, f) = \frac{\text{grad } F \cdot f}{|\text{grad } F| \cdot |f|},$$

it follows that grad $F \cdot f < 0$ at A. On the other hand,

$$\frac{d}{dt}F(x, y, z)|_{F=0} = F_x\dot{x} + F_y\dot{y} + F_z\dot{z}|_{F=0} = \text{grad}F \cdot f < 0, \text{at } A.$$

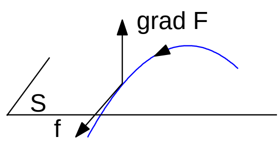

Figure 1.1: An orbit tangent to the vector field f at a point A, entering transversally the surface $S : F(x, y, z) = 0$ coming from the positive region $F > 0$, makes with the gradient vector grad$F$ at A an angle larger than 90^0.

As a conclusion, we have

Proposition 1.1. *If the derivative (for t increasing) along a trajectory of the system (1.1),*

$$\frac{d}{dt}F(x, y, z)|_{F=0} < 0,$$

then the trajectory hits the surface $S : F(x, y, z) = 0$ coming from the positive region $F(x, y, z) > 0$. If

$$\frac{d}{dt}F(x, y, z)|_{F=0} > 0,$$

then it crosses the surface $S : F(x, y, z) = 0$ coming from the negative region $F(x, y, z) < 0$. For t decreasing, the scenario is inversely, i.e. if

$$\frac{d}{dt}F(x, y, z)|_{F=0} < 0,$$

the trajectory hits the surface S coming from the negative region and if

$$\frac{d}{dt}F(x, y, z)|_{F=0} > 0,$$

the trajectory hits the surface S coming from the positive region.

Remark 1.1. If

$$\frac{d}{dt}F(x,y,z)|_{F=0} \neq 0,$$

the surface S is called without contact for trajectories of the system or a transversally section to the flow of the system. If

$$\frac{d}{dt}F(x,y,z)|_{F=0} = 0$$

then the trajectory is included in S or tangent to it.

2 Tracing the Unstable Manifold W_+^u

We will apply the method for detecting homoclinic orbits only but similarly it can be applied for heteroclinic orbits. The system considered in this work is a Shimizu-Morioka-like $3D$ model given by

$$\dot{x} = y,\ \dot{y} = (a+1)(1-z)x - ay,\ \dot{z} = x^2 - z. \tag{2.1}$$

Applications of Shimizu-Morioka models are suggested in [2]. In [3, 4] two types of Lorenz-like attractors of this system are shown. Relationships with the theory of normal forms are reported in [6, 8]. Computer–assisted results on the behavior of the system are obtained in [5].

The equilibrium points of the system are $O(0,0,0), O_1(1,0,1), O_2(-1,0,1)$. As the system is invariant under the transformation $x \to -x, y \to -y$, its orbits are symmetrically with respect to the Oz axis, so, with no loss of generality, we may restrict our investigations to $x > 0$. Since a homoclinic or heteroclinic orbit is bounded and for $z < 0$ the derivative $\dot{z} > 0$, they will lie in $z > 0$, so we may restrict further to $z > 0$. The Jacobian matrix associated to the system at $O(0,0,0)$ has the eigenvalues $1, -1, -a-1$ and the corresponding eigenvectors $(1,1,0)$, $(0,0,1)$, $(-1/(a+1), 1, 0)$, so $O(0,0,0)$ is a saddle. Assume $a+1 > 0$. Therefore the system has a one-dimensional unstable manifold W_0^u and a two-dimensional stable manifold W_0^s passing through $O(0,0,0)$. We have also the one-dimensional tangent space to W_0^u at $O(0,0,0)$,

$$TW_0^u = \{(x,y,z) : x = y, z = 0\}$$

and the two-dimensional tangent space to W_0^s at $O(0,0,0)$,

$$TW_0^s = \{(x,y,z) : y + x\,(a+1) = 0, z \in \mathbb{R}\}\,.$$

The Jacobian matrix at $O_{1,2}$ has the characteristic polynomial

$$p^3 + (a+1)\,p^2 + ap + 2a + 2 = 0.$$

From Routh-Hourwitz conditions, the characteristic polynomial has all roots with negative real part if and only if $a > 2$. Notice that the positive part of Oz axis, denoted by d^+, belongs to W_0^s.

Denote W_+^u the branch of W_0^u lying in $y > 0$ and W_-^u the branch of W_0^u lying in $y < 0$.

Consider the equation of W_+^u in the neighborhood of $O(0,0,0)$ given by

$$\begin{aligned} y &= a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ z &= b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \end{aligned} \tag{2.2}$$

As $W_0^u = W_+^u \cup W_-^u$ is invariant under the action of the flow of the system, after computing the coefficients, equations of W_+^u for x small enough, are given by

$$y = x + \cdots\ ,\ z = \frac{1}{3}x^2 + \cdots\ . \tag{2.3}$$

So for x small enough, W_+^u lies in the region

$$R = \Big\{(x,y,z)\colon 0 < y < x,\ \frac{1}{3}x^2 < z < x^2,\ x > 0\Big\}.$$

In [7] we proved that W_+^u lies in the region $x^2/3 < z < x^2$ not only for x small but for all $x > 0, y > 0$.

We want to estimate now the $x-$coordinate of the point where W_+^u crosses $y = 0$.

Consider the surface in $\mathbb{R}^3$, $y = x - bx^2, 0 < z < x^2, b > 0$. As $x - bx^2 < x, b > 0$ the surface $y = x - bx^2$ is below W_+^u for x small enough.

We are interested under what conditions, trajectories lying in $x^2/3 < z < x^2$ can cross this surface.

Proposition 2.1. *The surface in R^3, $y = x - bx^2, 0 < z < x^2, b > 0$ is a surface without contact for trajectories of the system. If t increases, trajectories of the system cross the surface on the side*

$$y < x - bx^2 \quad \textit{for all } 0 < x < \frac{b(a+3)}{a+1+2b^2},\ b > 0.$$

In particular, if $b = 1$ we obtain that the trajectories of the system (and implicitly W_+^u) for $0 < z < x^2$ cross $y = 0$ for $x \geq 1$.

Proof. if

$$x < \frac{b\,(a+3)}{a+1+2b^2} := x_u^1,$$

we have

$$\frac{d}{dt}\left(y - x + bx^2\right)\big|_{y=x-bx^2} > -x^2\left(x\left(a+1+2b^2\right) - b\,(a+3)\right) > 0.$$

We used $z < x^2$. So the surface $y = x - bx^2, 0 < z < x^2$ is without contact for trajectories of the system and for t increases, trajectories of the system cut the surface on the side

$y < x - bx^2$ for all $0 < x < x_u^1, b > 0$. Notice that $x_u^1 \geq 1/b$ for all $b \geq 1$ and all $a + 1 > 0$. Consequently, $x_u^1 = 1$ is the inferior (and maximum in this estimation) limit which is valid for all $a + 1 > 0$ such that trajectories cross the plane $y = 0$ for $x > 1$. $\square$

Consider now a curve above W_+^u, $y = cx - dx^2, c > 1, d > 0$. One may remark that for any $d > 0,\ c > 1$ we have $cx - dx^2 > x$ for x small enough, more exactly if $x < (c-1)/d$. So the surface $y = cx - dx^2$ is above W_+^u for x small enough.

Proposition 2.2. *There exists $c > 1, d > 0, c/d = \sqrt{3}$ such that the surface in $\mathbb{R}^3$,*

$$y = cx - dx^2,\ \frac{1}{3}x^2 < z$$

is a surface without contact for trajectories of the system for all $x > 0$. If t increases, trajectories of the system cross the surface for all $a > 2, a^2 - 8a - 8 > 0$ on the side $y > cx - dx^2$. Consequently, for $a > 2, a^2 - 8a - 8 > 0$ trajectories (and implicitly W_+^u) starting in

$$y < cx - dx^2, \quad \frac{1}{3}x^2 < z$$

cross $y = 0$ for $x \leq \sqrt{3}$.

Proof. We have

$$\frac{d}{dt}\left(y - cx + dx^2\right)\big|_{y=cx-dx^2} \tag{2.4}$$
$$< -x\left(x^2\left(2d^2 + \frac{a+1}{3}\right) - d\left(a + 3c\right)x + c^2 + ac - a - 1\right) < 0$$

for all $x > 0$ if

$$d^2\left(a + 3c\right)^2 - 4\left(2d^2 + \frac{a+1}{3}\right)\left(c^2 + ac - a - 1\right) \leq 0,$$

or equivalently

$$0 < d \leq \sqrt{\frac{4\left(a+1\right)\left(c-1\right)\left(a+c+1\right)}{3\left(\left(a-c\right)^2 + 8\left(a+1\right)\right)}} := d_m.$$

As the intersection of $y = cx - dx^2$ with Ox is 0 and c/d, we are interested in the maximum of d in order to see how low on Ox trajectories can cross $y = 0$. So take $d = d_m$ and estimate the minimum of c/d for $c > 1, a + 1 > 0$.

Denoting $p = c^2/d^2$, we obtain

$$\frac{\partial p}{\partial c} = \frac{3}{4}\frac{\left(2c^2 + 3ac - 8a - a^2 - 8\right)\left(c^2 - ac + 2a + 2\right)c}{\left(a+c+1\right)^2\left(a+1\right)\left(c-1\right)^2}.$$

Further, $\partial p/\partial c = 0$ implies $ac - 2a - c^2 - 2 = 0$ with

$$c_1 = \frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 8a - 8}, c_2 = \frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 8a - 8},\ a^2 - 8a - 8 \geq 0$$

and $8a - 3ac + a^2 - 2c^2 + 8 = 0$ with

$$c_3 = -\frac{3}{4}a + \frac{1}{4}\sqrt{64a + 17a^2 + 64}, c_4 = -\frac{3}{4}a - \frac{1}{4}\sqrt{64a + 17a^2 + 64}$$

for any $a + 1 > 0$.

Case 1. If $a^2 - 8a - 8 > 0$ and $a > 2$ then we have

$$c_4 < 1 < c_2 < c_3 < c_1.$$

Evaluating the sign of $\partial p/\partial c$ for c on $(1, +\infty)$ we get that c_1 and c_2 are the minimum points. Evaluating also p at c_1 and c_2 we get surprisingly,

$$p(c_1) = p(c_2) = 3, \text{ for any } a > 2, a^2 - 8a - 8 > 0.$$

So

$$x_u^2 := c/d = \sqrt{3}$$

is the limit for $a > 2, a^2 - 8a - 8 > 0$, with the property that trajectories can cross y=0 only for $x < x_u^2$.

Case 2. If $a^2 - 8a - 8 < 0$ then $c_{1,2}$ are not real anymore and $c_4 < 1 < c_3$ for any $a + 1 > 0$. So the minimum in this case is $c = c_3$ and one can show that the minimum of $x_u^2 = c/d$ is also $\sqrt{3}$.

Case 3. If $a + 1 > 0, a < 2$ and $a^2 - 8a + 8 > 0$ then $c_{1,2}$ are real again but $c_{1,2} < 1$. We keep still $c_4 < 1 < c_3$ for any $a + 1 > 0$ and again $c = c_3$ is the minimum. With these constraints of a, $x_u^2 = c/d$ is decreasing with respect to a, raging from $+\infty$ to $\sqrt{3}$. So these two last cases do not offer a better option for x_u^2. □

Remark 2.1. The above Propositions 2.1-2.2 say that, the separatrix W_+^u crosses the plane $y = 0$ in a point $K(x_u, 0, z_u)$ with $1 < x_u < \sqrt{3}$ if a is large enough (in fact for $a^2 - 8a - 8 > 0$ and $a > 2$).

3 Tracing the Stable W_0^s Manifold

As the positive part of the Oz axis, denoted d^+, belongs to W_0^s, we express W_0^s in a small neighborhood of some parts of d^+ in the form $y = -f(t)x, z = z_0 e^{-t}, t > 0$. Denote $TW^s|_A$ the tangent space to W_0^s at the point A. In [7] we proved the following assertions.

Proposition 3.1. *a) There exists a sequence*

$$z_0, z_1, \ldots, z_k, \ldots,\ z_{k+1} > z_k,\ z_0 > 1 + \frac{a^2}{4(a+1)}$$

such that the tangent space

$$TW^s\,|_{A_k} = \{(x, y, z) : y = 0\},$$

where $k = 0, 1, 2, \ldots$ *and* $A_k = (0, 0, z_k)$.
b) For $z < z_0$ *we express* $TW^s\,|_A = \{(x, y, z) : y = -f_0(t)x, z = z_0e^{-t}, t > 0\}$, *where* $A = (0, 0, z)$ *and* $f_0(t)$ *is a solution of the Riccati equation (generally unsolvable)*

$$f'(t) = f^2(t) - af(t) - (a+1)(1 - z_0e^{-t}), \tag{3.1}$$

which is bounded for $t > 0, f_0(0) = 0$ *and such that*

$$\frac{1}{2}\Big(a + \sqrt{(a+2)^2 - 4(a+1)z_0e^{-t}}\Big) < f_0(t) < a+1, t > t_0 > 0, \tag{3.2}$$

where

$$t_0 := \ln\frac{4(a+1)z_0}{(a+2)^2} > 0.$$

This result implies that in the neighborhood of the integral line $d^+ : \{x = y = 0, z > 0\}$, the two-dimensional manifold W^s for $z \leq z_0$ can be expressed as

$$\{y = -f_0(t)x, z = z_0e^{-t}, t \geq 0\}.$$

So for $0 < \varepsilon \ll 1$ we can find a curve

$$C = W^s \cap \{x^2 + y^2 = \varepsilon^2,\ 0 \leq z \leq z_0\}.$$

Proposition 3.2. *Consider the region in* $\mathbb{R}^3$, $R_1 = \{-(a+1)x < y < 0\}$ *where* $y = 0$ *and* $y = -(a+1)x$ *are seen as surfaces in* $\mathbb{R}^3$. *Then the curve* $C \subset R_1$ *for* ε *small enough. In addition, for decreasing* t*, each trajectory starting on* C *up to its intersection with the plane* $y = 0$*, remains in the region* R_1 *and will cross this plane only for* $z > 1$.

Proof. It is clear that the curve C lies in R_1 because $x > 0, f_0(t) > 0, t > 0$ so $y < 0$ and from $f_0(t) < a+1, t > 0$ we get $y + (a+1)x > 0$. Recall that $TW_0^s = \{(x, y, z) : y + x\,(a+1) = 0, z \in \mathbb{R}\}$. We have also $\frac{d}{dt}\,(y + (a+1)x)\,\big|_{y=-(a+1)x} = -(a+1)xz \neq$ for all $xz \neq 0$. Hence, the surface $y + (a+1)x = 0$ is a surface without contact for trajectories of the system and the trajectories, for t decreasing, cross the surface on the side $y + (a+1)x < 0$ for any $xz > 0$. From $\frac{d}{dt}y\,|_{y=0} = (a+1)(1-z)x < 0$ if $x > 0, z > 1$, we have that, for t decreasing, trajectories can cross $y = 0$ on the side $y < 0$ only if $x > 0, z > 1$. □

Consider in the following a surface in $\mathbb{R}^3$ given by $y = bx^2 - cx, b > 0, c > a+1$ with $z > 0$. As $bx^2 - cx < -(a+1)x$ for x small enough, this surface is below W_0^s for x small enough. Evaluate,

$$\begin{aligned}\frac{d}{dt}(y - bx^2 + cx)\,\big|_{y=bx^2-cx} &= -x\left(z - a - ac + az + abx - 3bcx + c^2 + 2b^2x^2 - 1\right)\\ &< 0\end{aligned} \tag{3.3}$$

as long as $x > 0$ and

$$2b^2x^2 + b(a-3c)x + z - a - ac + az + c^2 - 1 > 0. \tag{3.4}$$

Relation (3.4) holds true if and only if

$$b^2\left(8a - 8z + 2ac - 8az + a^2 + c^2 + 8\right) < 0,$$

that is, if $z > (8a + 2ac + a^2 + c^2 + 8)/(8 + 8a)$. As the minimum of $(8a + 2ac + a^2 + c^2 + 8)/(8 + 8a)$ for $c > a + 1 > 0$ is attained at $c = -a$, if $-1 < a < -1/2$ we can choose $c = -a > a + 1 > 0$. This leads to $z > 1$. So if $-1 < a < -1/2$, there exists $c = -a > a + 1$ such that $\frac{d}{dt}(y - bx^2 + cx)\big|_{y=bx^2-cx} < 0$ for any $x > 0, z > 1$ and for any $b > 0$. Therefore, the surface in $\mathbb{R}^3$, $y = bx^2 + ax,\ b > 0,\ -1 < a < -1/2$ with $z > 0,\ z \neq 1, x > 0$ is a surface without contact for trajectories of the system and the trajectories for t decreasing cross this surface for $z > 1$ on the side $y < bx^2 + ax$. As the curve C contains points below $z = 1$, some trajectories starting on it may cross the surface $y = bx^2 + ax$ before meeting $z = 1$, so for such trajectories we have no boundary. But, take $b = -a > 0$. The nonzero intersection of the curve $y = -ax^2 + ax$ with the Ox axis is $x = 1$. Make now the following assumption:

Assumption A. *Assume that a trajectory lying in $z > x^2$, starting on C and remaining in $z < 1$ before to cross the surface $y = -ax^2 + ax$ (if enters $z > 1$, it can not cross the surface anymore), will cross first the surface $z = x^2$.*

With this assumption at hand, such a trajectory after crossing the surface $z = x^2$, gives $z(t)$ decreasing and $x(t)$ increasing ($\dot{z} > 0, \dot{x} < 0$), that is, the trajectory remains in $z < 1$ and it can not cross $y = 0$ (see Prop. 3.2) so it may escape on the surface $z = 0$.

As the trajectories starting in $y > -ax^2 + ax,\ x > 0$ (in particular those starting on C) can not cross $y = -ax^2 + ax$ for any $x > 0, z > 1$ they will cross $y = 0$ in a point $(x, 0, z)$ such that $x < 1, z > 1$. A trajectory, for t decreasing, starting on C at a point such that $z > 1$ increases in $y(t)$ because $\dot{y} < 0$. Also it increases in $x(t)$ and in $z(t)$ as $\dot{x} < 0, \dot{z} < 0$ so it will cross surely the plane $y = 0$. It would decrease in $z(t)$ after crossing the plane $z = x^2$ which is not possible because the surface $z = x^2, x \in (0,1)$ is below the surface $z = 1, x \in (0,1)$. If $-1/2 \leq a \leq 0$, we can not find anymore c and b such that $\frac{d}{dt}(y - bx^2 + cx)\big|_{y=bx^2-cx} < 0$, so we only can infer that the curve C lies in $z > 1$ in this case. Summarizing, we have the following useful intuitive conjecture. We can not give it as a result since the trajectory may hit the surface $y = -ax^2 + ax$ first and then $z = x^2$.

Conjecture 1. *Consider $-1 < a \leq 0$. Then for t decreasing and $-1 < a < -1/2$, the* *trace of W_0^s through the flow of the system on the plane $y = 0$ is a curve lying completely in $x < 1,\ z > 1$ and for $-1/2 \le a \le 0$ lying in $z > 1$.*

The evolution of the curve C through the flow of the system for $a \ge 0$ is summarized in the following result.

Theorem 3.1. *If $a > 0$ the trace of the curve C through the flow of the system on the plane $y = 0$, is a curve Γ_1 joining $A_0(0, 0, z_0)$ with a point $B_1(x, 0, x^2)$, lying on the curve $z = x^2$, and a point $B_2(x, 0, 1),\ x > \sqrt{z_0}$ lying on $z = 1$.*

Proof. As $a > 0$ we have $z_0 > 1$. For t decreases, trajectories starting on C for $z > x^2$ and for $S := (a+1)(1-z)x-ay < 0$ will increase in all three directions as $\dot{x} < 0, \dot{y} < 0, \dot{z} < 0$ and some of them will cross the plane $y = 0$ and others could meet the surface $z = x^2$. Considering $z \ge 0$ arbitrarily fixed, the surface S is generated by the lines $y = mx$ with the slope $m = (a + 1)(1 - z)/a$, see Fig. 3.1. On the other hand, if $S > 0$ then they will decrease in $y(t)$, and if not meet $S = 0$ they will cross the surface $z = x^2$. So we obtained a curve Γ joining the points A_0, B_1 and O which belongs also to W_0^s. The part of the curve Γ which does not lie in $y = 0$ and which is above the plane $z = 1$ is further translated through the flow of the system in a curve Γ_1 on the plane $y = 0$. Recall that from Proposition 3.2 trajectories can cross the plane $y = 0$ only for $z > 1$. The curve Γ_1 ends in $y = 0, z = 1$ because at $B_2(x, 0, 1)$ the trajectory is tangent to the plane $y = 0$ as $\dot{y}\,|_{y=0,z=1} = 0$ and after the tangency it returns to the region where $y < 0$ if $x > 1$ or crosses the plane $y = 0$ if $x < 1$. If $x = 1$ it remains there as $(1, 0, 1)$ is an equilibrium point. A similar argument is used in [1] where also is proved that the point B_1 is unique. □

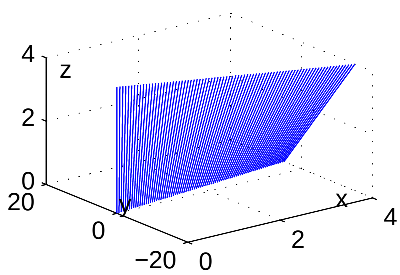

Figure 3.1: Crossing the surface S given by $y = (a+1)(1-z)x/a$, the flow of the system approaches or departs the plane $y = 0$.

Let us now describe the scenario leading to a homoclinic orbit considering the above Conjecture 1 is valid. Denotes in the following by D_+ the region on the plane $y = 0$ defined by the curve Γ_1 and the lines $z = 1,\ z = z_0,\ x = 0$ and by D_- the remaining region in the strip $1 < z < z_0,\ x > 0$.

Define the function $g(a) = \min_{x,z} d(x_u, \Gamma_1) \cdot i(C)$, where x_u is the Ox-coordinate of the point $P(x_u, 0, z_u)$ where W_+^u crosses the plane $y = 0$, $i(C)$ is $+1$ if the point P lies in D_+ and -1 otherwise and $d(x, y)$ is the Euclidian distance in the plane. From Conjecture 1 and Theorem 3.1 we have that the function $g(a)$ is continuous with respect to the parameter a for $a + 1 > 0$ and from Conjecture 1 and from Proposition 2.1 we have in addition $g(a) < 0$ for $a \in (-1, -1/2)$. For $a > 2,\ a^2 - 8a - 8 > 0$ and again from Theorem 3.1 we get that the intersection of the curve Γ_1 with the region on $y = 0$, given by $x^2 > z, z > 1$ is a curve entirely contained in $x > \sqrt{3}$ because $x > \sqrt{z_0} > \sqrt{1 + \frac{a^2}{4(a+1)}}$ while from Proposition 2.2 we have that P satisfies $x_u < \sqrt{3}$ so $P \in D_+$ and $g(a) > 0$. Therefore, there exists a number $a_0 \in (-1/2, 8.9)$ such that $g(a_0) = 0$.

Summarizing, we have the range of the parameter a where the system can have a homoclinic loop.

Conjecture 2. *There exists a value $a_0 \in (-1/2, 8.9)$ of intersection of the curve Γ_1 with W_+^u such that, the system (2.1) corresponding to this value, has a homoclinic loop to the saddle point $O(0, 0, 0)$.*

Numerical investigations reveal indeed a unique point $a_0 \simeq 1.718$ such that the system possesses a homoclinic orbit. In Fig. 3.2 is presented a single homoclinic orbit lying in $x > 0, y > 0, z > 0$ while Fig. 3.3 displays two symmetrical homoclinic orbits to $O(0, 0, 0)$ corresponding to a_0.

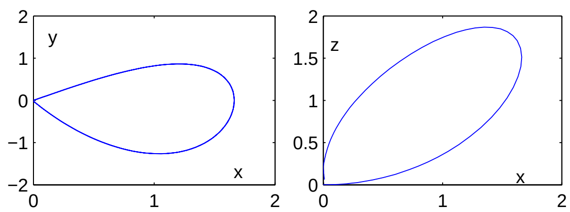

Figure 3.2: One homoclinic orbit of the system for a_0.

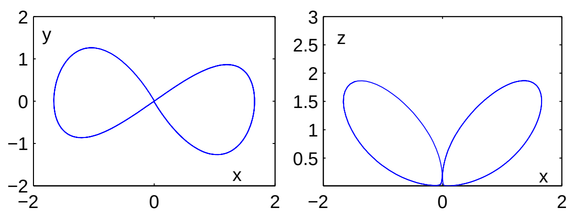

Figure 3.3: Two homoclinic orbits of the system for a_0.

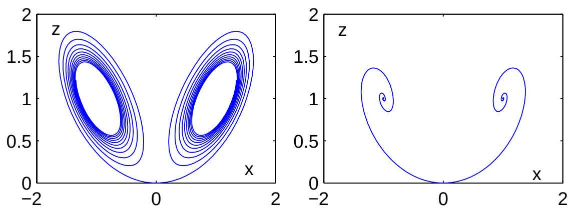

Figure 3.4: Separatrices of the system for $a = 2$ (left) and $a = 10$ (right).

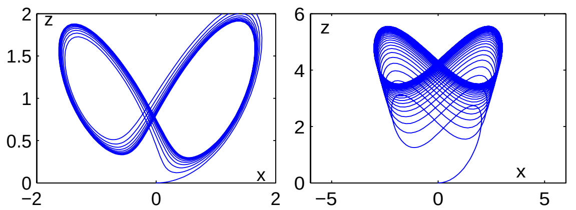

Figure 3.5: Orbits of the system for $a = 1.1$ (left) and $a = 0.5$ (right).

4 Conclusions

Finding out closed orbits in a multidimensional dynamical system is a challenging problem. Of this type of orbits, homoclinic and heteroclinic ones play an important role in understanding the behavior of the system. Under some conditions, homoclinic or heteroclinic bifurcations lead to periodic orbits or chaos in a system. We pointed out here details on a method of detecting homoclinic orbits in a three-dimensional dynamical system. The method is applicable for higher-order dynamical systems. Tracing the separatrices and their intersections with a given surface, we showed that there are conditions for them to meet one to another. Their intersections imply the existence of a homoclinic orbit. While in this paper we registered some analytical results, others we had to left as conjectures. Nevertheless, numerical results offer a good basis for these conjectures. In [7] we started to improve these results.

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