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On Generalized Interest Rate Dynamics

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Abstract: In this paper we consider stochastic interest rate and classify its deterministic part with respect to oscillation and monotonicity, latter according to Kiguradze. We illustrate usefulness of such classification on convenient examples, including feasibility ratio and choice of pension pillar. Interest rate follows a 2nd order quasilinear stochastic differential equation which generalizes a result of [13]. As a side result we obtain Kiguradze characterization of smooth solutions of deterministic part of Parker's stochastic differential equation. We show that Parker's model allows oscillations and better long term behavior of the interest rate in comparison of 1st order interest rate model. In such a setup we study the sensitivity of feasibility ratio to the dynamics of the underlying interest rate. We cover the wide spectrum of life time distributions including the generalized Gamma as well as the Gompertz-Makeham law. We apply obtained results to briefly discuss the situation typical for some Eastern-European countries when many people are members of a pension scheme operated under public management as well as a funded scheme financed by employees contributions.

Keywords: generalized Gamma distribution, Kiguradze class, pension system, quasilinear differential equation, stochastic interest rate

1 Introduction

Many economist as well as financial analysts will disagree with an assumption of constant interest rate. Stochastic interest rates play an important role considering long maturity contracts. This is well-known in the literature, see e.g. [1,19] or [16] and references therein. However, most of the literature uses one-factor interest rate models which do not capture long run trends. In this paper we consider stochastic interest rate, its classification and illustration of its effects on convenient examples, including feasibility ratio and choice of pension pillar. The interest rate is following a 2nd order quasilinear stochastic differential equation, which generalizes differential equation of [13]. This generalization allows interest rate to mimic oscillatory behavior, improve its long-term properties and give opportunity to study faster (slower) diffusion models for interest rate. Thus inclusion of Parker's model in its natural generalization by quasilinear SDE provides us possibility to better study main features of Parker's interest rate and behind. This also modifies the classical models and brings the novelty in the study of several actuarial problems. As a side result we obtain Kiguradze characterization of smooth solutions of deterministic part of Parker's stochastic differential equation. As far as we know, an analogous classification for stochastic differential equations (systems) has not been done yet. Such a classification is important for initial values problems in theory of ordinary differential equations (ODEs). This relates to solutions which can be extended on the semi-axis. Therefore the unique classification obtained by Kiguradze can be well employed to classify deterministic part of stochastic differential equations. From our numerical experiments this classification can play a significant role in an analogous classification for stochastic interest rates.

The paper is organized as follows. In the 2nd section we introduce the model of time-dependent interest rate following 2nd order quasilinear stochastic differential equation, which generalizes a result of [13]. In section 2.1 we characterize smooth solutions of deterministic part of Parker's stochastic differential equation according to Kiguradze. We also discuss main features of introduced

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interest rates. In section 3 we present life time distributions used in this paper, namely Gompertz-Makeham law and generalized Gamma distribution. We derive the feasibility ratios for these life time models and several interest rate models. We show that decreasing deterministic interest rate boosts feasibility ratio in comparison with u/v for constant interest rate, which makes investment in pension fund less attractive for individual person. In the section 3.5 the case of stochastic interest rate is discussed. Therein we will see that size of volatility is crucial for expected feasibility ratio $E(u/v)$ to be grater (lower) in comparison with u/v computed for constant interest rate. The pattern of decreasing u/v for increasing T is well visible also for non-constant (both deterministic and stochastic) interest rates. This justifies the fact (observed for a constant interest rate by [3] that given the age of the member, when the retirement age T increases, the fund can afford to pay a higher pension rate v to the member. In section 4 we illustrate the impact of both constant and non-constant interest rates on pensions with the real data from Slovakia. Technicalities and proofs are put in Appendix.

2 Model of interest rate

[13] assumes that interest rate (or force of interest) $r(t)$ can be modeled, for $t \geq 0$, by a linear second-order stochastic differential equation (SDE) of the form

$$\mathrm{d}\left(\frac{\mathrm{d}r_t}{\mathrm{d}t}\right) = \alpha_1\,\mathrm{d}r_t + \alpha_0 r_t\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t, \tag{1}$$

where σ is nonconstant volatility. One can divide this type of equation into deterministic part (ODE) and stochastic noise (Wiener process). For technical purposes, we rewrite deterministic part of (1) in the Sturm-Liouville form (as differential equation, DE)

$$(p(t)r')' - \alpha_0\,p(t)r = 0, \tag{2}$$

where $p(t)$ is defined as

$$p(t) = \mathrm{e}^{-\alpha_1 t}. \tag{3}$$

To be able to better classify Parker's interest rate we modify both (2) and (1) into a quasilinear DE and SDE, respectively, by adjustment of derivatives together with multiplying the volatility by powered interest rate term. It is formalized by means of a signed-power function $\Phi_\gamma(z) = |z|^{\gamma-1}z,\ \gamma > 0$.
Thus we arrive to the following two nonlinear DE and SDE for interest rate

$$\left(p(t)\,\Phi_\gamma(r')\right)' - \alpha_0\,p(t)\Phi_\gamma(r) = 0, \tag{4}$$

$$\mathrm{d}\left(p(t)\Phi_\gamma\left(\frac{\mathrm{d}r_t}{\mathrm{d}t}\right)\right) - \alpha_0 p(t)\,\Phi_\gamma(r_t)\,\mathrm{d}t -$$

$$-\sigma(t)\Phi_{\alpha_3}(r_t)\,\mathrm{d}W_t = 0. \tag{5}$$

Notice that (4) is quasilinear DE and (5) is quasilinear SDE which is apparently generalization of Parker model (1) obtained from (5) for $\gamma = 1$ and $\alpha_3 = 0$.

Definition 1 *By deterministic interest rate we consider a weak solution of differential equation (4), i.e. the derivative of interest rate exists up to 2nd order in a weak sense (see e.g. Chapter 5.4 of [6]). By stochastic interest rate we understand arbitrary solution of stochastic differential equation (5).*

Our model (5) can be expressed in the classical form of the stochastic differential equation (system)

$$\begin{aligned}\mathrm{d}r_t &= \Phi_\gamma^{-1}\left(\frac{s_t}{p(t)}\right)\mathrm{d}t\\ \mathrm{d}s_t &= \alpha_0 p(t)\Phi_\gamma(r_t)\,\mathrm{d}t + \sigma(t)\Phi_{\alpha_3}(r_t)\,\mathrm{d}W_t\end{aligned} \tag{6}$$

Using e.g. following theorem we can directly obtain uniqueness result. We write $|Z|^2 = \sum_{i,j}|Z_{ij}|^2$ for matrix Z.

Theorem 1 (12, Theorem. 5.2.1.) *Let $T > 0$ and $b(\cdot,\cdot) : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\Sigma(\cdot,\cdot) : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|b(t,x)| + |\Sigma(t,x)| \leq C(1+|x|);$$

$(x,t) \in \mathbb{R}^n \times [0,T]$ for some constant C, and such that

$$|b(t,x) - b(t,y)| + |\Sigma(t,x) - \Sigma(t,y)| \leq D|x-y|;$$

$x,y \in \mathbb{R}^n,\quad t \in [0,T]$ for some constant D. Let Z be be a random variable which is independent of the σ-algebra $\mathscr{F}_\infty^{(m)}$ generated by $W_s(\cdot),\ s \geq 0$ and such that

$$E\left[|Z|^2\right] < \infty.$$

Then the stochastic differential equation

$$\begin{aligned}\mathrm{d}X_t &= b(t,X_t)\,\mathrm{d}t + \Sigma(t,X_t)\,\mathrm{d}W_t, \quad 0 \leq t \leq T,\\ X_0 &= Z\end{aligned}$$

has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathscr{F}_t^Z$ generated by Z and $W_s(\cdot)$; $s \leq t$ and

$$E\left[\int_0^T |X_t|^2\mathrm{d}t < \infty\right].$$

In our case we have $n = 2,\ \ X_t = (r_t, s_t)$ and

$$b(t,X_t) = \left(\Phi_\gamma^{-1}\left(\frac{s_t}{p(t)}\right), \alpha_0 p(t)\Phi_\gamma(r_t)\right),$$

$$\Sigma(t,X_t) = \begin{pmatrix}0 & 0\\ 0 & \sigma(t)\Phi_{\alpha_3}(r_t)\end{pmatrix}.$$

In the linear case, i.e. $\gamma = 1,\ \alpha_3 = 1$, we have following values of constants in inequalities of previous theorem

$$C = D = 2\max\left\{S + P|\alpha_0|, \frac{1}{P}\right\},$$

$$S = \max_{t\in[0,T]}|\sigma(t)|, P = \max_{t\in[0,T]}p(t).$$

Existence is always secured, but changing values of γ or α_3 either violates uniqueness (< 1) or causes blow-up (> 1).

To characterize the oscillatoric interest rates, let us consider the analogue with the linear equation by looking for smooth solutions of (4) in the form $r(t) = \mathrm{e}^{\lambda t}$. Substituting into (4) we get the algebraic equation for λ in form

$$F(\lambda) := \gamma|\lambda|^{\gamma+1} - \alpha_1 \Phi_\gamma(\lambda) - \alpha_0 = 0. \tag{7}$$

Function F introduced by (7) is not convex but it has only one point of extremum (zero is inflection point) and hence the equation $F(\lambda) = 0$ has two, one or no (real) root according to the values of parameters. Using calculus we find that double root in (7) occurs only for $\lambda = \frac{\alpha_1}{\gamma+1}$ and $\alpha_0 = -\frac{|\alpha_1|^{\gamma+1}}{(\gamma+1)^{\gamma+1}}$. In the case

$$\alpha_0 > -\frac{|\alpha_1|^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \tag{8}$$

all classical solutions of (4) are oscillating (by oscillation we mean that function has infinite many roots). See [5], chapter 1.4.2 for detailed discussion of analogous case. Thus we know that if condition (8) holds, then only oscillatory solutions can occur (this immediately holds when α_0 is positive). The following example illustrates the case of oscillatory interest rate.

Example 1 ***Oscillatory interest rate-deterministic part***

Now, consider the special case of the equation (4) for α_0 satisfying (8). Be more specific we study equations of the type

$$\left(p(t)\,\Phi_\gamma(r')\right)' - \alpha_0\, p(t)\,\Phi_\gamma(r) = 0. \tag{9}$$

Exact solution can be expressed in the implicit form (with periodic continuation) similarly as half-linear trigonometric functions (generalized sine and cosine), see [5], chapter 1.1.2. (representative in the linear case is $c_1 \mathrm{e}^{at}\cos bt + c_2 \mathrm{e}^{at}\sin bt$).

The initial value problem with $\alpha_0 = -1, \alpha_1 = -0.2$

$$\begin{aligned}&\left(p(t)\,\Phi_\gamma(r)'\right)' + p(t)\,\Phi_\gamma(r) = 0,\\ &r(0) = 0.05,\\ &r'(0) = -0.05,\end{aligned} \tag{10}$$

has a solution, which is oscillatory decreasing, see Figure 2 for several values of parameter gamma (blue $\gamma = \frac{1}{3}$, red $\gamma = 1$, black $\gamma = \frac{4}{3}$), where a solution in the weak sense is taken.

By oscillatory interest rates we can obtain solutions of Parker's equation which are possibly negative. In the next section we focus on non-oscillatory interest rates which can be classified by the means of Kiguradze. This allows us for a better classification of such interest rates.

The next example illustrates the special case of interest rate $r(t) = t^\alpha, \alpha < 0$ as a solution of either Parker‘s equation (1) or generalized SDE (5).

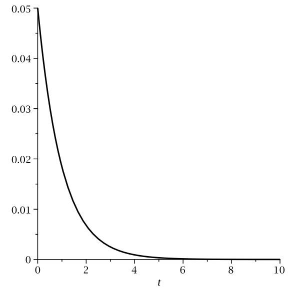

Figure 1: Behavior of monotonic $r(t)$

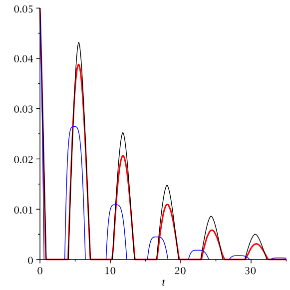

Figure 2: Behavior of oscillatory $r(t)$

Example 2 ***Stochastic process related to power interest rate***

Considering the original Parker‘s equation (1) and having $r(t) = t^\alpha, \alpha < 0$, $\sigma(t) = 1$ and integrating equation from 0 to $t_0, t_0 > 0$ we obtain

$$W(t_0) = W(0) + \alpha t_0^{\alpha-1} - \alpha_1 t_0^\alpha - \frac{\alpha_0}{\alpha+1} t_0^{\alpha+1}$$

which is the non-random function with random beginning. If $\sigma(t)$ is non-constant, we have after integration from 0 to $t_0, t_0 > 0$

$$\alpha t_0^{\alpha-1} - \alpha_1 t_0^\alpha - \frac{\alpha_0}{\alpha+1} t_0^{\alpha+1} - \int_0^{t_0} \sigma(t)\,dW(t) = 0$$

If we consider the generalized SDE (5) and use $r(t) = t^\alpha, \alpha < 0$, $\sigma(t) = 1$ and integrate from 0 to $t_0, t_0 > 0$ then we receive

$$\begin{aligned}&\int_0^{t_0} (|-\alpha|^\gamma \gamma(\alpha-1) t^{\gamma(\alpha-1)-1} p(t) - \alpha_0 p(t) t^{\gamma\alpha} +\\ &+|-\alpha|^\gamma \alpha_1 p(t) t^{\gamma(\alpha-1)})dt = \int_0^{t_0} t^{\alpha\alpha_3} dW(t)\end{aligned} \tag{11}$$

The right side of equation (11) could be written as $\int_0^{t_0} (\overline{t} - s)^{\alpha\alpha_3} dW(t)$ where $t = \overline{t} - s$ which is a fractional Brownian motion. In the case of non-constant $\sigma(t)$ the right side of equation (11) will become $\int_0^{t_0} \sigma(t) t^{\alpha\alpha_3} dW(t)$.

2.1 Kiguradze classification of deterministic interest rates

To be able to classify interest rates defined in Definition 1 by means of Kiguradze classes, we focus on smooth solutions with $p(t) > 0$, $\alpha_0 < 0$ and such that the following condition holds (in the Lebesgue sense)

$$\int_0^\infty p(t)^{-\frac{1}{\gamma}}\,\mathrm{d}t = \infty. \tag{12}$$

Notice that from (12) we have $\alpha_1 > 0$ in (3). [13] considered negative constant α_1 to obtain stable and revertible stochastic process. Negative α_1 can occur in oscillatory solutions (see previous Section).

Taking into account the corresponding homogeneous equation one can see that its solution space is generated by

$$J_0(t) = 1,$$
$$J_1(t) = \int_0^t (J_0(s)\,p(s))^{-\frac{1}{\gamma}}\,\mathrm{d}s = \frac{\gamma}{\alpha_1}\left(\mathrm{e}^{\frac{\alpha_1 t}{\gamma}} - 1\right)$$

(in linear case it is called a principal system). It can be shown (a generalized lemma of Kiguradze (see [8], also see [20] for an English treatment on generalization for the nelinear case) that if r is nonoscillatory solution $(\mathcal{N})$ of (4) then it belongs into the one of the following classes

$$\mathcal{N}_0 = \left\{r \in \mathcal{N} : \exists\, t_r : r r' < 0,\ r\left(p(t)\,\Phi_\gamma(r')\right)' > 0,\ t \geq t_r\right\},$$
$$\mathcal{N}_1 = \left\{r \in \mathcal{N} : \exists\, t_r : r r' > 0,\ r\left(p(t)\,\Phi_\gamma(r')\right)' < 0,\ t \geq t_r\right\},$$
$$\mathcal{N}_2 = \left\{r \in \mathcal{N} : \exists\, t_r : r r' > 0,\ r\left(p(t)\,\Phi_\gamma(r')\right)' > 0,\ t \geq t_r\right\}.$$

Important consequence of mentioned lemma is the fact that, in view of inequalities valid for $r \in \mathcal{N}_1$, there exist positive constants c_0, c_1 and T such that

$$c_0\,J_0(t) \leq |r(t)| \leq c_1\,J_1(t), \quad \text{for} \quad t \geq T.$$

The following properties can be found in [5] (chapter 4). It is known that equation (4) with arbitrary initial conditions is a well-posed problem. Further, all solutions of the (4) must be nonoscillatory since the equation $\left(p(t)\,\Phi_\gamma(r')\right)' = 0$ is its nonoscillatory majorant. For such equations we know even more about nonoscillatory solutions. If $p(t)$ is different from zero for large t then all solutions of nonoscillatory equation are eventually monotone. Thus, it is possible, a-priori, to divide the solutions into the following classes:

$$M^+ = \left\{r : \exists\, t_r : r r' > 0,\ t \geq t_r\right\},$$
$$M^- = \left\{r : \exists\, t_r : r r' < 0,\ t \geq t_r\right\}.$$

It is shown that M^+ (either positive increasing or negative decreasing), M^- (either positive decreasing or negative increasing) are not empty for (4). In addition these classes can be divided into mutually disjoint subclasses depending on the limit of $r(t)$ near infinity. M^- can contain only bounded solutions tending to zero on nonzero constant. Moreover we need the following notations

$$I_1 := \lim_{T\to\infty} \int_0^T p(t)^{\frac{1}{\gamma}} \left(\int_0^t |p(s)|\,\mathrm{d}s\right)^{\frac{1}{\gamma}} \mathrm{d}t \tag{13}$$
$$I_2 := \lim_{T\to\infty} \int_0^T p(t)^{\frac{1}{\gamma}} \left(\int_t^T |p(s)|\,\mathrm{d}s\right)^{\frac{1}{\gamma}} \mathrm{d}t. \tag{14}$$

The convergence or divergence of these integrals characterize the classes above, i.e. decide about qualitative behavior of solution of equation (4).

Theorem 2 *If both integrals (13),(14) converge, then the equation (4) has solutions in class M^-.*

We know that in our case

$$I_1 = \lim_{T\to\infty} \int_0^T \mathrm{e}^{-\alpha_1 t} \left(\int_0^t \alpha_0\,\mathrm{e}^{-\alpha_1 s}\,\mathrm{d}s\right)^{\frac{1}{\gamma}} \mathrm{d}t \leq$$
$$\leq \Phi_{\frac{1}{\gamma}}(\alpha_0)\,\gamma\,\alpha_1^{-\frac{1+\gamma}{\gamma}}$$

and

$$I_2 = \lim_{T\to\infty} \int_0^T p(t)^{\frac{1}{\gamma}} \left(\int_t^T |p(s)|\,\mathrm{d}s\right)^{\frac{1}{\gamma}} \mathrm{d}t \leq$$
$$\leq \frac{\Phi_{\frac{1}{\gamma}}(\alpha_0)\,\gamma\,\alpha_1^{-\frac{1+\gamma}{\gamma}}}{2}.$$

Example 3 ***Nonoscillatory interest rate - deterministic part***

Let us consider $\alpha_0 \leq -\alpha_1^2/4$ (which is the condition considered by [13] and negation of condition (8) from previous section). In the case of $\gamma = 1$, it is not hard to show that the initial value problem

$$\begin{aligned} &\left(p(t)\,r'\right)' - \alpha_0\,p(t)\,r = 0,\\ &r(0) = 0.05,\\ &r'(0) = 0.05\,c, \end{aligned} \tag{15}$$

where $c = \frac{1}{2}(\alpha_1 - \sqrt{\alpha_1^2 + 4\,\alpha_0})$, has a solution $r(t) = 0.05\,\mathrm{e}^{ct}$. The quotient of the parameters α_1, α_0 determine how fast the interest rate will decrease in time. The higher is $\dfrac{\alpha_1}{\alpha_0}$ the slower is the decreasing and vice versa. Now consider the nonlinear case $\gamma = 3$ under negation of condition (8) from previous section, i.e. the initial value problem

$$\begin{aligned} &\left(p(t)\,(r')^3\right)' - \alpha_0\,p(t)\,r^3 = 0,\\ &r(0) = 0.05,\\ &r'(0) = 0.05\,c. \end{aligned} \tag{16}$$

Assume that there exist solution in form $r(t) = \mathrm{e}^{kt}$. Then after substituting we obtain quartic algebraic equation $p(k) = k^4 - \frac{\alpha_1}{3}k^3 - \frac{\alpha_0}{3} = 0$. With the change of variable $k = l + \frac{\alpha_1}{12}$ it can be reduced to an incomplete equation and then the so-called cubic resolvent can be written. The roots of the incomplete quartic equation are determined by the roots of the cubic resolvent. One can check that discriminant of the cubic resolvent is always positive and therefore it has one real root and two complex conjugate. Thus our related quartic equation must have two real and two complex conjugate roots. We also found the local minima $-\frac{\alpha_1^4}{256} - \alpha_0$ in $k_m = \frac{\alpha_1}{4}$. But obviously

$p(0) = -\frac{\alpha_0}{3}$. So there must always exist negative root as $p(k) \to \infty$ for $k \to -\infty$. If we denote it as c, then we have found the solution of the problem (16), namely $r(t) = 0.05\,\mathrm{e}^{ct}$. Notice that such a solution belongs to M^-, also it is decreasing (asymptotically 0), which is a convenient model of interest rate. The example of monotonic interest rate is displayed on Figure 1.

2.2 Properties of introduced interest rates

The usage of introduced interest rates allows us to model several periods of diffusion driven interest rate, in which the speed of diffusion changes. Thus the main advantage of 2nd order SDE model is the flexible model which can be used for forecasts with cycles. In particular, it may have a tendency to continue its recent trend before reverting to its long term mean. Moreover, the meanings of the parameters α_0 and α_1 are interesting (see also [11]): α_0 represents the restoring force bringing the process back to the equilibrium position and α_1 is a damping force which, for large values of t, brings the process back to its equilibrium position. Then we can argue that the process has a mean reverting property stronger than the one we find in first order models.

Beside that we have advantage of nonlinearity. The operator $\Delta_\gamma u = \operatorname{div}|\nabla u|^{\gamma-1}\nabla u$, often used in heat-type equation like Black-Scholes PDE etc., describes the type of diffusion with density-dependent diffusivity (depending on the gradient of the main unknown), which also has a strong connection with fast diffusion equations ($\gamma < \frac{1}{n}$) or slow diffusion ($\gamma > \frac{1}{n}$). Thus one can effectively regulate speed of the diffusion. In some sense it is the simplest nonlinear modification of the heat equation in the area of diffusion. In our case with dimension $n = 1$ we have generalized Parker model in order to obtain better diffusion-control. Model can be understood as a one dimensional analogue of quasilinear PDE (SDE). These type of equations might be even more ”nonlinear”. There are many models including p-Laplacian, but also models including other types of nonlinearity, see e.g. Nualart and Ouknine [10].

3 Life time distributions and feasibility condition

[2] derived a feasibility (equilibrium) condition on the ratio of contribution and pension rates which makes the model equally convenient for both the fund and its members. In [2] however, the exact form of the feasibility ratio is given only for the exponentially distributed death time and an approximation is given in the case of the Weibull distribution while [3] derived the exact form of feasibility ratio for Gompertz-Makeham distribution. But in reality, the life-time distribution can be more complex. Here we present an analytical study in the case of a generalized Gamma or Gompertz-Makeham distribution modeling the death time. It is worth mentioning that several different approaches dealing with the problem of contribution rate exist. For example some authors used the approach based on Lee-Carter models or Lee-Yang approach based on Kalman filter-like processes.

3.1 The generalized Gamma

We assume that τ follows a generalized Gamma distribution (ggd), i.e. its density has the form

$$f(y|\vartheta) = \frac{\alpha}{\sigma\Gamma(\frac{1+\beta}{\alpha})}(\frac{y}{\sigma})^{\beta}\exp(-(\frac{y}{\sigma})^{\alpha}),$$

for $y > 0$, and $\vartheta = (\alpha, \beta, \sigma)$. The ggd has many applications in life sciences since many of the important nondiscrete density functions can be derived from it. For example, $f(y|(2,0,\sqrt{2\sigma}))$ is the one-sided normal distribution, and $f(y|(1,n/2-1,2))$ is the χ^2_n-distribution. In the special case of $\beta = \alpha - 1$ the Gamma distribution is called the Weibull distribution and in case of $\alpha = 1$ we obtain the Gamma distribution. While not as frequently used for modeling life data the generalized Gamma distribution does have the ability to mimic the attributes of other distributions such as the Weibull or lognormal, based on the values of the distribution's parameters and therefore is sometimes used to model life data by itself. From now on we use $\gamma = 1/\sigma$ as a parameter of our distribution.

The Weibull distribution is a versatile distribution that can take on the characteristics of other types of distributions, based on the value of the shape parameter $\beta > 0$. It has the density of the form

$$f(y|\vartheta) = \beta\gamma^{\beta}y^{\beta-1}\exp(-(\gamma y)^{\beta}), y > 0,$$

where $\vartheta = (\beta, \gamma)$.

Relationship between ggd and Lee-Carter model

Girosi and King has shown in (2007) that the Lee-Carter model is equivalent to a special type of multivariate random walk with drift (RWD) model, in which the covariance matrix depends on the drift vector. These observations suggest that, since the RWD does not make any assumption about the structure of the covariance matrix, while the Lee-Carter approach does, the Lee-Carter estimator will be preferable to the RWD only when we have high confidence in its underlying assumptions. Such a model have a form

$$X_{t+1} = X_t + \theta + \varepsilon_{t+1}, \varepsilon_t \sim N(0, \sigma^2). \quad (17)$$

Let us generalize (17) by

$$X_{t+1} = X_t + g(X_t) + \eta_{t+1}, \quad (18)$$

where g is a positive function and $\{\eta_t\}$ is a square-integrable martingale difference sequence, the second conditional moments of which depend only on the present state of the process X_t. It is known that a large class of processes (18) diverges with positive probability, and when properly normalized converges almost surely or converges in distribution to a normal or a lognormal distribution (here notice that one sided normal and lognormal distributions are special cases of a ggd). Klebaner [9] has found a class of processes such that when properly normalized converges in distribution to a ggd. Applications of this result to state dependent random walks and population size-dependent branching processes yield new results and reprove some of the known results. In such a setup usage of generalized Gamma distribution is properly justified and Lee-Carter model could be a properly specified special case, if we are sure about its underlying assumptions. Notice, that beside this random walk justification, feasible estimation and testing procedures for ggd are developed, see e.g. Stehlík [17] and references therein.

3.2 The feasibility condition

We consider a deterministic pension scheme where the total amount $U(t)$ of contributions to the fund follows the differential equation $dU(t) = udt$, and the total amount $V(t)$ of pensions paid by the fund follows the differential equation $dV(t) = vdt$, where u and v are positive constants. These two rates are linked by the so-called feasibility condition for u/v which will be discussed in this section. It follows that such a pension scheme is both of a defined-benefit pension plan type and of a defined-contribution pension plan type. The retirement date T for a member is assumed to be imposed by the law. Also is assumed that until subscriber death time life annuities are paid not depending on fund performances. Many Eastern European countries prefer some form of compulsory annuitization. The problem of optimal asset allocation in this case is discussed in [3] under the assumption of a constant interest rate r. We emphasize that our paper does not deal with an optimal allocation problem but with the dynamics of feasibility ratio imposed by a condition on contribution and pension rates insuring that the model is equally convenient for both the fund and its members.

We suppose the constant level of the contribution and the pension rates (u and v respectively). Equating the expected present value of contributions and pensions then leads to the equation (3) in [3]:

$$\frac{u}{v} = \frac{\int_T^\infty \exp(-rs)q(s)ds}{\int_{t_0}^T \exp(-rs)q(s)ds}, \tag{19}$$

where at time t_0 the member enters the fund, T is the (deterministic) retirement age and q is the survival function. Finally we may recall the Definition 1 from [3], that a pair of contribution and pension rates (u,v), $u,v > 0$ in case of constant interest rate is said to be feasible if it satisfies (19). But this definition is not suitable for the case of nonconstant r since it is not true anymore. Thus we consider the following analogy

Definition 2 *A pair of contribution and pension rates* (u,v), $u,v > 0$ *in case of nonconstant interest rate is said to be feasible if*

$$\frac{u}{v} = \frac{\int_T^\infty \exp(-\int_{t_0}^s r(u)du)q(s)ds}{\int_{t_0}^T \exp(-\int_{t_0}^s r(u)du)q(s)ds}. \tag{20}$$

Since lifetime density functions have nonnegative support and

$$E_{t_0}^\tau\left[\int_{t_0}^\tau I(s \le T)\mathrm{e}^{-\int_{t_0}^s r(u)\,\mathrm{d}u}\,\mathrm{d}s\right] = E_{t_0}^\tau\left[\int_{t_0}^\tau g(s)\,\mathrm{d}s\right] =$$

$$= \int_{t_0}^\infty f(\tau)\int_{t_0}^\tau g(s)\,\mathrm{d}s\,\mathrm{d}\tau = \int_{t_0}^\infty g(\tau)\int_\tau^\infty f(s)\,\mathrm{d}s\,\mathrm{d}\tau =$$

$$= \int_{t_0}^\infty I(\tau \le T)\,\mathrm{e}^{-\int_{t_0}^\tau r(u)\,\mathrm{d}u}\int_\tau^\infty f(s)\,\mathrm{d}s\,\mathrm{d}\tau,$$

this formulation is equivalent to

$$\frac{u}{v} = \frac{\lim_{T\to\infty} R_f(T)}{R_f(T)} - 1,$$

where $R_f(T) = \int_{t_0}^T \mathrm{e}^{-\int_{t_0}^\tau r(u)\,\mathrm{d}u}(1-F(\tau))\,\mathrm{d}\tau$ and F is cdf related to pdf f. Both integrals are defined properly as they converge (numerator is less or equal to $E(X)$ and we consider distribution with existing mean value). It is thus obvious that ratio is decreasing function of T bounded by 0 from below (remind here that r is decreasing function to zero). Here we give $R_f(T)$ for certain type of lifetime distributions.

Exponential distribution:$R_f(T) = \int_{t_0}^T \mathrm{e}^{-\int_{t_0}^\tau r(u)\,\mathrm{d}u - \lambda\tau}\,\mathrm{d}\tau$

Gamma distribution:
$R_f(T) = \frac{1}{\Gamma(\beta)}\int_{t_0}^T \mathrm{e}^{-\int_{t_0}^\tau r(u)\,\mathrm{d}u}\,\Gamma(\beta,\gamma\tau)\,\mathrm{d}\tau$

Gompertz-Makeham distribution:The Gompertz-Makeham law states that the death rate is the sum of an age-independent component and an age-dependent component which increases exponentially with age.
$R_f(T) = \mathrm{e}^{\frac{\alpha}{\beta}}\int_{t_0}^T \mathrm{e}^{-\int_{t_0}^\tau r(u)\,\mathrm{d}u - \lambda\tau - \frac{\alpha}{\beta}\mathrm{e}^{\beta\tau}}\,\mathrm{d}\tau$

3.3 Feasibility ratio for constant r

First let us consider the Gamma distribution of the form

$$f(\tau|\gamma) = \gamma^\beta \frac{\tau^{\beta-1}}{\Gamma(\beta)}e^{-\gamma\tau}, \text{ for } \tau > 0, \tag{21}$$

where $\gamma > 0$ is scale parameter and $\beta > 0$ is shape parameter. Following theorem provides the feasibility ratio for the Gamma distributed life time.

Theorem 3 *Let τ follow the Gamma distribution (21). Then the feasibility ratio has the form*

$$\frac{u}{v} = \frac{(\gamma+r)^{\beta} - \gamma^{\beta}}{A} - 1,$$

where

$$A = (\gamma+r)^{\beta}\left(1-\exp(-rT)\frac{\Gamma(\beta,\gamma T)}{\Gamma(\beta)}\right) -$$

$$-\gamma^{\beta}\left(1-\frac{\Gamma(\beta,(\gamma+r)T)}{\Gamma(\beta)}\right),$$

$\Gamma(\beta,t) := \int_t^{+\infty}\exp(-s)s^{\beta-1}ds$ is an incomplete Gamma function. Feasibility ratios for $\beta = 1$ and various γ's are in Table 1.

The next theorem provides the exact feasibility ratio for τ distributed according to generalized Gamma distribution.

Theorem 4 *Feasibility ratio for Generalized Gamma Distribution has form*

$$\frac{u}{v} = \frac{\tilde{u}}{\tilde{v}} - 1,$$

where

$$\tilde{u} = \Gamma\left(\frac{\beta+1}{\alpha}\right) - \alpha\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\beta+k\alpha+1)}{k!(\sigma r)^{\beta+1+k\alpha}},$$

$$\tilde{v} = \Gamma\left(\frac{\beta+1}{\alpha}\right) - \Gamma\left(\frac{\beta+1}{\alpha},\left(\frac{T}{\sigma}\right)^{\alpha}\right)e^{-rT} -$$

$$-\alpha\sum_{k=0}^{\infty}\frac{(-1)^k\,[\Gamma(\beta+k\alpha+1)-\Gamma(\beta+k\alpha+1,Tr)]}{k!(\sigma r)^{\beta+1+k\alpha}}.$$

Table 3 provides the comparison between the exact and approximative feasibility ratio, where Batt. denotes [2] approach. We can observe a severe bias for specific values of r,α,β,T. Therefore we recommend the exact feasibility ratio for its practical assessment and we used entirely exact formulas in the paper. In the section A1 of appendix we illustrate inaccuracy of approximative feasibility ratio when the parameters of Weibull distribution are estimated by maximum likelihood, which also justify usage of exact formulas.

Feasibility ratio in the case of a logistic model for τ The force of mortality $\mu_{x+t} = p + \frac{s}{1+re^{s(x+t)}}$ leads to ${}_tp_x = \exp[-(p+s)t]\,\frac{1+re^{s(x+t)}}{1+re^{sx}}$

Theorem 5 *The feasibility ratio $\frac{u}{v}$ for the above logistic model is*

$$u = \exp((p+s)(t_0-T))\,[(p+r)+r(p+s+r)\exp(sT)]$$

$$v = \exp(-r(t_0-T))\,[(p+r)+r(p+s+r)\exp(st_0)] -$$

$$-\exp(p+s)(t_0-T)\,[(p+r)+r(p+s+r)\exp(sT)].$$

Table 1: u/v for Exponential distribution, constant, oscillatory and exponentially decreasing rates. $\hat{r}(t) = \frac{1}{20}e^{-\frac{t}{100}}\cos(\frac{t}{100})$.

r	γ	T	$e^{(-\frac{\sqrt{20t}}{\sqrt{r}})}$	const. r	$\hat{r}(t)$
0.02	0.013	50	6.2137	0.2377	0.1839
0.02	0.013	30	15.9806	0.5912	0.3918
0.02	0.013	20	34.1186	1.0697	0.6843
0.005	0.013	50	6.213736	0.6851	0.1839
0.01	0.013	50	6.213739	0.4633	0.1839
0.05	0.013	50	6.213762	0.0477	0.1839
0.02	0.02	50	2.7844	0.1565	0.1023
0.02	0.025	50	1.8140	0.1178	0.0723
0.02	0.03	50	1.2615	0.0894	0.0525

Table 2: Feasibility ratios for Gompertz-Makeham distribution. In nonconstant case $c = -\frac{\sqrt{20}}{\sqrt{r}}$.

r	α	β	λ	T	$r(t)$	const. r
0.02	0.000007	0.11807	0.0006	50	0.5431	0.235
0.02	0.000314	0.08564	0.0006	50	0.2423	0.115
0.02	0.00062	0.0532	0.0006	50	0.5328	0.213
0.005	0.00062	0.0532	0.0006	50	0.5329	0.426
0.01	0.00062	0.0532	0.0006	50	0.53288	0.340
0.05	0.00062	0.0532	0.0006	50	0.53288	0.054
0.005	0.000007	0.11807	0.0006	50	8.90776	2.896
0.01	0.000007	0.11807	0.0006	50	8.90776	1.417
0.05	0.000007	0.11807	0.0006	50	8.90778	0.086
0.02	0.000007	0.11807	0.001	50	7.9660	0.537
0.02	0.000007	0.11807	0.002	50	6.1380	0.498
0.02	0.000007	0.11807	0.01	50	1.5232	0.2869

Feasibility ratio for Gompertz-Makeham distribution is given in [3]. Some special cases are shown in Table 2. Tables 1 and 2 show in particular that non-oscillatory decreasing deterministic interest rates increase feasibility ratio in comparison with u/v for constant interest rate (which makes investment in pension fund less attractive for individual person). The difference for Gompertz-Makeham distribution (Table 2) is lower. However, notice that oscillatory decreasing interest rate $r(t) = 0.05e^{-t/100}\cos(t/100)$ decreases feasibility ratio with respect to u/v for constant interest rate. In section 3.5 the case of stochastic interest rate is discussed. Therein we will see that size of volatility is crucial for expected feasibility ratio $E(u/v)$ to be grater (lower) in comparison with u/v for constant interest rate.

3.4 Feasibility ratio for r following 2nd order ODE

We compare [3] results for the case of distribution

$$q(t_0,t) = \exp\left\{-l(t-t_0) + e^{\frac{t_0-m}{b}}\left(1-e^{\frac{t-t_0}{b}}\right)\right\}$$

and parameters $\lambda = 0.01,\ \ m = 88.18,\ \ b = 10.5.$

Table 3: Values of feasibility ratio for different values of parameters and Weibull distribution

r	α	β	T	Series	Batt.
0.02	0.01	1.5	50	0.274725	0.258447
0.02	0.01	1.5	30	0.697062	0.627378
0.02	0.01	1.5	20	1.26771	1.09244
0.02	0.01	1.3	50	0.277487	0.270992
0.02	0.01	1.7	50	0.273807	0.244817
0.02	0.01	1.9	50	0.27415	0.230378
0.02	0.005	1.5	50	0.42889	0.397525
0.02	0.008	1.5	50	0.330223	0.308385
0.02	0.02	1.5	50	0.103901	0.0912553
0.01	0.01	1.5	50	0.512524	0.461475
0.03	0.01	1.5	50	0.155897	0.148266
0.04	0.01	1.5	50	0.0912556	0.087176
0.02	0.1	1.5	50	$1,77\times10^{-6}$	0.0400618
0.02	0.01	4	50	0.298932	0.0787496
0.02	0.06	1.8	50	$7,42\times10^{-5}$	0.076965
0.02	0.01	10	50	0.339487	-0.12023

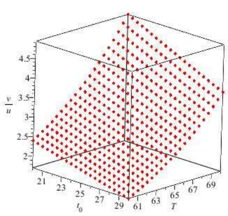

Figure 3: Behavior of v/u for $r(t) = \frac{1}{1+t^2}$

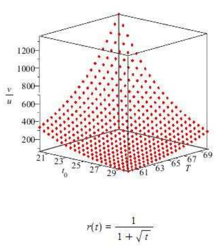

Figure 4: Behavior of v/u

Thus we obtain the following ratio

$$\frac{v}{u} = \frac{\int_{t_0}^{T} q(t_0,t)\, \mathrm{e}^{\int_{t_0}^{t} r(s)\,\mathrm{d}s}\,\mathrm{d}t}{\int_{T}^{\infty} q(t_0,t)\, \mathrm{e}^{\int_{t_0}^{t} r(s)\,\mathrm{d}s}\,\mathrm{d}t} = \frac{\int_{t_0}^{\infty} q(t_0,t)\, \mathrm{e}^{\int_{t_0}^{t} r(s)\,\mathrm{d}s}\,\mathrm{d}t}{\int_{T}^{\infty} q(t_0,t)\, \mathrm{e}^{\int_{t_0}^{t} r(s)\,\mathrm{d}s}\,\mathrm{d}t} - 1$$

since, even for the simplest case, $r(t) = exp(ct)$ the integrals are not in explicit form.

As we can see from Figure 3, the choice of $r(t) = \frac{1}{1+t^2}$ does not change substantially behavior of v/u in comparison with [3], the similar happens for $r(t) = \exp(-10t)$. However, slowly decreasing $r(t)$ leads to a substantial differences of v/u in comparison with [3] (see e.g. Figure 4 for $r(t) = \frac{1}{1+\sqrt{t}}$). Both Figures 3 and 4 are computed from the numerical solution of v/u. Notice that all 3 interest rates considered here are from Kiguradze classes as a solutions of specific deterministic part of generalized Parker equation with nonconstant coefficients.

3.5 Feasibility ratio for r following SDE

In this section we study feasibility ratio for r following SDE. Notice that in such cases feasibility ratio is also random variable. We are interested in both SDE of 1st and 2nd order which are treated separately in sections 3.5.1 and 3.5.2. The reason is better comparison of different features of feasibility ratios for interest rates following 1st and 2nd order SDE. Let us recall that 1st order SDE interest rates are widespread in insurance and finance,however, it is well known fact that they are not capturing oscillations and long-term behavior satisfactorily. Also in the case of 1st order SDE interest rates we know explicit solution, however, in the case of 2nd order SDE interest rate we must rely on numerical methods. Unlike the scalar homogeneous linear equations, it is generally not possible to solve linear SDE explicitly for even when all of the involved matrices are constant (see e.g. [4]). Here we consider only one realization of E(u/v). For numerical solution of interest rate we have used strong numerical scheme of EulerMaruyama (`Euler()`) from The MAPLE Stochastic Package (version 5.1 by S. Cyganowski). This type of scheme is of strong order of convergence $p = 1$, i.e. mean of the error $\sup_{0\le t_n\le T} E[|X_n - X(t_n)|] \le K(\Delta t)^p$, with Δt being the maximum time increment of the discretization. Such convergence is sufficient for our purposes. It is in fact the strong stochastic Taylor scheme of order 0.5. For simulations we have used numerical approximations of integrals in feasibility ratio quotient. Several computations have been made also in software R [15].

3.5.1 Exponential life time distribution and SDE of 1st order

Here we consider exponential life time distribution, thus we have $R_f(T) = \int_0^T \mathrm{e}^{-\lambda\tau - \int_0^\tau r(u)\,\mathrm{d}u}\,\mathrm{d}\tau$, where r is deterministic or stochastic process (both plotted on Figure 10), respectively:

$$r(t) = 0.05\mathrm{e}^{\mu t},$$

$$r_t = 0.05\mathrm{e}^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \tag{22}$$

with $\sigma = 0.1,\ \mu = -0.1$. The following Table 4 compares u/v and $E(u/v)$ for both cases.

Figures compare case of $T = 20$ (see Figure 5) with case of $T = 50$ (see Figure 6). In both cases we have $\lambda = 0.15$. The pattern of decreasing u/v for increasing T is well visible also for stochastic interest rate. This justifies the fact (observed for a constant interest rate by [3] that given the age of the member, when the retirement age T increases, the fund can afford to pay a higher pension rate v to the member.

Table 4: Values of feasibility ratios u/v and $E(u/v)$ for deterministic and stochastic rate

λ	T	$\frac{u}{v}$ for determ. $r(t)$	$E(u/v)$ for r_t
0.05	20	0.4597413456	0.1856195632
0.1	20	0.120974967	0.1196360311
0.15	20	0.039603543	0.038107873
0.1	50	0.005229213	0.03090041290
0.15	50	0.000407282	0.004006664665

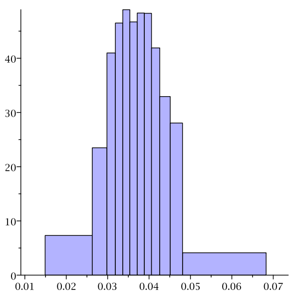

Figure 5: Histogram for $T = 20, \lambda = 0.15$

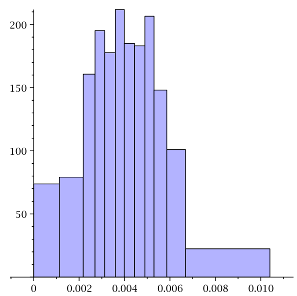

Figure 6: Histogram for $T = 50, \lambda = 0.15$

3.5.2 Exponential life time distribution and SDE of 2nd order

Here we consider exponential life time distribution, deterministic interest rate

$$r(t) = \max\left(0, 0.05\,\mathrm{e}^{-\frac{1}{10}}\cos\left(\frac{\sqrt{19}}{10}t\right)\right), \tag{23}$$

and stochastic interest rate r_t^i, respectively, where r_t^i solves modified Parker equation

$$\begin{aligned} \mathrm{d}p_t &= (\alpha_1 p_t + \alpha_0 r_t)\,\mathrm{d}t + \sigma(t)\,\Phi_{\alpha_3}(r_t)\,\mathrm{d}W_t \\ \mathrm{d}r_t &= p_t\,\mathrm{d}t, \end{aligned} \tag{24}$$

with $\alpha_0 = \alpha_1 = -0.2, \sigma = 0.1,\ \mu = 0,\ \gamma = 1$, here $\alpha_3 = 0.1$ for $i = 0$ and $\alpha_3 = 1$ for $i = 1$. Solution of 2nd order SDE is compared with the deterministic solution and their match can be seen at Figure 9.

The following figure compares case of $T = 20$ (see Figure 11) with case of $T = 50$ (see Figure 12). In both cases we have $\lambda = 0.15$. The pattern of decreasing u/v for increasing T is well visible also for stochastic interest rate.

Table 5 compares feasibility ratios for deterministic and five different stochastic rates (Where not stated explicitly, we consider $\alpha_3 = 1, \gamma = 1$. Volatility σ is written in brackets.) Notice that $E(u/v)$ is closer to u/v for r_t^1 (than for r_t^0) since process has a regularized volatility part by the multiplication of higher exponent of r_t. Also notice that $E(u/v)$ for slow(fast) diffusion $\gamma = \frac{4}{3} > 1$ ($\gamma = \frac{1}{3} < 1$) provides less (more) convenient, i.e. greater (lower) values of expected feasibility ratio

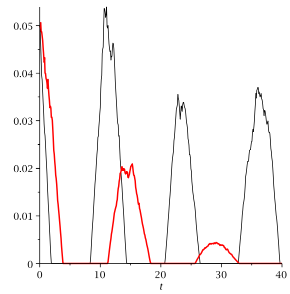

Figure 7: Comparison of 2nd order SDE ($\alpha_3 = 1$) with normal diffusion (red line:$\gamma = 1$) and slow diffusion (black line:$\gamma = 4/3$)

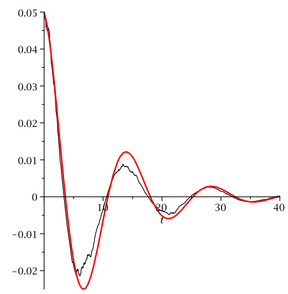

Figure 8: Oscillating interest rate

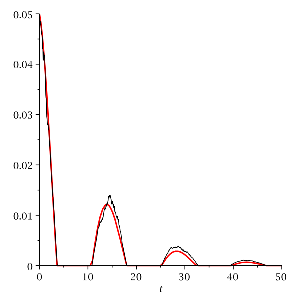

Figure 9: 2nd order SDE with $\alpha_3 = 1$

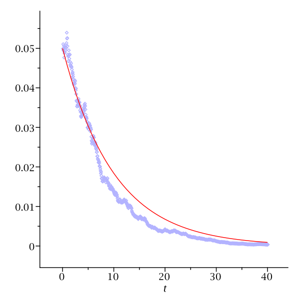

Figure 10: Geometric Brownian Motion

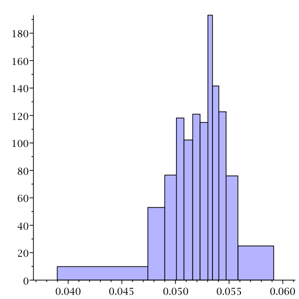

Figure 11: Histogram for $T = 20, \lambda = 0.15$

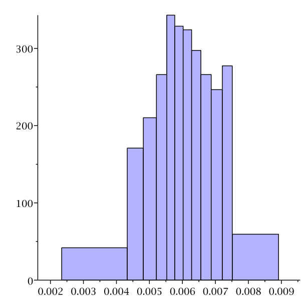

Figure 12: Histogram for $T = 50, \lambda = 0.15$

than r_t^1 at the same value of volatility (for graphical comparison see Figure 7). Also higher volatility increases expected feasibility ratio thus making pension fund less attractive for the individual investment. Figure 8 shows deterministic (red line) and stochastic (black line) oscillating interest rate.

4 Impact of interest rates on pensions: Real data illustration

In this section we illustrate the impact of both constant and non-constant interest rates on pensions with the real

Table 5: Values of expected feasibility ratio $E(u/v)$ for various volatilities σ

λ	T	$r(t)$	$r_t^1(0.1)$	$r_t^1(0.2)$	$r_t^0(0.01)$
0.05	20	0.5334	0.4180	0.5028	0.3965
0.1	20	0.1570	0.1491	0.1810	0.3546
0.15	20	0.0531	0.4931	0.7539	0.6541
0.05	50	0.0533	0.0291	0.0431	0.0413
0.1	50	0.0057	0.0727	0.1701	0.2368
0.15	50	0.0005	0.3434	0.5972	0.2245
λ	T	$r_t^0(0.1)$	$\gamma=\frac{4}{3},(0.1)$	$\gamma=\frac{1}{3}(0.1)$	oscil (0.1)
0.05	20	0.5561	0.5059	0.1458	0.4828
0.1	20	0.6320	0.3357	0.2000	0.1399
0.15	20	1.0007	0.6268	0.4347	0.2578
0.05	50	0.0459	0.0675	0.0068	0.0308
0.1	50	0.6095	0.1127	0.1078	0.0001
0.15	50	0.4095	0.4013	0.3074	0.1665

data from Slovakia. In the World Bank formulation, the first pillar is a mandatory pension scheme operated under public management. The second pillar is a fully funded mandatory scheme financed by employees contributions. In many countries, this is a defined contribution system managed by private companies. We consider the problem of pension membership from an individual perspective: a person can be involved in two pillars, namely a state pension pillar(obligatory) and another based on a pension fund (voluntary). This situation is typical for many Eastern European countries. While conditions in the state pension pillar are given by law there is more freedom in the pillar based on a private pension fund. We use the interest rate defined in section 2 and illustrate its effects on pensions. The recent development of pension funds not only in Slovakia is well described,e.g. in Whitehouse [21]. We consider two closed groups of Slovakian people (opposite with respect to their incomes),all entering pension fund in the year 1993. The salaries are taken from [18](see Labour Market,III.3-10,Structure of average gross nominal monthly wage of employees in the economy of the Slovak Republic) and are shown in Table 6.

In order to avoid the situation that pensions from the first pillar become inadequate (to guarantee an acceptable state of living) people are recommended to join also the second pillar. Of course, some caution is necessary when a person considers whether to choose the first pillar solely or to prefer a mix of both pillars. In what follows ${}_tp_x$ means the probability of a person aged x to survive the next t years. We suppose that people with monthly wage given in Table 6 obey mortality published in Statistical Yearbook of Slovak Republic (2009). It is well-known (see,e.g. [14]) that Weibull, Gompertz-Makeham, logistic (and also Gamma) distributions fit the life data in Slovakia well.In order to find out the parameters of these distributions the method of moments was used. If a person belongs to the first pillar only, the contribution rate is 18% of his (her) salary. If, however, he (she) chooses both the state and private pillar,the rate becomes 9% for each of these two pillars.

Table 6: Month Mean Brutto Salary, 1993-2009

Year	Max Salary	Min Salary
1993	9882	4579
1994	11592	5179
1995	13537	5975
1996	19377	6326
1997	24824	5639
1998	24 233	6208
1999	26 862	6421
2000	30 021	6785
2001	31 825	7262
2002	34 041	8533
2003	34443	8840
2004	39452	9446
2005	42544	10199
2006	45349	10947
2007	51154	12945
2008	57186	14614
2009	60736	15428

In what follows we concentrate on the second pillar only. We suppose that after reaching the retirement age of 62 years (given by law) the member of the second pillar buys a life annuity. We compute his (her) monthly pension under different assumptions on life time distributions as well as on interest rates,both constant and stochastic. Using the equation of equivalence we have

$$\sum_{t=1}^{17} 0,95 \times 0.09 \times X_{t+1992} \times (1+r_t)^{17-t} {}_tp_{45} =$$

$$= \sum_{t=0}^{\infty} 0,95 \times v \times P \times (1+r_t)^{-t} {}_tp_{62},$$

where the contribution rate $u = 0.09$ (given by law), X_{t+1992} is salary in year $t+1992$, P stands for pension, 0.95 stands for 5% costs which is the standard actuarial practice and we assume that people entered into the fund at the age 45 years. We recall that in Slovakia only defined contribution plans exist and the contribution rate is given by law.

Table 7 provides the estimation of pension at age 62 for 45 years old male for Weibull, Gamma, Logistic and Makeham life time distributions. Since Makeham does not have an explicit expressions for moments, we have employed [7] and obtained (from empirical data): $\lambda = 0., \alpha = 0.116e-3, \beta = 0.84e-1$. We employ both constant and non-constant interest rate. It follows from this table that the ratio v/u is approximately 3 for Weibull and similarly for Gamma but it is too high for the logistic model. Comparing the results obtained by using the mortality from Statistical Yearbook with the results from above mentioned distributions (with parameters obtained by the method of moments) confirmed that the logistic distribution is not suitable. As to the rest it is difficult to say which model outperforms the others.

Table 7: Estimation of pension at age 62 for 45 years old male, $r(t) = 0.05\exp(-t/10)$

distribution	r	v	P_{max}	P_{min}
Weibul	0.005	0.271543	3056.08	842.30
$\hat{\gamma} = 0.033$	0.01	0.301346	3298.08	913.12
$\hat{\beta} = 1.548$	0.02	0.369141	3827.27	1069.53
	$r(t)$	0.244068	2292.01	620.108
Gamma	0.005	0.217238	3056.07	842.30
$\hat{\gamma} = 0.084$	0.01	0.243699	3298.09	913.12
$\hat{\beta} = 2.297$	0.02	0.303896	3827.27	1069.54
	$r(t)$	0.192834	2179.20	589.59
Logistic	0.005	0.416747	3056.08	842.30
$\hat{\mu} = 27.35$	0.01	0.460883	3298.08	913.12
$\hat{\sigma} = 9.948$	0.02	0.561155	3827.27	1069.53
	$r(t)$	0.37605	2331.00	630.657
Makeham	0.02	0.1580	3855.45	1021.24
	0.01	0.1311	3310.29	872.22
	0.05	0.1193	3062.15	804.80
	$r(t)$	0.1085	2400.89	649.566

Taking into account that the minimal pension in Slovakia is approximately 7500 Slovakian crowns we come to the conclusion that people from the lowest income group should stay in the first pay-as-you-go) pillar exclusively because the benefit from the second pillar is smaller than the loss from leaving the first pillar. On the other hand people with the highest incomes are recommended to become members of both the first and second pillars. Being the member of the second pillar only is not possible because of the law.

Table 8 illustrates impact of volatility and different stochastic rates on pension. Therein r_t is given by (22), $r^o(t)$ is defined by (23), $r_t^o := \max(0, r_t^*)$ where r_t^* solves (5) with parameters $\alpha_0 = -1/5$, $\alpha_1 = -1/5$, $\gamma = 1$, $\alpha_3 = 1$, $r^{os} := \max(0, r_t^{**}), r_t^{os} := \max(0, r_t^{***})$ where oscillatory decreasing interest rates r_t^{**}, r_t^{***} solve (4) and (5) respectively, both with parameters $\alpha_0 = -1/5$, $\alpha_1 = -1/50$, $\gamma = 1$, $\alpha_3 = 1$. We computed v from lifetime distribution and deterministic r (constant or following ODE). The same value of v was used also for stochastic interest rate. It is clear that for all cases of stochastic interest rates both decrease of α_1 and increase of volatility have negative impact on value of the pension in comparison with deterministic interest rate.

5 Conclusions

Owing to turbulence of financial markets many economists as well as financial analysts agree that the assumption of constant interest rate is not realistic. The main novelty of the paper is that force of interest follows 2nd order quasilinear stochastic differential equation, which generalizes model introduced in [13]. As a side result we obtain Kiguradze characterization of smooth solutions of deterministic part of Parker's stochastic differential equation. This allows interest rate to mimic oscillatory behavior and improve its long-term properties. The last mentioned case modifies the classical models and brings the novelty in the study of several actuarial problems. As far as we know, analogous classification for stochastic differential equations (systems) has not been done yet. Such a classification is important for initial values problems in theory of ODEs. Also we demonstrate that the unique classification obtained by Kiguradze can be well employed to classify deterministic part of stochastic differential equations. From our numerical experiments this classification plays a significant role in analogous classification for stochastic interest rates

Table 8: Estimation of pension at age 62 for 45 years old male, $r(t) = 0.05\exp(-t/10)$, r_t, r_t^o. Weibull distr. with $\hat{\gamma} = 0.033, \hat{\beta} = 1.548$.

r	v	P_{max}	P_{min}
$r(t)$	0.244068	2292.01	620.108
r_t, $\sigma = 0.01$	0.244068	2955.73	781.066
r_t, $\sigma = 0.2$	0.244068	2945.23	777.05
$r^o(t)$	0.244133	2435.95	653.254
r_t^o, $\sigma = 0.01$	0.244133	3048.93	817.513
r_t^o, $\sigma = 0.2$	0.244133	3000.40	793.98
$r^{os}(t)$	0.0954801	2825.47	752.75
r_t^{os}, $\sigma = 0.01$	0.0954801	3576.71	945.19
r_t^{os}, $\sigma = 0.2$	0.0954801	3030.76	825.81

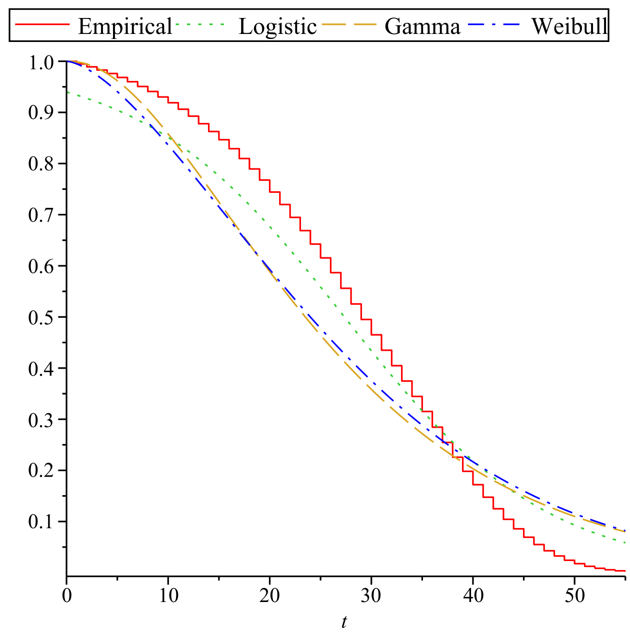

Figure 13: Fit of empirical life time data by Logistic, Gamma and Weibull

Beside that we also pay attention to different types of mortality and present an analytical study in the case of a generalized Gamma or Gompertz-Makeham distributions modeling the death time. We derive the feasibility ratios for these life time models. We show that decreasing deterministic interest rate increases feasibility ratio in comparison with u/v for constant interest rate which makes investment in pension fund less attractive for individual person. We show that for the case of stochastic interest rate the size of volatility is decisive factor whether expected feasibility ratio $E(u/v)$ is grater or lower in comparison with u/v for constant interest rate. The pattern of decreasing u/v for increasing T is well visible also for non-constant (both deterministic and stochastic) interest rates. This justifies the fact that given the age of the member, when the retirement age T increases, the fund can afford to pay a higher pension rate v to the member. We also observed that $E(u/v)$ for slow(fast) diffusion $\gamma = \frac{4}{3} > 1(\gamma = \frac{1}{3} < 1)$ provides less(more) convenient, i.e. grater(lower) values of expected feasibility ratio than r_t^1 at the same value of volatility. Also it is visible, that stochastic part of interest rate has higher influence on feasibility ratios for $\gamma \neq 1, \alpha_3 \neq 1$ (i.e. not Parker's case).

Finally, in section 4 we illustrate the impact of both constant and non-constant interest rates on pensions with the real data from Slovakia. For a fixed contribution rate which is linked to income, we illustrated the implied pension rate taking account of constant interest rates, an interest rate following the suggested 2nd order differential equation and the different mortality laws. The results obtained for a real data agree with our intuitive expectation that being the member of both pension schemes is good for a person belonging to higher income group but it can be too risky for people with low salaries. It is clear that for all cases of stochastic interest rates both decrease of α_1 and increase of volatility have negative impact on value of the pension in comparison with deterministic interest rate.

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Appendix

Appendix A: On accuracy of feasibility ratio

In this section we illustrate accuracy of the feasibility ratio with respect to the maximum likelihood estimation (MLE) of the parameters of Weibull distribution. Table 9 compares the exact solution (given in section 3) and the approximative solution given by [2]. The simulation study is conducted and Weibull distribution with known shape and scale parameters ($shape = 1, 2, 10$, $scale = 0.1, 1, 10$) is simulated. Then we estimate parameters by MLE from the simulated data. We have computed the corresponding exact feasibility ratios based on true shape and scale parameters (i.e. those, used in the simulation), the exact feasibility ratios based on estimated parameters and finally the approximated feasibility ratios given by [2] with estimated parameters. We can conclude that the results edited in the Table 9 support the exact approach conducted in this paper.

Appendix B: Proofs

Proof of Theorem 3

The direct computation leads to

$$\mathbf{P}(\tau \geq T) = \frac{\gamma^\beta}{\Gamma(\beta)} \int_T^\infty \tau^{\beta-1} e^{-\gamma\tau} d\tau =$$
$$= \frac{1}{\Gamma(\beta)} \int_{\gamma T}^\infty \xi^{\beta-1} e^{-\xi} d\xi = \frac{\Gamma(\beta,\gamma T)}{\Gamma(\beta)},$$

$$\mathbf{E}_0^\tau[e^{-r\tau}] = \frac{1}{\Gamma(\beta)} \left(\frac{\gamma}{r+\gamma}\right)^\beta \int_0^\infty \xi^{\beta-1} e^{-\xi} d\xi = \left(\frac{\gamma}{r+\gamma}\right)^\beta,$$

$$\mathbf{E}_0^\tau[e^{-r\tau}|_{\tau<T}] = \frac{\gamma^\beta}{\Gamma(\beta)} \int_0^T \tau^{\beta-1} e^{-(\gamma+r)\tau} d\tau =$$
$$= \left(\frac{\gamma}{\gamma+r}\right)^\beta - \frac{\gamma^\beta}{\Gamma(\beta)} \int_T^\infty \tau^{\beta-1} e^{-(\gamma+r)\tau} d\tau =$$

$$= \left(\frac{\gamma}{\gamma+r}\right)^\beta - \frac{\gamma^\beta}{\Gamma(\beta)} \int_{T(\gamma+r)}^\infty \left(\frac{\xi}{\gamma+r}\right)^{\beta-1} e^{-\xi} \frac{d\xi}{\gamma+r} =$$
$$= \left(\frac{\gamma}{\gamma+r}\right)^\beta - \left(\frac{\gamma}{\gamma+r}\right)^\beta \frac{\Gamma(\beta, T(\gamma+r))}{\Gamma(\beta)}.$$

Using summarization we get (3).
□

Proof of Theorem 4

The density function has form

$$f(t) = \frac{\alpha \left(\frac{t}{\sigma}\right)^\beta e^{-\left(\frac{t}{\sigma}\right)^\alpha}}{\Gamma\left(\frac{\beta+1}{\alpha}\right)\sigma}, \ \alpha, \beta, \sigma > 0.$$

So the direct substitution gives $\mathbf{P}(t \geq T) = \int_T^\infty f(t)dt$. We use substitution $z = e^{-\left(\frac{t}{\sigma}\right)^\alpha}$, then $t = \sigma \sqrt[\alpha]{-\ln(z)}$ and $dt = \frac{\sigma \sqrt[\alpha]{-\ln(z)}}{z\ln(z)\alpha} dz$. So direct substitution gives us

$$\mathbf{P}(t \geq T) = \frac{\alpha}{\Gamma\left(\frac{\beta+1}{\alpha}\right)\sigma} \int_{c_1}^0 \frac{z(-\ln(z))^{\frac{\beta}{\alpha}} \sigma \sqrt[\alpha]{-\ln(z)}}{z\ln(z)\alpha} dz =$$

Table 9: Comparison for f.r. for Weibull distribution: exact, estimated and [2]

α	β	exact	Battocchio	MLE (exact)
0,1	1	0,03829	0,03829	0,02064
0,1	2	$1,7\times10^{-5}$	-11,483	0,000018236
0,1	10	$-1,7\times10^{-14}$	-0,8545	$-1,37\times10^{-13}$
1	1	$6,72\times10^{-14}$	$7,17\times10^{-14}$	$2,42\times10^{-12}$
1	2	0	10^{-9}	$3,40\times10^{-11}$
1	10	$6,6\times10^{-9}$	$-2,4\times10^{-9}$	$2,57\times10^{-9}$
10	1	0	0	$-3,30\times10^{-12}$
10	2	$-1,11\times10^{-16}$	0	$6,26\times10^{-11}$
10	10	$1,27\times10^{-11}$	0	$8,867\times10^{-11}$

$\hat{\alpha}$	0,1	0,09	0,1	1,16	1,01	1,01	9,57	9,82	10,05
$\hat{\beta}$	1,13	2,1	10,0	1,01	2,3	10,2	1,10	1,95	9,48

$$= \frac{1}{\Gamma\left(\frac{\beta+1}{\alpha}\right)}\int_0^{c_1}[-\ln(z)]^{\frac{\beta-\alpha+1}{\alpha}}dz,$$

where $c_1 = e^{-\left(\frac{T}{\sigma}\right)^\alpha}$. Further we use integration by parts. Denote as $k = \frac{\beta-\alpha+1}{\alpha}$ and $I(k) = \int_0^{c_1}[-\ln(z)]^k dz$. Then we have

$$I(k) = (-1)^k\left(\ln(z)^k z|_0^{c_1} - k\int_0^{c_1}\ln(z)^{k-1}dz\right).$$

We obtain recurrence relation

$$I(k) = (-1)^k c_1\ln(c_1)^{k-1} + kI(k-1).$$

The solution of this difference equation has form

$$I(k) = \Gamma(k+1,-\ln(c_1))$$

(We know that $I(0)=c_1$). So, finally

$$\mathbf{P}(t\geq T) = \frac{\Gamma(k+1,-\ln(c_1))}{\Gamma\left(\frac{\beta+1}{\alpha}\right)} = \frac{\Gamma\left[\frac{\beta+1}{\alpha},\left(\frac{T}{\sigma}\right)^\alpha\right]}{\Gamma\left(\frac{\beta+1}{\alpha}\right)}.$$

Now we consider $\mathbf{E}[e^{-rt}]$.

$$\mathbf{E}[e^{-rt}] = \int_0^\infty \frac{\alpha\left(\frac{t}{\sigma}\right)^\beta e^{-\left(\frac{t}{\sigma}\right)^\alpha}e^{-rt}}{\Gamma\left(\frac{\beta+1}{\alpha}\right)\sigma}dt =$$

$$= \frac{\alpha}{(\sigma r)^{\beta+1}\Gamma(\frac{\beta+1}{\alpha})}\int_0^\infty z^\beta e^{-z}e^{-\left(\frac{z}{r\sigma}\right)^\alpha}dz,$$

But

$$e^{-\left(\frac{z}{\sigma}\right)^\alpha} = \sum_{k=0}^\infty \frac{(-1)^k\left(\frac{z}{r\sigma}\right)^{k\alpha}}{k!},$$

so

$$\mathbf{E}[e^{-rt}] = \frac{\alpha}{(\sigma r)^{\beta+1}\Gamma(\frac{\beta+1}{\alpha})}\int_0^\infty z^\beta e^{-z}\sum_{k=0}^\infty\frac{(-1)^k\left(\frac{z}{r\sigma}\right)^{k\alpha}}{k!}dz =$$

$$= \frac{\alpha}{\Gamma(\frac{\beta+1}{\alpha})}\sum_{k=0}^\infty\frac{(-1)^k}{k!(\sigma r)^{\beta+1+k\alpha}}\int_0^\infty z^{\beta+k\alpha}e^{-z}dz =$$

$$= \frac{\alpha}{\Gamma(\frac{\beta+1}{\alpha})}\sum_{k=0}^\infty\frac{(-1)^k\,\Gamma(\beta+k\alpha+1)}{k!(\sigma r)^{\beta+1+k\alpha}}.$$

For $\mathbf{E}[e^{-rt}|t<T]$ we use similar procedure.

$$\mathbf{E}[e^{-rt}|t<T] = \int_0^T\frac{\alpha\left(\frac{t}{\sigma}\right)^\beta e^{-\left(\frac{t}{\sigma}\right)^\alpha}e^{-rt}}{\Gamma\left(\frac{\beta+1}{\alpha}\right)\sigma}dt =$$

$$= \frac{\alpha}{\Gamma(\frac{\beta+1}{\alpha})}\sum_{k=0}^\infty\frac{(-1)^k}{k!(\sigma r)^{\beta+1+k\alpha}}\int_0^{Tr}z^{\beta+k\alpha}e^{-z}dz =$$

$$= \frac{\alpha}{\Gamma(\frac{\beta+1}{\alpha})}\sum_{k=0}^\infty L(k)\left\{\frac{1}{\Gamma(\beta+k\alpha+1)}\int_0^{Tr}z^{\beta+k\alpha}e^{-z}dz\right\} =$$

$$= \frac{\alpha}{\Gamma(\frac{\beta+1}{\alpha})}\sum_{k=0}^\infty L(k)\left\{1-\frac{1}{\Gamma(\beta+k\alpha+1)}\int_{Tr}^\infty z^{\beta+k\alpha}e^{-z}dz\right\},$$

where $L(k) = \frac{(-1)^k\,\Gamma(\beta+k\alpha+1)}{k!(\sigma r)^{\beta+1+k\alpha}}$. Thus finally we have

$$\frac{\alpha}{\Gamma(\frac{\beta+1}{\alpha})}\sum_{k=0}^\infty\frac{(-1)^k\,[\Gamma(\beta+k\alpha+1)-\Gamma(\beta+k\alpha+1,Tr)]}{k!(\sigma r)^{\beta+1+k\alpha}}.$$

□

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