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Malcev Algebras and Combinatorial Structures

Manuel Ceballos[1], *Juan Núñez*[1] *and Ángel F. Tenorio*[2,*]

[1] Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad de Sevilla, Aptdo. 1160, 41080, Seville, Spain.
[2] Dpto. de Economía, Métodos Cuantitativos e Historia Económica, Escuela Politécnica Superior, Universidad Pablo de Olavide, Ctra. Utrera km. 1, 41013 Seville, Spain

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Abstract: In this work, we design an algorithmic method to associate combinatorial structures with finite-dimensional Malcev algebras. In addition to its theoretical study, we have performed the implementation of procedures to construct the digraph associated with a given Malcev algebra (if its associated combinatorial structure is a digraph) and, conversely, a second procedure to test if a given digraph is associated with some Malcev algebra.

Keywords: Digraph, Combinatorial structure, Malcev algebra, Combinatorial operations, Algorithm

1 Introduction

At the present time, finding and exploring new links and relations between different fields is one of the most exciting research lines in Sciences and, in particular, Mathematics. Thanks to alternative techniques and methods, researchers can work out many open problems, achieving improvements for known theories and revealing other new ones. In this paper, we deal with the relation between Graph Theory and Malcev algebras. More concretely, our goal is to make progress on the research line started in [1], where a mapping between Lie algebras and combinatorial structures was introduced in order to translate properties of Lie algebras into the language of Graph Theory and vice versa. Now, we want to obtain an analogous mapping for Malcev algebras.

Non-associative algebras have been profusely studied due to both its own theoretical relevance and its multiple applications to many different fields, like Engineering, Physics or Applied Mathematics. A particular type of these algebras consists of Malcev algebras, which are the purpose of this paper. They were introduced by Malcev [6] as tangent algebras of analytic Moufang loops and are related to alternative algebras in the same way that Lie algebras are related to associative algebras; i.e. if A is an alternative algebra, then the algebra A^- with the operator $[a,b] = ab - ba$ is a Malcev one. As happens with any class of non-associative algebras, there exist many general questions to be solved and these questions (as, for example, the classification of Malcev algebras) require alternative techniques since the traditional ones are not sufficient.

In turn, Graph Theory is nowadays a fundamental tool for solving wide range of problems in most of research fields. In this way, graphs and simplicial complexes (its generalization to higher dimensions) may be used as a helpful tool in the study of non-associative algebras, providing new ways to solve many open problems like the above-mentioned classification problem of Malcev algebras.

Hence, our main goal is to study the link between combinatorial structures and Malcev algebras, giving the generalization of the techniques introduced in [1] and developed in [2,3,4] to the case of Malcev algebras instead of considering Lie algebras.

This paper is structured as follows: after reviewing some well-known results on Graph Theory and Malcev algebras in Section 2, Section 3 is devoted to define the method to associate combinatorial structures with Malcev algebras. Next, Section 4 proves the main theoretical results in this paper about the structure of Malcev algebras starting from the association given in the previous section. Finally, Section 5 shows an algorithm to evaluate Malcev identities and determine the restrictions over the structure constants, in order to return the list of allowed and forbidden configurations for combinatorial structures associated with Malcev algebras. In addition,

* Corresponding author e-mail: aftenorio@upo.es

we also show an algorithm to draw these configurations when they are digraphs. All this goes with a brief computational study, showing that the complexity order of the procedures here presented is polynomial.

In our opinion, the tools and results shown in this paper may be useful and helpful for understanding the relation between Malcev algebras and simplicial complexes. Moreover, the classification of combinatorial structures may involve easier methods to classify Malcev algebras by means of the classification of their associated combinatorial structures.

2 Preliminaries

For a general overview on Malcev algebras and Graph Theory, the reader can consult [7,5]. We only consider finite-dimensional Malcev algebras over the complex number field $\mathbb{C}$.

Definition 1. *A* Malcev algebra $\mathscr{M}$ *is a vector space with a second bilinear inner composition law* ($[\cdot,\cdot]$) *called the* bracket product *or* commutator, *which satisfies*

1. $[X,Y]=-[Y,X]$, $\forall X\in\mathscr{M}$; *and*
2. $[[X,Y],[X,Z]] = [[[X,Y],Z],X] + [[[Y,Z],X],X] + [[[Z,X],X],Y]$, $\forall X,Y,Z\in\mathscr{M}$.

The second constraint is named the Malcev identity. *From now on, we use the notation* $M(X,Y,Z)=[[X,Y],[X,Z]]-[[[X,Y],Z],X]-[[[Y,Z],X],X]-[[[Z,X],X],Y]$.

Given a basis $\{e_i\}_{i=1}^n$ *of* $\mathscr{M}$, *its* structure (or Maurer-Cartan) constants *are defined by* $[e_i,e_j]=\sum c_{i,j}^h e_h$, *for* $1\leq i<j\leq n$.

Remark. Since we are considering a field of characteristic different from 2, the first constraint in Definition 1 is equivalent to $[X,X]=0, \forall X\in\mathscr{M}$.

Definition 2. *Given a Malcev algebra* $\mathscr{M}$, *its* center *is* $Z(\mathscr{M})=\{X\in\mathscr{M}\mid[X,Y]=0,\ \forall Y\in\mathscr{M}\}$.

Definition 3. *Given a finite-dimensional Malcev algebra* $\mathscr{M}$, *its* derived series *is*

$$\mathscr{M}_1=\mathscr{M},\ \mathscr{M}_2=[\mathscr{M},\mathscr{M}],\ \ldots,\ \mathscr{M}_k=[\mathscr{M}_{k-1},\mathscr{M}_{k-1}],\ \ldots$$

Thus, $\mathscr{M}$ *is called* solvable *if there exists* $m\in\mathbb{N}$ *such that* $\mathscr{M}_m=\{0\}$. *In addition, if* $\mathscr{M}_{m-1}\neq\{0\}$ *also holds, then* $\mathscr{M}$ *is* $(m-1)$*-step solvable.*

Definition 4. *Given a finite-dimensional Lie algebra* $\mathscr{M}$, *its* central series *is*

$$\mathscr{M}^1=\mathscr{M},\ \mathscr{M}^2=[\mathscr{M},\mathscr{M}],\ \ldots,\ \mathscr{M}^k=[\mathscr{M}^{k-1},\mathscr{M}],\ \ldots$$

Thus, $\mathscr{M}$ *is called* nilpotent *if there exists* $m\in\mathbb{N}$ *such that* $\mathscr{M}^m=\{0\}$. *In addition, if* $\mathscr{M}^{m-1}\neq\{0\}$ *also holds, then* $\mathscr{M}$ *is* $(m-1)$*-step nilpotent.*

Remark. Every nilpotent algebra is trivially solvable, because $\mathscr{M}_i\subseteq\mathscr{M}^i$, for all $i\in\mathbb{N}$.

Definition 5. *A Malcev algebra* $\mathscr{M}$ *is perfect if* $\mathscr{M}$ *and* $\mathscr{M}_2$ *are isomorphic.*

Although the reader can consult [5] as an introductory reference to Graph Theory, some notions are recalled next in this section.

Definition 6. *A* graph *is an ordered pair* $G=(V,E)$, *where* V *is a non-empty set of vertices and* E *is a set of unordered pairs (edges) of two vertices. If the edges are ordered pairs of vertices, then the graph is named* digraph.

Throughout the paper, we consider digraphs admitting double edges.

Definition 7. *Given a digraph* G, *then a subdigraph* H *is said to be* induced *by a vertex-subset* $V(H)$ *in* G *if the edge-set of* H *consists of all the edges of* G *between two vertices in* $V(H)$.

3 Associating combinatorial structures with Malcev algebras

Let $\mathscr{M}$ be a n-dimensional Malcev algebra with basis $\mathscr{B}=\{e_i\}_{i=1}^n$. The structure constants are given by $[e_i,e_j]=\sum_{k=1}^n c_{i,j}^k e_k$. Due to the skew-symmetry of the bracket product and the remark to Definition 1, the pair $(\mathscr{M},\mathscr{B})$ can be associated with a combinatorial structure obtained according to the following steps, which are similar to those introduced in [1]

a) Draw vertex i for each $e_i\in\mathscr{B}$.

b) Given three vertices $i<j<k$, draw the full triangle ijk if and only if $(c_{i,j}^k,c_{j,k}^i,c_{i,k}^j)\neq(0,0,0)$. Then, the edges ij, jk and ik have weights $c_{i,j}^k$, $c_{j,k}^i$ and $c_{i,k}^j$, respectively.

  b1) Use a discontinuous line (named *ghost edge*) for edges with weight zero.

  b2) If two triangles ijk and ijl with $1\leq i<j<k<l\leq n$ satisfy $c_{i,j}^k=c_{i,j}^l$, draw only one edge between vertices i and j shared by both triangles; see Figure 1.

c) Given two vertices i and j with $1\leq i<j\leq n$ and such that $c_{i,j}^i\neq 0$ (resp. $c_{i,j}^j\neq 0$), draw a directed edge from j to i (resp. from i to j), as can be seen in Figure 2.

Consequently, every Malcev algebra with a given basis is associated with a combinatorial structure of this type, which turns out to be a simplicial complex of dimension less than 3.

Example 1. The 3-dimensional Malcev algebra with law $[e_1,e_2]=e_1+e_3$, $[e_1,e_3]=-e_1$, $[e_2,e_3]=e_1+e_2+e_3$, is associated with the combinatorial structure in Figure 3.

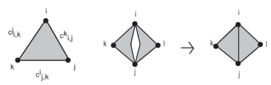

Fig. 1: Full triangle and two triangles sharing an edge.

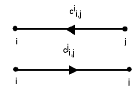

Fig. 2: Directed edges.

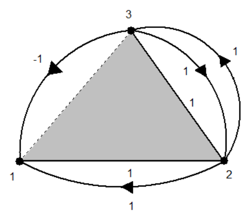

Fig. 3: Combinatorial structure associated with a 3-dimensional Malcev algebra.

4 Theoretical results

Next, we state and prove some general properties arising from the association between Malcev algebras and combinatorial structures and corresponding to topological properties of the combinatorial structure.

Proposition 1. *Let G be the combinatorial structure associated with a Malcev algebra $\mathscr{M}$ with basis $\mathscr{B}$. If v is an isolated vertex of G, then the basis vector $e_v \in \mathscr{B}$ associated with v belongs to the center $Z(\mathscr{M})$.*

Proof. If vertex v is isolated in G, then there are no edges incident with e_v and, hence, $[e_v, e] = 0$, for every basis vertex $e \in \mathscr{B}$, which concludes the proof. □

Proposition 2. *Let G be the combinatorial structure associated with a Malcev algebra $\mathscr{M}$. Each connected component of G is associated with a Malcev subalgebra of $\mathscr{M}$. Moreover, if G is non-connected, then $\mathscr{M}$ is the direct sum of the Malcev subalgebras associated with the connected components of G.*

Proof. Let C be a connected component of G and $\mathscr{B}$ be the basis of $\mathscr{M}$ corresponding to the configuration G. We consider the vector space $\mathscr{M}' = \mathrm{span}(\mathscr{B}')$, where $\mathscr{B}'$ consists of the basis vector in $\mathscr{B}$ corresponding to the vertices of C. Since there are no edges from $V(C)$ to $V(G) \setminus V(C)$, then we can conclude two facts:

i) The brackets between two basis vectors of $\mathscr{B}'$ do not contain coordinates corresponding to basis vectors in $\mathscr{B} \setminus \mathscr{B}'$; i.e. $[\mathscr{B}', \mathscr{B}'] \subseteq \mathscr{B}'$ and $\mathscr{M}'$ is a Malcev subalgebra of $\mathscr{M}$.
ii) The basis vectors in $\mathscr{B}'$ commutes with those in $\mathscr{B} \setminus \mathscr{B}'$ and *vice versa*; i.e. $\mathscr{M} = \mathscr{M}' \oplus \mathrm{span}(\mathscr{B} \setminus \mathscr{B}')$. □

According to Proposition 2, we only need to study the connected configurations associated with Malcev algebras.

4.1 Malcev algebras and digraphs

In this section, we study which weighted digraphs are associated with Malcev algebras; i.e. we only consider the case of non-existence of full triangles in the combinatorial structure. This assertion is equivalent to take into consideration a Malcev algebra $\mathscr{M}$ with basis $\mathscr{B} = \{e_i\}_{i=1}^n$ and law

$$[e_i, e_j] = c_{i,j}^i e_i + c_{i,j}^j e_j, \; 1 \le i < j \le n. \qquad (1)$$

Proposition 3. *If G is a connected digraph with 3 vertices associated with a Malcev algebra, then G must be isomorphic to some of the configurations shown in Figure 4.*

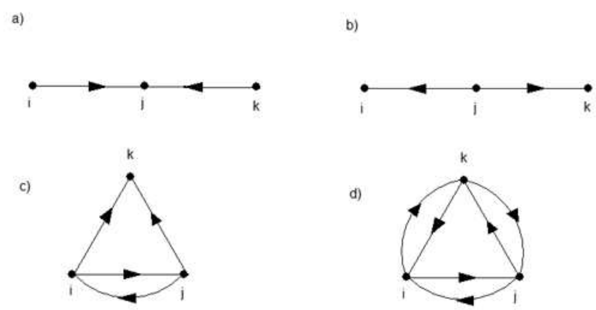

Fig. 4: Connected digraphs with 3 vertices associated with Malcev algebras.

Moreover, Configurations a) and b) are always associated with Malcev algebras, independently on the values of edge weights. In turn, the rest of configurations are associated with Malcev algebras if and only if the following sets of restrictions hold

Configuration c): $\{c_{i,j}^i c_{i,k}^k + c_{i,j}^j c_{j,k}^k = 0\}$ *or* $\{c_{i,j}^i = c_{j,k}^k,\; c_{i,j}^j = -c_{i,k}^k\}$.
Configuration d): $\{c_{i,j}^i = \mp c_{j,k}^k, c_{i,j}^j = \pm c_{i,k}^k, c_{j,k}^j = \pm c_{i,k}^i\}$.

Proof. Let G be a connected digraph with 3 vertices with vertex-set $V(G) = \{i, j, k\}$. We define the vectors e_i, e_j and e_k corresponding to the vertices i, j and k, respectively. The vector space $V = \mathrm{span}(e_i, e_j, e_k)$ endowed with brackets

$$[e_i, e_j] = c_{i,j}^i e_i + c_{i,j}^j e_j, \; [e_i, e_k] = c_{i,k}^i e_i + c_{i,k}^k e_k,$$

$$[e_j, e_k] = c_{j,k}^j e_j + c_{j,k}^k e_k$$

is a Malcev algebra if and only if the Malcev identities hold. After imposing these identities, we obtain the following possible solutions for the coefficients (up to permutation of the subindexes)

1) $\{c_{i,j}^i c_{i,k}^k + c_{i,j}^j c_{j,k}^k = 0\}$;
2) $\{c_{i,j}^i = c_{j,k}^k,\; c_{i,j}^j = -c_{i,k}^k\}$;

3) $\{c_{i,j}^{i} = c_{j,k}^{k}, c_{i,j}^{j} = -c_{i,k}^{k}, c_{j,k}^{j} = -c_{i,k}^{i}\}$;
4) $\{c_{i,j}^{i} = -c_{j,k}^{k}, c_{i,j}^{j} = c_{i,k}^{k}, c_{j,k}^{j} = c_{i,k}^{i}\}$.

Let us note that solutions 1) and 2) correspond to Configuration c) in Figure 4 when all the coefficients are non-null from these conditions. If some of these coefficients are null in these solutions, we obtain configurations a) or b).

When considering solutions 3) and 4) with all the coefficients being non-null, we obtain Configuration d). For these two solutions, we obtain solution 2) if there are only two null coefficients and they both are in the same restriction. Obviously, if some of the remaining coefficients is zero, then we obtain Configurations a) or b). $\square$

Corollary 1. *The connected digraphs with 3 vertices shown in Figure 5 cannot be contained as induced subdigraphs in digraphs associated with Malcev algebras of arbitrary dimension (i.e. they are forbidden configurations).*

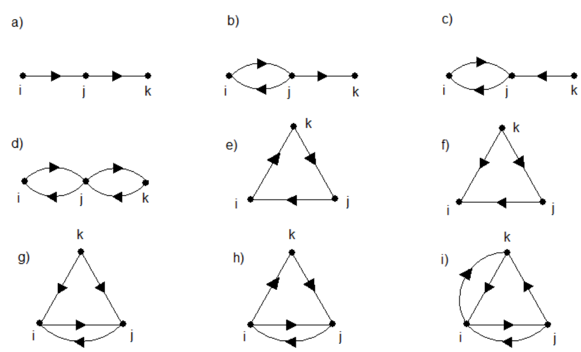

Fig. 5: Forbidden configurations in digraphs associated with Malcev algebras.

Proof. Suffice it to prove that some Malcev identity is not zero. Effectively, if we consider the basis vectors e_i, e_j and e_k, we obtain

Configuration a) and b): $M(e_i,e_k,e_j) = -(c_{i,j}^{j})^2 c_{j,k}^{k} e_k \neq 0$;
Configuration c): $M(e_i,e_k,e_j) = c_{i,j}^{i} c_{i,j}^{j} c_{j,k}^{j} e_i \neq 0$;
Configuration d):
$M(e_i,e_k,e_j) = c_{i,j}^{i} c_{i,j}^{j} c_{j,k}^{j} e_i - (c_{i,j}^{j})^2 c_{j,k}^{k} e_k \neq 0$;
Configuration e):
$M(e_i,e_j,e_k) = (c_{i,k}^{k})^2 c_{j,k}^{j} e_j + 2c_{i,j}^{i} (c_{i,k}^{k})^2 e_k \neq 0$;
Configuration f): $M(e_j,e_i,e_k) = -(c_{i,j}^{i})^2 c_{j,k}^{j} e_i \neq 0$;
Configuration g):
$M(e_i,e_j,e_k) = -c_{i,j}^{i} c_{i,j}^{j} c_{i,k}^{i} e_i - (c_{i,j}^{j})^2 c_{i,k}^{i} e_j \neq 0$;
Configuration h):
$M(e_k,e_i,e_j) = c_{i,j}^{i} (c_{j,k}^{j})^2 e_i - 2c_{i,k}^{k} (c_{j,k}^{j})^2 e_j \neq 0$.

However, we cannot trivially find a non-zero bracket for Configuration i). On the contrary, we need to consider the system of non-linear equations arising from the Malcev identities $M(e_k,e_j,e_i) = 0$, $M(e_i,e_k,e_j) = 0$, $M(e_i,e_j,e_k) = 0$ and $M(e_j,e_k,e_i) = 0$. Every solution of this system contains some coefficient being null although all of them must be non-zero. $\square$

Proposition 4. *Under the restrictions of Proposition 3, it is verified that*

– *Configurations a) to c) in Figure 4 are associated with 2-step solvable, non-nilpotent Malcev algebras.*
– *Configuration d) in Figure 4 is associated with a 2-step solvable, non-nilpotent Malcev algebra if $\{c_{i,j}^{i} = -c_{j,k}^{k},\ c_{i,j}^{j} = c_{i,k}^{k},\ c_{i,k}^{i} = c_{j,k}^{j}\}$ holds; otherwise, the configuration is associated with a perfect Malcev algebra.*

Proof. For every configuration in Figure 4, we can prove that $\mathscr{M}^3 = \mathscr{M}^2$, where $\mathscr{M}$ denotes the Malcev algebra associated with some of these configurations. Therefore, $\mathscr{M}$ is not nilpotent.

Moreover, for Configurations a) to c), $\mathscr{M}^2 = \mathscr{M}_2$ is an abelian ideal of $\mathscr{M}$, which involves $\mathscr{M}_3 = \{0\}$ (i. e. $\mathscr{M}$ is 2-step solvable). This assertion can be trivially proved for Configurations a) and b); whereas the proof for Configuration c) requires to take into account the restrictions over the weights obtained in Proposition 3.

Finally, for Configuration d), we consider the two possible sets of weights given in Proposition 3 to assure the existence of the associated Malcev algebra $\mathscr{M}$. The first option corresponds to define the law of $\mathscr{M}$ as

$$[e_1,e_2] = -c_{2,3}^{3} e_1 + c_{1,3}^{3} e_2,\ [e_1,e_3] = c_{2,3}^{2} e_1 + c_{1,3}^{3} e_3,$$

$$[e_2,e_3] = c_{2,3}^{2} e_2 + c_{2,3}^{3} e_3.$$

If we take the matrix A defined by taking as rows the coordinate vectors of the brackets with respect to the basis $\{e_1,e_2,e_3\}$, then $\det(A) = c_{i,k}^{k} c_{j,k}^{j} c_{j,k}^{k} - c_{j,k}^{k} c_{j,k}^{j} c_{i,k}^{k} = 0$. In addition, the leading principal minor D_2 is non-zero and hence, $\mathscr{M}_2 = \operatorname{span}(-c_{j,k}^{k} e_i + c_{i,k}^{k} e_j, c_{j,k}^{j} e_i + c_{i,k}^{k} e_k)$ and is a 2-dimensional ideal. Since $[-c_{j,k}^{k} e_i + c_{i,k}^{k} e_j, c_{j,k}^{j} e_i + c_{i,k}^{k} e_k] = 0$, then $\mathscr{M}$ is 2-step solvable.

If we take the second set of weights for Configuration d), then the law of $\mathscr{M}$ consists of

$$[e_1,e_2] = c_{2,3}^{3} e_1 - c_{1,3}^{3} e_2,\ [e_1,e_3] = -c_{2,3}^{2} e_1 + c_{1,3}^{3} e_3,$$

$$[e_2,e_3] = c_{2,3}^{2} e_2 + c_{2,3}^{3} e_3.$$

By using an analogous reasoning, we construct the matrix A again, but now $\det(A) = -2c_{i,k}^{k} c_{j,k}^{j} c_{j,k}^{k} - c_{j,k}^{k} c_{j,k}^{j} c_{i,k}^{k} \neq 0$. Hence, $\mathscr{M}_2 = \mathscr{M}$ because it is a 3-dimensional ideal of $\mathscr{M}$. Consequently, $\mathscr{M}$ is perfect. $\square$

Example 2. Now we show two examples of Configuration d) in Figure 4: one being 2-step solvable and non-nilpotent and other being perfect. First, we consider the 3-dimensional Malcev algebra $\mathscr{M}$ with law $[e_1,e_2]=e_1+e_2, [e_1,e_3]=e_1+e_3, [e_2,e_3]=e_2-e_3$. In this case, $\mathscr{M}_2=\mathscr{M}^2=\mathscr{M}^3=\mathrm{span}(e_1+e_2,e_1+e_3)$ and $\mathscr{M}_3=\{0\}$; i. e. $\mathscr{M}$ is 2-step solvable, non-nilpotent.

If we consider the 3-dimensional Malcev algebra $\mathscr{M}$ with law $[e_1,e_2]=e_1-e_2$, $[e_1,e_3]=-e_1+e_3$, $[e_2,e_3]=e_2+e_3$, then $\mathscr{M}_2=\mathrm{span}(e_1,e_2,e_3)=\mathscr{M}$ and hence, $\mathscr{M}$ is perfect.

5 Algorithm to obtain digraph associated with Malcev algebra

This section is devoted to introduce an algorithm which computes the digraph associated with a given finite-dimensional Malcev algebra starting from its law.

Under the same notation as in Section 4, we consider a n-dimensional Malcev algebra $\mathscr{M}$ with basis $\mathscr{B}$. In this way, we consider a law consisting only of brackets $[e_i,e_j]=c^i_{i,j}e_i+c^j_{i,j}e_j$, avoiding full triangles and dealing only with digraphs.

We have designed the following algorithm to obtain the digraph associated with $\mathscr{M}$, structured in four steps

1. Computing the bracket product between two arbitrary basis vectors in $\mathscr{B}$.
2. Evaluating the bracket between two vectors expressed as a linear combination of vectors from basis $\mathscr{B}$.
3. Imposing the Malcev identity and solving the corresponding system of equations.
4. Drawing the digraph associated with the Malcev algebra $\mathscr{M}$.

To implement the algorithm, we have used the symbolic computation package MAPLE 12, loading the libraries `linalg`, `combinat`, `GraphTheory` and `Maplets[Elements]`. The first three libraries allow us to apply commands of Linear Algebra, Combinatorics and Graph Theory, respectively; whereas the last is used to display a message so that the user introduces the required input in the first subprocedure, corresponding to the definition of the law of the algebra $\mathscr{M}$.

The first subprocedure, named `law`, receives two natural numbers as inputs. These numbers represent the subindexes of two basis vectors in $\mathscr{B}$. The subprocedure returns the result of the bracket between these two vectors. In addition, conditional sentences are inserted to determine the non-zero brackets and the skew-symmetry property. Since the user has to complete the subprocedure inserting the non-zero brackets of $\mathscr{M}$, we have also added a sentence at the beginning of the implementation, reminding this fact. Note that before running any other sentence, we must restart all the variables and delete all the computations saved for previous law. Additionally, we must update the value of variable `dim` with the dimension of $\mathscr{M}$.

```
> restart:
> maplet:=Maplet(AlertDialog("Don’t forget
to introduce non-zero brackets of the
algebra and its dimension in subprocedure
law",’onapprove’=Shutdown("Continue"),
’oncancel’=Shutdown("Aborted"))):
> Maplets[Display](maplet):
> assign(dim,...):
> law:=proc(i,j)
>  if i=j then 0; end if;
>  if i>j then -law(j,i); end if;
>  if (i,j)=... then ...; end if;
>  if ....
>  else 0; end if;
> end proc;
```

The ellipsis in command `assign` corresponds to write the dimension of $\mathscr{M}$. The following two suspension points are associated with the computation of $[e_i,e_j]$: first, the value of the subindexes (i,j) and second, the result of $[e_i,e_j]$ with respect to $\mathscr{B}$. The last ellipsis denotes the rest of non-zero brackets. For each non-zero bracket, a new sentence `if` has to be included in the cluster.

Next, we have implemented the subprocedure called `bracket`which computes the product between two arbitrary vectors of $\mathscr{M}$, being expressed as linear combinations of the vectors in $\mathscr{B}$. The subprocedure `law` is called in the implementation.

```
> bracket:=proc(u,v,n)
>  local exp; exp:=0;
>  for i from 1 to n do
>     for j from 1 to n do
>      exp:=exp +
       coeff(u,e[i])*coeff(v,e[j])*law(i,j);
>     end do;
>  end do;
>  exp;
> end proc:
```

Now, we implement the main procedure `Malcev`, which checks if the vector space $\mathscr{M}$ is or is not a Malcev algebra. This procedure receives as input the dimension n of the vector space $\mathscr{M}$ and returns the solution of a system of equations obtained from imposing all the Malcev identities for $\mathscr{M}$ (this is done by using all the permutations of three basis vectors). If there are no solutions for the system, then the vector space $\mathscr{M}$ is not a Malcev algebra. Otherwise, we obtain the set of conditions over the structure constants $c^k_{i,j}$ so that $\mathscr{M}$ is a Malcev algebra.

```
>Malcev:=proc(n)
> local L,M,N,P;
> L:=[];M:=[];N:=[];P:=[];
> for i from 1 to n do
>   L:=[op(L),i];
> end do;
> M:=permute(L,3);
> for j from 1 to nops(M) do
> eq[j]:=bracket(bracket(e[M[j][1]],e[M[j][2]],n),
bracket(e[M[j][1]],e[M[j][3]],n),n)-
bracket(bracket(bracket(e[M[j][1]],e[M[j][2]],n),
e[M[j][3]],n),e[M[j][1]],n)-
bracket(bracket(bracket(e[M[j][2]],e[M[j][3]],n),
e[M[j][1]],n),e[M[j][1]],n)
-bracket(bracket(bracket(e[M[j][3]],e[M[j][1]],n),
e[M[j][1]],n),e[M[j][2]],n);
> end do;
> N:=[seq(eq[k], k=1..nops(M))];
> for k from 1 to nops(N) do
>  for h from 1 to n do
>    P:=[op(P),coeff(N[k],e[h])=0];
>  end do;
> end do;
```

```
> for k from 1 to n do
>  for h from k+1 to n do
>    if coeff(law(k,h),e[h])<>0 then
     P:=[op(P),coeff(law(k,h),e[h])<>0];
>    if coeff(law(k,h),e[k])<>0 then
     P:=[op(P),coeff(law(k,h),e[k])<>0];
>  end do;
>end do;
>solve(P);
>end proc:
```

Finally, the last step of our algorithm is implemented to represent the digraph associated with the Malcev algebras obtained in the previous step. To do so, we have defined two sets: vertex-set V containing all the natural numbers up to n; and edge-set E consisting of each edge determined by a non-zero weight, which must be included in the definition of E according to the brackets defined in (1).

```
> V:=[seq(i,i=1..dim)];
> E:={[[i,j],c_{i,j}^j],[[j,i],c_{i,j}^i],...};
```

and the following sentence provides us the representation of the digraph

```
> G:=Digraph(V,E): DrawGraph(G);
```

Example 3. Now, we show an example with Configuration c) from Figure 4 with the 3-dimensional Malcev algebra given by the law

$$[e_1,e_2] = c_{1,2}^1 e_1 + c_{1,2}^2 e_2;\ [e_1,e_3] = c_{1,3}^3 e_3;\ [e_2,e_3] = c_{2,3}^3 e_3.$$

First, we have to complete the implementation of the subprocedure `law` as follows

```
> if (i,j)=(1,2) then c121*e[1]+c122*e[2];
> end if;
> if (i,j)=(1,3) then c133*e[3];
> end if;
> if (i,j)=(2,3) then c233*e[3];
> else 0;
```

After that, we must run the subprocedure `bracket` and the procedure `Malcev`. Now, if we evaluate the main procedure over the variable `dim`, we obtain the restrictions

```
{c121=-c122*c233/c133, c122=c122, c133=c133,
c233=c233},{c121=c233,c122=-c133,c233=c233,
c133=c133}
```

According to the previous output, we consider one of the Malcev algebras associated with this digraph in order to obtain its representation

$$[e_1,e_2] = e_1 - e_2;\ [e_1,e_3] = e_3;\ [e_2,e_3] = e_3$$

Now, this digraph can be easily represented by using the following orders

```
> V:=[1,2,3];
> E:={[[1,2],-1],[[2,1],1],[[1,3],1],[[2,3],1]};
> G:=Digraph(V,E): DrawGraph(G);
```

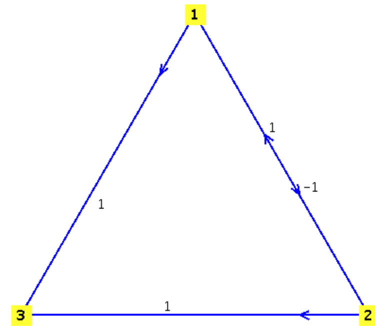

Fig. 6: Digraph corresponding to Configuration c).

5.1 Computational and complexity study

Next, we show a computational study of the previous algorithm, which has been implemented with MAPLE 12, in an Intel Core 2 Duo T 5600 with a 1.83 GHz processor and 2.00 GB of RAM. Table 1 shows some computational data about both the computing time and the memory used to return the output of the main procedure according to the value of the dimension n of the algebra.

To do the computational study, we have considered the family of Malcev algebras $\mathfrak{s}_n$ spanned by $\{e_i\}_{i=1}^n$, with law

$$[e_i,e_n] = e_i, \quad \forall i < n.$$

This family has been chosen because it constitutes a special subclass of non-nilpotent, solvable Malcev algebras, which allow us to check empirically the computational data given for both the computing time and the used memory.

Table 1: Computing time and used memory for `Malcev`.

Input	Computing time	Used memory
$n=2$	0 s	0 MB
$n=3$	0 s	0 MB
$n=4$	0.11 s	3.13 MB
$n=5$	0.15 s	5.06 MB
$n=6$	0.43 s	5.38 MB
$n=7$	1.05 s	5.56 MB
$n=8$	2.67 s	6.06 MB
$n=9$	6.98 s	7.06 MB
$n=10$	20.27 s	8.25 MB
$n=11$	61.17 s	11.50 MB
$n=12$	187.89 s	13.87 MB
$n=13$	804.73 s	51.93 MB

Next we show some brief statistics about the relation between the computing time and the memory used by the implementation of the main procedure `Malcev`.

In this sense, Figure 7 shows the behavior of the computing time (C.T.) for the procedure `Malcev` with respect to the dimension n. In turn, Figure 8 graphically represents the behavior of the used memory (U.M.) with respect to the dimension n. Note that the computing time increases more quickly than the used memory and both of them fit a positive exponential model.

Next, we have studied the quotients between used memory and computing time, obtaining the frequency diagram shown in Figure 9. In this case, the behavior also fits an exponential model, but being negative this time.

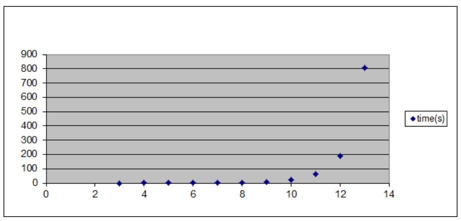

Fig. 7: Graph for the C.T. with respect to dimension.

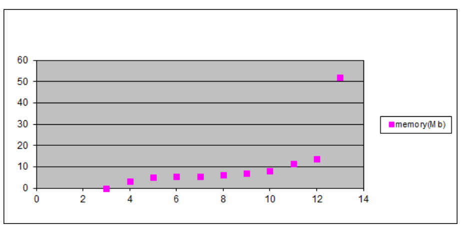

Fig. 8: Graph for the U.M. with respect to dimension.

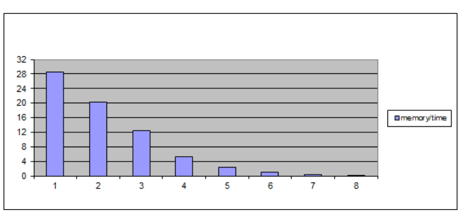

Fig. 9: Graph for quotients U.M./C.T. with respect to dimension.

Finally, we compute the complexity of the algorithm taking into account the number of operations carried out in the worst case. We have used the big O notation to express the complexity. To recall the big O notation, the reader can consult [8]: given two functions $f,g:\mathbb{R}\to\mathbb{R}$, we could say that $f(x)=O(g(x))$ if and only if there exist $M\in\mathbb{R}^+$ and $x_0\in\mathbb{R}$ such that $|f(x)|<M\cdot g(x)$, for all $x>x_0$.

We denote by $N_i(n)$ the number of operations when considering the step i. This function depends on the dimension n of the Malcev algebra. Table 2 shows the number of computations and the complexity of each step, as well as indicating the name of the procedure corresponding to each step.

Table 2: Complexity and number of operations.

Step	Routine	Complexity	Operations
1	`law`	$O(n^2)$	$N_1(n)=2+\frac{n(n-1)}{2}$
2	`bracket`	$O(n^4)$	$N_2(n)=\sum_{i=1}^{n}\sum_{j=1}^{n}N_1(n)$
3	`Malcev`	$O(n^7)$	$N_3(n)=O(n)+O(n^3)+2\sum_{i=1}^{n^3}N_2(n)+\sum_{j=1}^{n^3}\sum_{k=1}^{n}1$

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Manuel Ceballos B.Sc., M.Sc. and Ph.D. in Mathematics from the University of Seville. After studying a Master in Advanced Studies on Mathematics, he held a five-year Research Fellowship to complete his PhD at the Department of Geometry and Topology of this University. He has worked as Substitute Lecturer at the Department of Economics, Quantitative Methods and Economic History from Pablo de Olavide University and at the Department of Algebra from the University of Seville. At present, he is working as Substitute Lecturer in the Department of Applied Mathematics of this University. His contributions are related to Lie and Graph Theory.

Juan Núñez B.Sc. and Ph.D. in Mathematics from the University of Seville, where he works as a "Profesor Titular de Universidad" in the Department of Geometry and Topology (placed in the Faculty of Mathematics). His research is mainly referred to Lie groups and algebras and Discrete Mathematics, having published papers about both of them in several impact factor journals. He has also written papers about Recreational Mathematics, as well as the History and Popularization of Mathematics.

Ángel F. Tenorio B.Sc., M.Sc. and Ph.D. in Mathematics from the University of Seville. At present, he is an Associate Professor of Applied Mathematics in the Department of Economics, Quantitative Methods and Economic History at Pablo de Olavide University. His contributions are related to Lie Theory, Computer Algebra, Graph Theory, Applications to Economics, and History, Popularization and Didactics of Mathematics, with more than 100 papers about these topics.