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Two Fractal Versions of Newton's Law of Cooling

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Abstract: Newton cooling-law equation in terms of a fractional non-local time Caputo derivative of order $0 < \alpha \leq 1$ is solved analytically by the conventional Laplace transform. Smooth solutions in terms of Mittag-Leffler function show two different behaviors when compared to the exponential decay solution from the classical integer-order model: 1) fast heat dissipation at short times, this is characterized by transient solutions showing faster cooling as α tends to 0; 2) slow heat dissipation at medium-large times, solutions in this regime exhibit slower cooling as α approaches 0. Moreover, for $\alpha < 1$ and as time tends to infinity, the temperature decays algebraically with time rather than exponentially. On the other hand, we used the fractional complex transform method to derive the local fractional Newton's law of cooling differential equation of order α. This model defined on Cantor sets, is analytically solved via the Laplace transform. Our staircase shaped solutions are compared with those from the model with Caputo derivative; similarities and differences between these two approaches are pointed out. Hopefully, this generalization of Newton's law of cooling will allow both gaining a better insight into heat convection processes through fractal media and developing a wide variety of new applications.

Keywords: Newton's law of cooling Caputo derivative, Non-local fractal derivative, Laplace transform.

1 Introduction

Fractional calculus is an area of classical mathematics which deals with the generalization of derivatives and integrals to arbitrary orders. Fractional calculus is an old concept that has gained importance during the last few decades in various fields of science and engineering [1]. Recently, some researchers have been studying the fractal version of simple physical models found in classical mechanics. These types of analyses are performed from a heuristic point of view, the idea behind this is to replace the integer order derivatives of an ordinary differential equation for fractional derivatives. The resulting fractional differential equation can be solved by transform methods. Some examples of this procedure can be found in the study of projectile motion in a resisting medium [2], mechanical oscillations [3], relaxation phenomena in viscoelastic materials [4], particle falling through a resisting medium [5,6], and so on.

In this article we study a simple thermal model for convection cooling commonly known in literature as Newton's law of cooling. The main purpose is to extend this cooling model by including fractal properties as power law long-term memory observed in many natural and artificial systems. Consequently, two approaches based on fractional calculus were developed. Typically, Newton's law of cooling is stated in terms of a first order differential equation whose integer order derivative was replaced by a nonlocal fractional Caputo time derivative. The resulting fractional differential equation of order α was solved by the traditional Laplace transform technique. In this way we obtained a variety of smooth solutions for different values of α in the range $0 < \alpha \leq 1$. These solutions include the typical case $\alpha = 1$ in which the cooling process is characterize by an exponential decay. In contrast, when $\alpha < 1$ the solutions show an algebraic decay associated with memory effects.

Furthermore, we obtained a fractal version of Newton's law of cooling in terms of a local fractional derivative which generalizes the usual derivative to fractional order keeping their local nature intact [7]. This local fractional operator of order α is introduced with the motivation of studying the local properties of fractal structures and processes. The fractal

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model with local derivative is solved by Laplace transform [7]. Our staircase shaped non-differentiable solutions closely follow the trend of the solutions from the model with Caputo derivative especially for values of α close to one.

In some sense, the results from the two above mentioned approaches represent a generalization of the classical Newton's law of cooling that hopefully can be used to describe a wide variety of thermal physical situations important for practical applications, which until now have not been attempted by the limitations of the original integer order model.

The plan of this manuscript is as follows. In the next section the traditional Newton's law of cooling stated in terms of an integer order differential equation is analyzed briefly. Then, the cooling law of Newton is expressed in terms of a fractal differential equation with non-local Caputo time derivative; the solutions of this model are presented and discussed. Likewise the cooling law model is derived in terms of a local fractional derivative; solutions and discussions of this model are given. In Section 3, some solutions from cooling models with local and non-local derivatives are compared. Finally, some concluding remarks are drawn in Section 4.

2 Newton's law of cooling

"The rate at which the temperature, $T(t)$, changes in a cooling body at time t is proportional to the difference between the temperature of the body, $T(t)$, and the temperature of the surrounding medium T_m" [8].

Newton's law of cooling is usually modeled with the first-order initial-value problem

$$\left.\begin{array}{l}\frac{dT}{dt} = k(T - T_m) \\ T(0) = T_0\end{array}\right\}, \tag{1}$$

where T_0 is the initial temperature of the body and k is the constant of proportionality. If T_m is constant, the differential equation (1) is separable, resulting in [9]:

$$T(t) = (T_0 - T_m)e^{kt} + T_m. \tag{2}$$

Recall that if $k < 0$, $lim_{t\to\infty} e^{kt} = 0$. Hence, $lim_{t\to\infty} T(t) = T_m$, the temperature of the body approaches that of its surroundings.

2.1 Newton's law of cooling with non-local derivative

In this section, we use the Caputo fractional derivative for a function of time $f(t)$; this operator is defined as [10]

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{3}$$

where $\Gamma(\cdot)$ is the Euler Gamma function, $n = 1, 2, ... \in \aleph$ and $n - 1 < \alpha \leq n$. We consider the case when $n = 1$, i. e., in the integrand there is only a first-order derivative. So, in this case the order of the fractional derivative α is defined in the interval $0 < \alpha \leq 1$. The Caputo fractional derivative fulfills the following properties:

$${}_0^C D_t^\alpha c = 0, \tag{4}$$

$${}_0^C D_t^\alpha [f(t) + g(t)] = {}_0^C D_t^\alpha f(t) + {}_0^C D_t^\alpha g(t). \tag{5}$$

Equation (4) represents the derivative of a constant c, and equation (5) is the linearity property. It should be remarked that the Caputo fractional derivative is defined using an integral, so it is a non-local operator. The fractional derivative in time defined by equation (3) contains information about the function at earlier points, so it allows modeling a memory effect [11]. As in [2,3,5], we replace the integer time derivative by the fractional operator

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\alpha}} \frac{d^\alpha}{dt^\alpha}, \tag{6}$$

where $\frac{d^\alpha}{dt^\alpha} = D$, and α represents the order of the fractional time derivative operator. To assure dimensional homogeneity on both sides of the differential equation (1), σ must have dimension of seconds, $[\sigma] = s$. The time parameter $\sigma^{1-\alpha}$ is associated with the fractional time components of the system [12], of course its dimensionality is

$s^{1-\alpha}$. This non-local time is called in the literature the cosmic time [13]. Thus, we have the following fractional differential equation of order $0 < \alpha \leq 1$

$$\frac{1}{\sigma^{1-\alpha}} \frac{d^{\alpha}T}{dt^{\alpha}} = -k\left(T - T_m\right), \tag{7}$$

subject to the initial condition

$$T(0) = T_0. \tag{8}$$

It should be recalled that the minus sign of the constant k in equation (7) is introduced to model a cooling process. In order to reduce the number of parameters in the problem, the following dimensionless variables are introduced:

$$\Theta = \frac{T}{[T]}, \tag{9}$$

$$\tau = \frac{t}{[t]}, \tag{10}$$

where Θ and τ represent dimensionless temperature and time, respectively; and the reference scales are set as

$$[T] = T_m, \tag{11}$$

$$[t] = \frac{1}{k^{\frac{1}{\alpha}}\,\sigma^{\frac{1-\alpha}{\alpha}}}. \tag{12}$$

It is worthy of note that the time reference scale, equation (12), represents the fractal time constant of the system. Indeed, for $\alpha = 1$, equation (12) reduces to $\tau_c = 1/k$, which is the time constant of the ordinary case [14].

Substituting equations (11) and (12) into equations (9) and (10) and the result into equations (7) and (8), yields:

$$\left.\begin{array}{l} \frac{d^{\alpha}\Theta}{d\tau^{\alpha}} = 1 - \Theta \\ \Theta(0) = \beta \end{array}\right\}. \tag{13}$$

The behavior of the system (13) only depends on the values of the dimensionless parameters $\beta = T_0/T_m$ and α . It should be noted that $\beta > 1$, since the cooling process only occurs for $T_0 > T_m$. Also important is to note that the value of β has to be determined in function of the relative contributions of convective and radiative heat-transfer rates [15]. We set as an upper limit $\beta = 4.5$ which is reasonable if we consider standard conditions of temperature at $T_m = 20°\text{C}$. The initial value problem represented by equations (13) is solved by applying the Laplace transform [16]. Thus, the solution is expressed as follows:

$$\Theta(\tau) = 1 + E_{\alpha}\left(-\tau^{\alpha}\right)\,[\beta - 1]. \tag{14}$$

The analytic solution (14) contains the Mittag-Leffler function $E_{\alpha}(-\tau^{\alpha})$, which is a generalization of the natural exponential function and represents attenuation with strong memory effect. Figure 1 shows graphs of equation (14) for various values of α and for a fixed value of the initial condition β. The thick dashed green line represents the solution for the ordinary case which is obtained by setting $\alpha = 1$. Clearly, the steady state condition for the ordinary case is reached at around five time constants. However, as the order of the time derivative decreases, the solutions show an algebraic decay. This behavior is explained by taking into account the asymptotic behavior of the Mittag-Leffler function for large values of arguments [17]:

$$E_{\alpha}\left(-\tau^{\alpha}\right) \approx \frac{1}{\Gamma(1-\alpha)} \frac{1}{t^{\alpha}}. \tag{15}$$

Furthermore, Fig. 1 shows that as α decreases the steady state solution is reached at longer times. This, of course, is a consequence of the algebraic decay of solutions. Figure 2 shows the time constants to reach the steady state for different values of α. Each point of this graph was calculated by enforcing the following condition:

$$E_{\alpha}\left(-\tau^{\alpha}\right)\,[\beta - 1] \approx \phi, \tag{16}$$

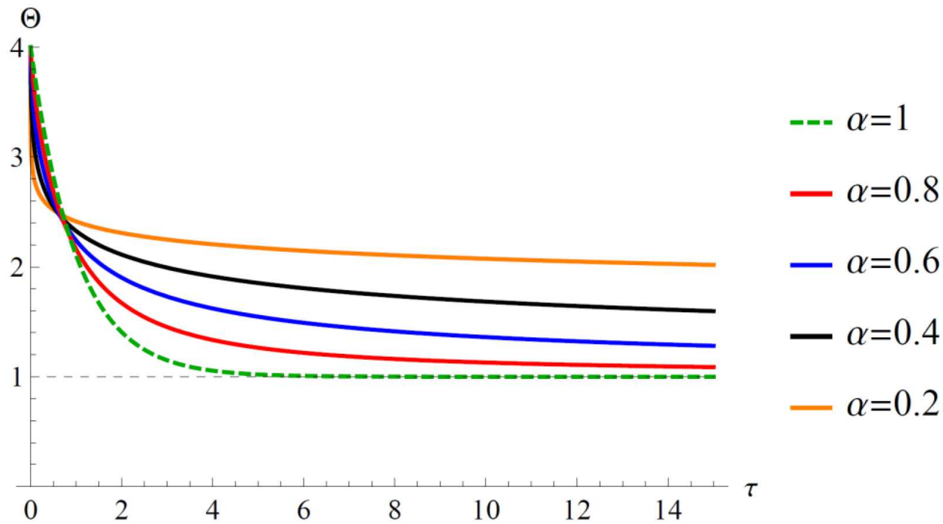

Fig. 1: Newton's law of cooling with time Caputo derivative for different values of α and for $\beta = 4$. The thin dashed line represents the steady state environmental temperature.

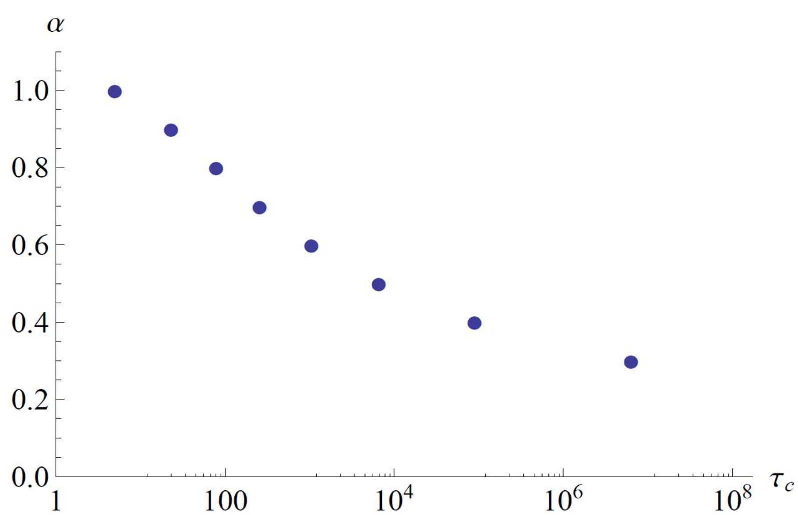

Fig. 2: Order of the time derivative (α) vs. time constants (τ_c) to reach the steady state. This curve was computed from Eq. (17) with $\beta = 4$ and $\phi = 0.02$.

where ϕ is a small positive number; according to Eq. (14), the expression (16) represents the transient response of the fractal model. Thus, by combining Eqs. (15) and (16) is readily found

$$\tau_c \approx \left(\frac{\beta - 1}{\phi\, \Gamma(1-\alpha)} \right)^{\frac{1}{\alpha}}. \tag{17}$$

Note that all calculations were performed by setting $\phi = 0.02$.

Solutions from some analogous models to our thermal system have been published in the past. Very similar curves as those presented in Fig. 1 can be found in [4]. In this article, the fractional generalization of the first-order differential equation governing the phenomenon of viscoelastic relaxation is treated. Another analogous system with similar solutions as those from Fig. 1 is represented by a fractional R-C circuit (with resistance R and capacity C), which is used to simulate the aging of alkaline batteries after repeated charge/discharge cycles [18]. Figure 3a depicts a zoom of Fig. 1 of the transient response of cooling law with time Caputo derivative for different values of α and for short times.

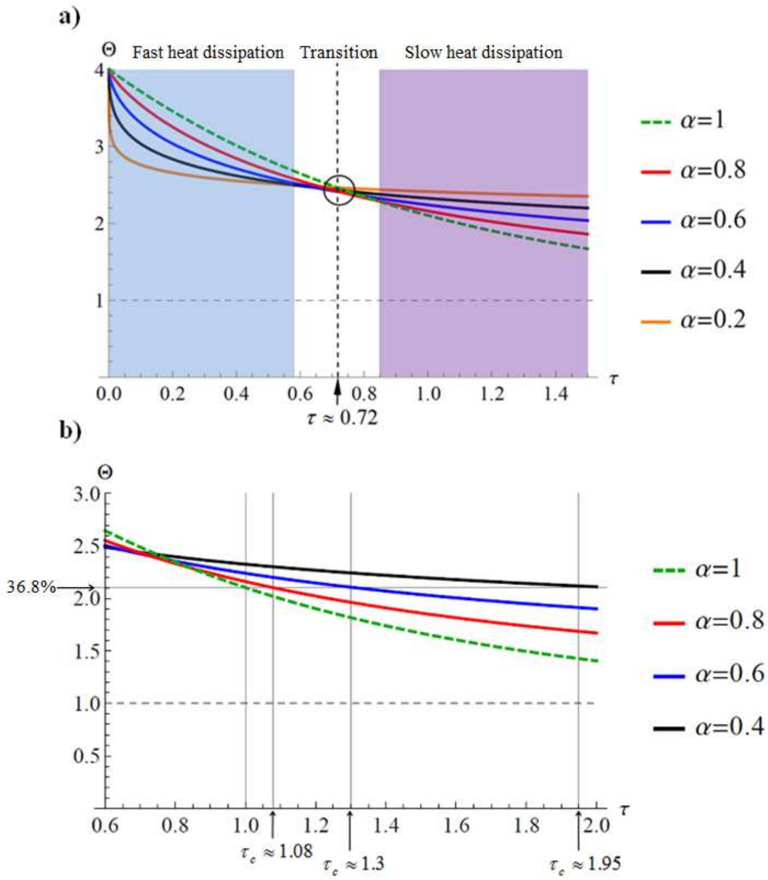

Fig. 3: a) Transient solutions of Newton's law of cooling with time Caputo derivative. The values α and β as well as the colors used in curves correspond with those in Fig. 1. b) Transient solutions after the transition zone, time constants for each α are indicated with arrows; $\tau_c\,|_{\alpha=1}=1$,$\tau_c\,|_{\alpha=0.8}\approx 1.08$, $\tau_c\,|_{\alpha=0.6}\approx 1.3$,$\tau_c\,|_{\alpha=0.4}\approx 1.95$, $\tau_c\,|_{\alpha=0.2}\approx 7.2$.

Three regions of different behaviors are identified. The region in blue,$0 \leq \tau \leq 0.58$, reveals faster heat dissipation as α decreases from 1 to 0. The no-color region, $0.58 < \tau \leq 0.85$, can be seen as a transition zone where the solutions are overlapped intersecting each other at $\tau = 0.72$. After this overlapping zone, the model predicts an opposite behavior than that observed in the blue region. Namely, in the domain $0.85 < \tau \leq 1.5$, the third region in purple is characterized by slower heat dissipation as α decreases from 1 to 0; this of course is the expected thermal memory effect. Figure (3b) shows transient solutions for different values of α. When the time constant is of course $\tau_c = 1$, which physically represents the time required for the temperature Θ to fall 63.2 % from the initial temperature β to the limiting environmental temperature $\Theta = 1$. The time constant for any value of α can be found by solving numerically the following equation

$$E_\alpha\left(-\tau^\alpha\right) \approx 0.368. \tag{18}$$

It is clearly observed that the time constant τ_c increases as α decreases; again, this behavior can be associated with a memory effect.

Figure 4a shows transient solutions of cooling model with time Caputo derivative for different values of α and β . Interestingly, the transition point at $\tau \approx 0.72$ seems to be independent of the value of β. On the other hand; as expected, the exponential decay characterized by $\alpha = 1$ implies reaching the steady state solution at around five time constants independently of the value of β. Nevertheless, for $\alpha < 1$, the stationary solutions strongly depend on the value of β. Each curve from Figure 4b shows the time needed to reach the steady state depending on the values of α and β; clearly, as β increases, longer times to reach the stationary solutions are required. Of course, this last result is a consequence of the algebraic decay of solutions for $\alpha < 1$.

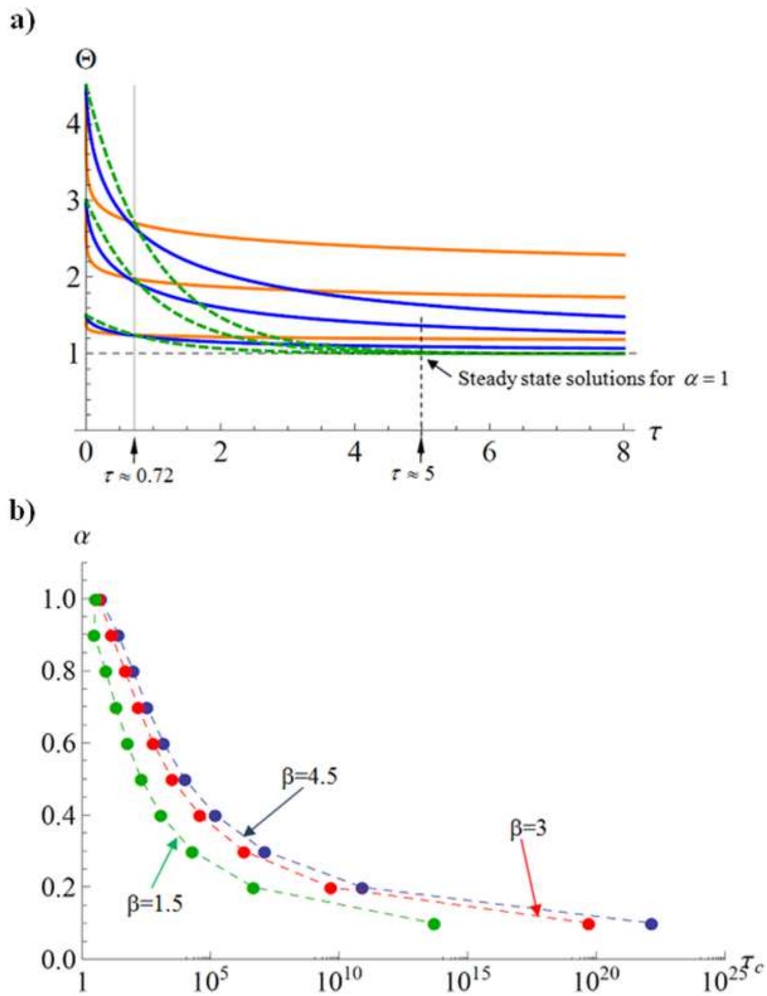

Fig. 4: a) Transient solutions of Newton's law of cooling with time Caputo derivative for $\beta = (1.5, 3 \text{ and } 4.5)$. The values of α as well as the colors used in curves correspond with those used in Fig. 1. Note that for the sake of clarity, the curves for $\alpha = 0.2$ and $\alpha = 0.4$ were removed. b) Each of the three curves depicts the order of the time derivative (α) vs. time constants (τ_c) to reach the steady state (Eq. (17) was used to compute curves).

2.2 Newton's law of cooling with local fractional derivative

The fractal complex transform [19], expressed as:

$$t = \frac{(p\xi)^{\alpha}}{\Gamma(1+\alpha)}, \tag{19}$$

is applied to switch the conventional differential equation of Newton's law of cooling into local fractional differential equation. So, substituting Eq. (19) into Eq. (1) yields:

$$\frac{1}{p^{\alpha}} \frac{d^{\alpha}T(\xi)}{d\xi^{\alpha}} = -k\left(T(\xi) - T_m\right), \tag{20}$$

with the constant parameter $p^{\alpha} = \frac{d^{\alpha}t}{d\xi^{\alpha}}$.

Here we argue that parameter p plays the same role as σ used in equation (7); i. e., the units in equation (20) are balanced by p.

After renaming ξ as t , equation (20) becomes

$$\frac{1}{p^{\alpha}}\frac{d^{\alpha}T}{dt^{\alpha}}=-k\left(T-T_m\right). \tag{21}$$

Equation (21) subject to the initial condition $T(0)=T_0$ represents the fractional cooling model with local derivative. This local fractional derivative operator on Cantor sets is defined by [7]

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{\left(x-x_0\right)^{\alpha}}, \tag{22}$$

with $\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(1+\alpha)\Delta(f(x)-f(x_0))$. Now, the parameters in equation (21) can be reduced using dimensionless variables as those defined by equations (9) and (10). In this case the reference scale for temperature is also set as $[T]=T_m$; however, the reference scale for time is defined as:

$$[t]=\frac{1}{k\,p^{\alpha}}. \tag{23}$$

After substituting equations (11) and (23) into equations (9) and (10) and the result into equation (21), and taking into account the initial condition $T(0)=T_0$, the dimensionless model becomes

$$\left.\begin{array}{l}\frac{d^{\alpha}\Theta}{d\tau^{\alpha}}=1-\Theta\\ \Theta(0)=\frac{T_0}{T_m}=\beta\end{array}\right\}. \tag{24}$$

Now, the behavior of the initial value problem (24) only depends on two dimensionless parameters, β and α. Equation (24) can be solved by the Laplace transform, which is defined as [7]

$$\tilde{L}_{\alpha}\left\{f(x)\right\}=f_s^{\tilde{L},\alpha}(s)=\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}E_{\alpha}(-s^{\alpha}x^{\alpha})f(x)(dx)^{\alpha},\ 0<\alpha\leq 1, \tag{25}$$

where $f(x)$ is a local fractional continuous function.

Finally, after applying the Laplace transform to the initial value problem (24), the solution in terms of the Mittag-Leffler function E_{α}, is:

$$\Theta(\tau)=1-E_{\alpha}\left(-\tau^{\alpha}\right)+\beta\,E_{\alpha}\left(-\tau^{\alpha}\right). \tag{26}$$

The exact solution (26) for $\alpha=ln2/ln3=0.631$ is shown in Fig. (5). It is worth noticing that $\alpha=0.631$ corresponds to the fractal dimension of the Cantor middle-1/3 set. This nonintegrable staircase shaped curve is only plotted in the range $0\leq\tau\leq 1$, since τ^{α} is closely related to the Cantor-Lebesgue function defined on [0, 1]. The geometrical characteristics of this solution are not only interesting from a physical point of view, but also of great practical importance if one wishes to exploit the fractal configurations in engineering devices. In relation to the first aspect and in some analogy with quantum systems, the steps of the staircase shaped curves may be related to the energy states of a submicroscopic thermal convection system.

Figure 6 depicts graphs of Eq. (26) for different values of the order of the derivative α . As α approaches 1, solutions appear smoother tending to the classical integer order solution for $\alpha=1$ (dashed curve in the figure). If α is very close to unity (i. e., when $\alpha=ln2/(ln2001-ln1000)=0.999$, see inset on Fig. 8), the curve will be continuous to the naked eye, but when zooming in, the staircase shaped structure will emerge.

3 Comparisons between models with local and non-local derivative operators

Figure 7 shows a comparison between solutions from equations (14) and (26) corresponding to models with non-local and local derivatives, respectively. Clearly, both curves show a similar trend; nevertheless, the staircase shaped curve from the model with local derivative indicates faster heat dissipation than that shown by the smooth continuous solution from the model with Caputo derivative. This behavior is observed for $\tau<5$; conversely, for $\tau>5$, the smooth solution indicates faster heat dissipation. This interesting behavior could be of practical interest in developing thermal convection devices using fractal configurations defined on Cantor sets. Indeed, this idea needs to mature before giving concrete examples.

As shown in Fig. 8, a comparison between solutions (14) and (26) for different values of α is shown. The approach with local fractional derivative predicts higher heat dissipation at short times ($\tau<5$) than that predicted with Caputo derivative; conversely, for $\tau>5$ the heat dissipation from model with local derivative is less than that predicted from the

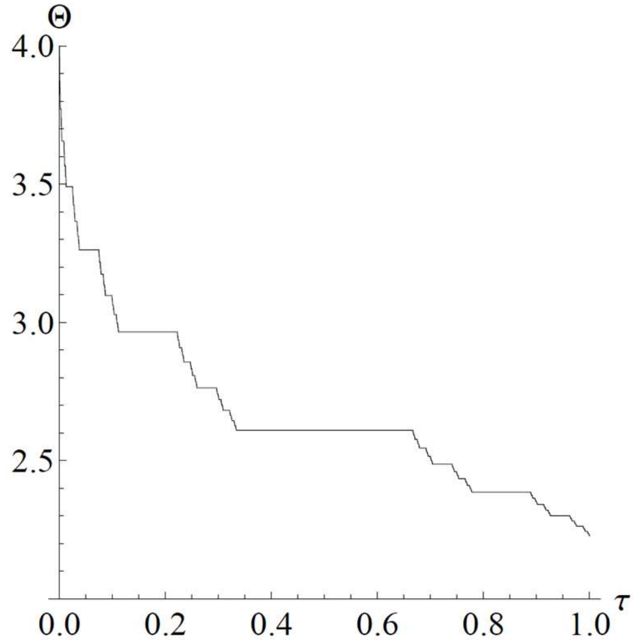

Fig. 5: The plot of nondifferentiable solution (26) with the parameter $\alpha = ln2/ln3 = 0.631$. The initial condition in this case was set to $\beta = 4$.

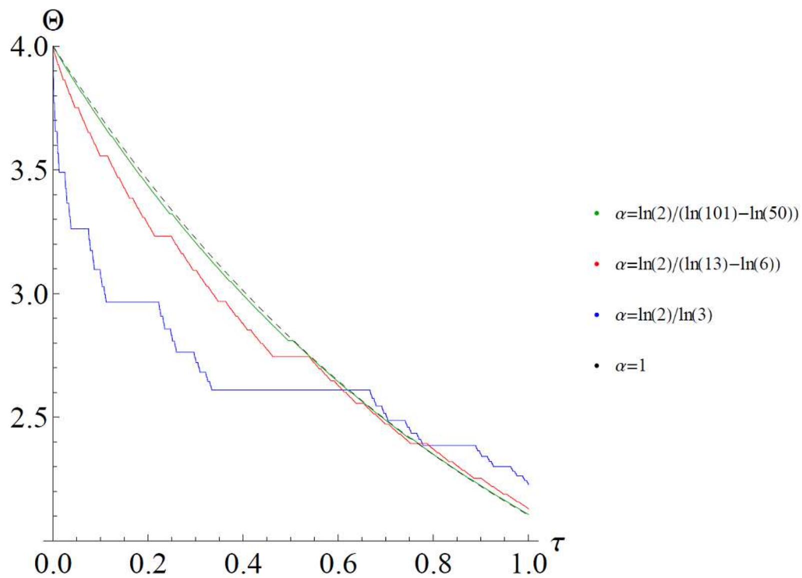

Fig. 6: The plot of nondifferentiable solution (26) for different values of α. $\alpha = ln2/ln3 = 0.631$ for a Cantor middle-1/3 set; $\alpha = ln2/(ln13 - ln6) = 0.896$ for a Cantor middle-1/13 set; and $\alpha = ln2/(ln101 - ln50) = 0.986$ for a Cantor middle-1/101 set [20].

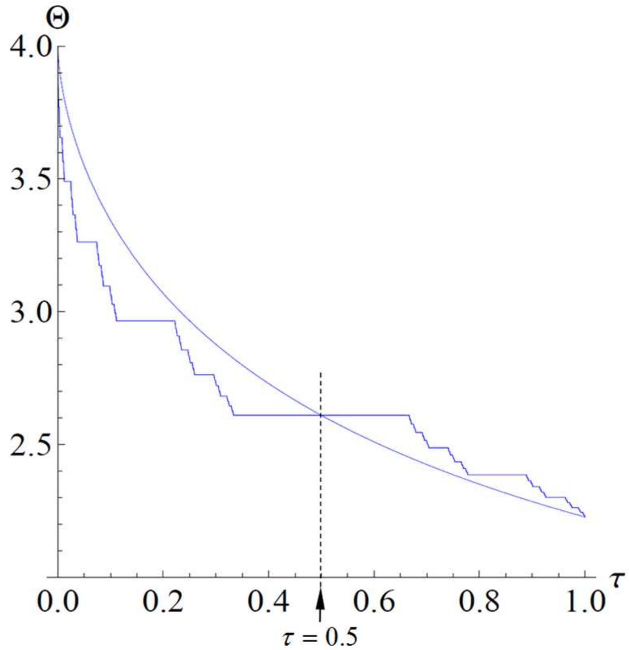

Fig. 7: Comparison between the exact solutions (14) and (26) of Newton's law of cooling with non-local and local fractional derivatives. Both curves were computed for $\alpha = ln2/ln3 = 0.631$, which corresponds with the fractal dimension of the Cantor middle-1/3 set.

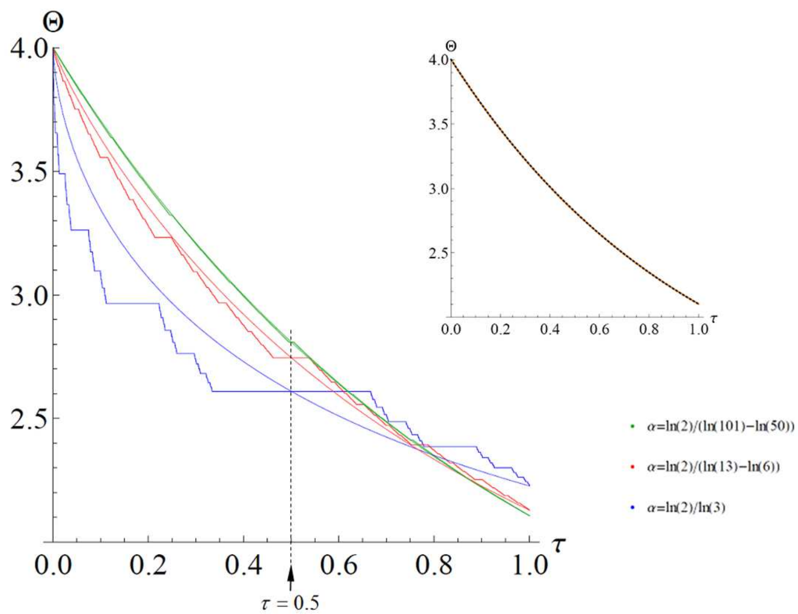

Fig. 8: Comparison between the exact solutions (14) and (26) of Newton's law of cooling with nonlocal and local fractional derivatives for different values of α.

model with Caputo derivative. Although this behavior is observed for three different values of α, it becomes less visible as α tends to unity. The inset in Figure 8 shows graphs of solutions (14) (dashed line) and (26) (solid orange line) for a Cantor middle-1/2001 set; clearly, it is not observed any difference between both solutions.

4 Conclusions

Two fractal versions of Newton's law of cooling were developed. One of them is expressed in terms of non-local Caputo time derivative. The behavior of the smooth curve solutions from this approach strongly depend on two parameters: the order of the fractal derivative α and the initial condition β. Our solutions in terms of Mittag-Leffler function show two different behaviors when compared to the exponential decay solution from the classical integer-order model (this is observed for $\beta > 1$): 1) *Fast heat dissipation* at short times, this regime is characterized by transient responses showing faster cooling as α tends to 0. 2) *Slow heat dissipation* at medium-large times, solutions in this regime exhibit slower cooling as α approaches 0. As α decreases (from 1 to 0) the stationary solutions of the thermal system are reached at longer times, this is true for $\beta > 1$. In other words, for $\alpha < 1$ and as time tends to infinity, the temperature decays algebraically with time rather than exponentially. Even this behavior is accentuated when increasing the initial temperature, i. e., the time to reach the steady state increases with β. In few words the approach using Caputo derivative quite well incorporates and describes long term memory effects which are related to an algebraic decay clearly seen in the solutions. In the other approach we replaced the integer order derivative by a local fractional derivative defined on Cantor sets. Curves from this model show a peculiar staircase shape and are similar in trend with those obtained from the model with Caputo derivative. In particular, when the Cantor middle-1/3 set is considered, the staircase shaped curve from model with local derivative indicates faster heat dissipation than that shown by the smooth continuous solution from model with Caputo derivative. Although this behavior is observed for three different values of α, it becomes less visible as α tends to unity. It is important to point out that the classical solution of Newton's law of cooling is recovered from the two above-mentioned approaches when $\alpha = 1$. So, in this sense, the two fractal versions of the thermal convection model developed here represent a generalization of Newton's law of cooling. Hopefully, these new results will allow to understand better the convection cooling processes through fractal media, to extend the range of applicability of Newton's law of cooling model and to develop novel practical engineering technologies based on convection cooling.

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