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The Modified Simplest Equation Method for Finding the Exact Solutions of Nonlinear PDEs in Mathematical Physics

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Abstract: In this present work, the modified simplest equation method is used to construct exact solutions of (2+1)-dimensional nonlinear Schrödinger equation, the Schrödinger-Hirota equation and the perturbed nonlinear Schrödinger equation (NLSE) with Kerr law nonlinearity. The modified simplest equation method is powerful method for obtaining exact solutions of nonlinear partial differential equations.

Keywords: modified simplest equation method, (2+1)-dimensional nonlinear Schrödinger equation,the Schrödinger-Hirota equation, NLSE with Kerr law nonlinearity.

1 Introduction

Research on solutions of nonlinear partial differential equations is popular. So, the powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. Many efficient methods have been presented so far. Recently, seeking the exact solutions of nonlinear equations has getting more and more popular. Many approaches have been presented so far such as Bcklund transformation method [1], Hirotas direct method [2,3] tanh-sech method [4,5], extended tanh method [6,7], sine-cosine method [8], homogeneous balance method [9], $\frac{G'}{G}$-expansion method [10] and so on.

In this paper, we proposed a modified simplest equation method, and present applications for this method to nonlinear partial differential equations. The rest of the paper is organized as follows. In section 2, we describe the modified simplest equation method for finding traveling wave solutions of nonlinear partial differential equations, and give the main steps of the method. In the subsequent sections, we will apply the method to find exact traveling wave solutions of the nonlinear (2+1)-dimensional nonlinear Schrödinger equation, the Schrödinger-Hirota equation and the perturbed nonlinear Schrödinger equation (NLSE) with Kerr law nonlinearity. In the last section, some conclusions are presented.

2 Description of the modified simplest equation method

The modified simplest equation method is based on the assumption that the exact solutions can be expressed by a polynomial in $\frac{F'}{F}$, such that $F = F(\xi)$ is an unknown linear ordinary equation to be determined later. This method consists of the following steps:

Step 1. Consider a general form of nonlinear partial differential equation (PDE)

$$P(u, u_x, u_t, u_{xx}, u_{tx}, \ldots) = 0. \tag{1}$$

Assume that the solution is given by $u(x,t) = U(\xi)$ where $\xi = x + ct$. Hence, we use the following changes:

$$\begin{aligned} \frac{\partial}{\partial t}(.) &= c\frac{\partial}{\partial \xi}(.), \\ \frac{\partial}{\partial x}(.) &= \frac{\partial}{\partial \xi}(.), \\ \frac{\partial^2}{\partial x^2}(.) &= \frac{\partial^2}{\partial \xi^2}(.). \end{aligned} \tag{2}$$

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and so on for other derivatives. Using (2) changes the PDE (1) to an ODE

$$Q(U,U',U'',\ldots)=0. \tag{3}$$

where $U=U(\xi)$ is an unknown function, Q is a polynomial in the variable U and its derivatives.

Step 2. We suppose that Eq. (3) has the following formal solution:

$$U(\xi)=\sum_{i=0}^{N}A_i\left(\frac{F'}{F}\right)^i, \tag{4}$$

where A_i are arbitrary constants to be determined such that $A_N\neq 0$, while $F(\xi)$ is an unknown function to be determined later.

Step 3. We determine the positive integer N in (4) by balancing the highest order derivatives and the nonlinear terms in Eq.(3).

Step 4. We substitute (4) into (3), we calculate all the necessary derivatives $U,U',U'',\ldots$ and then we account the function $F(\xi)$. As a result of this substitution, we get a polynomial of $\frac{F'(\xi)}{F(\xi)}$ and its derivatives. In this polynomial, we equate with zero all the coefficients of it. This operation yields a system of equations which can be solved to find A_i and $F(\xi)$. Consequently, we can get the exact solution of Eq.(1).

3 Application the modified simplest equation method

In this section, we study the (2+1)-dimensional nonlinear Schrödinger equation, the Schrödinger-Hirota equation and NLSE with Kerr law nonlinearity using the modified simplest equation method.

3.1 The modified simplest equation method to the (2+1)-dimensional nonlinear Schrödinger equation

Let us first the (2+1)-dimensional nonlinear Schrödinger equation that [11] that reads:

$$iu_t+au_{xx}-bu_{yy}+c|u|^2u=0 \tag{5}$$

where a,b and c are nonzero constants. Firstly, we introduce the transformations

$$u(x,y,t)=\exp(i(\alpha x+\omega y+\delta t))\phi(\xi),\quad \xi=k(x+ly-\lambda t) \tag{6}$$

where $\alpha,\omega,\delta,k,l,$ and λ are real constants. Substituting (6) into Eq. (5) we obtain the $\lambda=2(\alpha a-b\omega l)$ and $\phi(\xi)$ satisfy into ODE:

$$-(\delta+a\alpha^2-b\omega^2)\phi(\xi)+(a-bl^2)k^2\phi''(\xi)+c\phi^3(\xi)=0 \tag{7}$$

Rewrite this second-order ordinary differential equation as follows:

$$\phi''(\xi)+k_1\phi(\xi)+k_3\phi^3(\xi)=0 \tag{8}$$

Where $k_1=-\frac{(\delta+a\alpha^2-b\omega^2)}{(a-bl^2)k^2}$ and $k_3=\frac{c}{(a-bl^2)k^2}$.

By balancing the highest order derivative term ϕ'' with the nonlinear term ϕ^3 in (8), we obtain $N=1$ in (4). So we assume that Eq. (4) has solution in the form

$$\phi(\xi)=A_0+A_1\left(\frac{F'}{F}\right),\quad A_1\neq 0. \tag{9}$$

Using (9), we obtain

$$\phi^3=A_0^3+3A_0^2A_1\left(\frac{F'}{F}\right)+3A_0A_1^2\left(\frac{F'}{F}\right)^2+A_1^3\left(\frac{F'}{F}\right)^3 \tag{10}$$

$$\phi''=A_1\left(\frac{F'''}{F}-\frac{F'F''}{F^2}+2\left(\frac{F'}{F}\right)^3\right). \tag{11}$$

Substituting (9) to (11) into Eq. (8) and setting the coefficients of $F^j(j=0,-1,-2)$ to zero, we obtain

$$k_1A_0+k_3A_0^3=0, \tag{12}$$
$$A_1F'''+k_1A_1F'+3k_3A_0^2A_1F'=0, \tag{13}$$
$$-3A_1F'F''+3k_3A_0A_1^2F'^2=0, \tag{14}$$
$$2A_1F'^3+k_3A_1^3F'^3=0. \tag{15}$$

Eqs. (12) and (15) directly imply following solutions:

$$A_0=\pm\sqrt{-\frac{k_1}{k_3}},\quad A_1=\pm\sqrt{\frac{-2}{k_3}},\quad k_1>0,\ k_3<0.$$

Thus, Eqs. (13) and (14) become

$$F'''-2k_1F'=0, \tag{16}$$
$$-F''+\sqrt{2k_1}F'=0. \tag{17}$$

By substituting Eq. (17) into Eq. (16) we get

$$-\sqrt{2k_1}F''+F'''=0. \tag{18}$$

The general solution of Eq. (18) is

$$F(\xi)=a_0+a_1\xi+a_2\exp(\sqrt{2k_1}\xi)$$

where $a_i(i=0,1,2)$ are arbitrary constants.
Thus, we have

$$\phi(\xi)=\pm\sqrt{-\frac{k_1}{k_3}}\pm\sqrt{-\frac{2}{k_3}}\left(\frac{a_1+\sqrt{2k_1}a_2\exp(\sqrt{2k_1}\xi)}{a_0+a_1\xi+a_2\exp(\sqrt{2k_1}\xi)}\right)$$

Now, the exact solution of Eq.(5) have the form

$$u(x,y,t)=\pm\sqrt{-\frac{k_1}{k_3}}\pm\sqrt{-\frac{2}{k_3}}\left(\frac{a_1+\sqrt{2k_1}a_2\exp(\sqrt{2k_1}(k(x+ly-\lambda t)))}{a_0+a_1(k(x+ly-\lambda t))+a_2\exp(\sqrt{2k_1}(k(x+ly-\lambda t)))}\right)$$
$$\times\exp(i(\alpha x+\omega y+\delta t))$$

3.2 The modified simplest equation method to the Schrödinger-Hirota equation

Let us consider the nonlinear the Schrödinger-Hirota equation which governs the propagation of optical solitons in a dispersive optical fiber:

$$iu_t+\frac{1}{2}u_{xx}+|u|^2u+i\lambda u_{xxx}=0 \tag{19}$$

This equation studied [12] by the ansatz method for bright and dark 1-soliton solution. The power law nonlinearity was assumed. Introduce the transformations

$$u(x,t)=\exp(i(\alpha x+\beta t))\phi(\xi),\quad \xi=k(x-2\alpha t) \tag{20}$$

where α,β and k are real constants. Substituting (20) into Eq.(19) we obtain that $\alpha=\frac{-1}{3\lambda}$ and $\phi(\xi)$ satisfy into the ODE:

$$-(\frac{5}{54\lambda^2}+\beta)\phi(\xi)+\frac{3}{2}k^2\phi''(\xi)+\phi^3(\xi)=0 \tag{21}$$

Then we can write the following equation:

$$\phi''(\xi)+k_1\phi(\xi)+k_3\phi^3(\xi)=0 \tag{22}$$

Where $k_1=-\frac{(\frac{5}{54\lambda^2}+\beta)}{\frac{3}{2}k^2}$ and $k_3=\frac{1}{\frac{3}{2}k^2}$.

By balancing the highest order derivative term ϕ'' with the nonlinear term ϕ^3 in (22), we obtain $N=1$ in (4). So we assume that Eq. (4) has solution in the form

$$\phi(\xi)=A_0+A_1(\frac{F'}{F}),\quad A_1\neq 0. \tag{23}$$

Using (23), we obtain

$$\phi^3=A_0^3+3A_0^2A_1(\frac{F'}{F})+3A_0A_1^2(\frac{F'}{F})^2+A_1^3(\frac{F'}{F})^3 \tag{24}$$

$$\phi''=A_1(\frac{F'''}{F}-\frac{F'F''}{F^2}+2(\frac{F'}{F})^3). \tag{25}$$

Substituting (23) to (25) into Eq. (22) and setting the coefficients of $F^j(j=0,-1,-2)$ to zero, we obtain

$$k_1A_0+k_3A_0^3=0, \tag{26}$$
$$A_1F'''+k_1A_1F'+3k_3A_0^2A_1F'=0, \tag{27}$$
$$-3A_1F'F''+3k_3A_0A_1^2F'^2=0, \tag{28}$$
$$2A_1F'^3+k_3A_1^3F'^3=0. \tag{29}$$

Eqs. (26) and (29) directly imply following solutions:

$$A_0=\pm\sqrt{-\frac{k_1}{k_3}},\quad A_1=\pm\sqrt{\frac{-2}{k_3}},\quad k_1>0,\ k_3<0.$$

Thus, Eqs. (27) and (28) become

$$F'''-2k_1F'=0, \tag{30}$$
$$-F''+\sqrt{2k_1}F'=0. \tag{31}$$

By substituting Eq. (31) into Eq. (30) we get

$$-\sqrt{2k_1}F''+F'''=0. \tag{32}$$

The general solution of Eq. (19) is

$$F(\xi)=a_0+a_1\xi+a_2\exp(\sqrt{2k_1}\xi)$$

where $a_i(i=0,1,2)$ are arbitrary constants.
Thus, we have

$$\phi(\xi)=\pm\sqrt{-\frac{k_1}{k_3}}\pm\sqrt{-\frac{2}{k_3}}(\frac{a_1+\sqrt{2k_1}a_2\exp(\sqrt{2k_1}\xi)}{a_0+a_1\xi+a_2\exp(\sqrt{2k_1}\xi)})$$

Now, the exact solution of Eq.(19) have the form

$$u(x,t)=\pm\sqrt{-\frac{k_1}{k_3}}\pm\sqrt{-\frac{2}{k_3}}(\frac{a_1+\sqrt{2k_1}a_2\exp(\sqrt{2k_1}(k(x-2\alpha t)))}{a_0+a_1(k(x-2\alpha t))+a_2\exp(\sqrt{2k_1}(k(x-2\alpha t)))})\times\exp(i(\alpha x+\beta t))$$

3.3 The modified simplest equation method to the NLSE with Kerr law nonlinearity equation

In this section we consider the NLSE with Kerr law nonlinearity equation

$$iu_t+u_{xx}+\alpha|u|^2u+i[\gamma_1u_{xxx}+\gamma_2|u|^2u_x+\gamma_3(|u|^2)_xu]=0 \tag{33}$$

where γ_1 is third order dispersion, γ_2 is the nonlinear dispersion, while γ_3 is a also a version of nonlinear dispersion [13,14]. Eq.(33) describes the propagation of optical solitons in nonlinear optical fibers that exhibits a Kerr law nonlinearity. Eq. (33) has important application in various fields, such as semiconductor materials, optical fiber communications, plasma physics, fluid and solid mechanics. More details are presented [13,14,15] .

We seek its traveling wave solution of the form

$$u(x,t)=\phi(\xi)\exp(i(kx-\Omega t)),\quad \xi=x-ct \tag{34}$$

Substituting equation (34) into equation (33), we have

$$\begin{aligned}&i(\gamma_1\phi'''-3\gamma_1k^2\phi'+\gamma_2\phi^2\phi'+2\gamma_3\phi^2\phi'-c\phi'+2k\phi')\\&+(\Omega\phi+\phi''-k^2\phi+\alpha\phi^3+3\gamma_1k\phi''+\gamma_1k^3\phi-\gamma_2k\phi^3)=0,\end{aligned} \tag{35}$$

where $\gamma_i(i=1,2,3),\alpha$ are positive constants and prime meaning differentiation with respect to ξ. Then we have [16]:

$$A\phi''(\xi)+B\phi(\xi)+C\phi^3(\xi)=0.$$

Where $A=\gamma_2\gamma_1, B=2k-c-3\gamma_1k^2, C=\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3$. This equation can be also be written in more simplified form as

$$\phi''(\xi)+k_1\phi(\xi)+k_3\phi^3(\xi)=0. \tag{36}$$

where $k_1=\frac{2k-c-3\gamma_1k^2}{\gamma_2\gamma_1}$ and $k_3=\frac{\frac{1}{3}\gamma_2+\frac{2}{3}\gamma_3}{\gamma_2\gamma_1}$.

By balancing the highest order derivative term ϕ'' with the nonlinear term ϕ^3 in (36), we obtain $N=1$ in (4). So we assume that Eq. (4) has solution in the form

$$\phi(\xi)=A_0+A_1(\frac{F'}{F}),\quad A_1\neq 0. \tag{37}$$

Using (37), we obtain

$$\phi^3=A_0^3+3A_0^2A_1(\frac{F'}{F})+3A_0A_1^2(\frac{F'}{F})^2+A_1^3(\frac{F'}{F})^3 \tag{38}$$

$$\phi''=A_1(\frac{F'''}{F}-\frac{F'F''}{F^2}+2(\frac{F'}{F})^3). \tag{39}$$

Substituting (37) to (39) into Eq. (36) and setting the coefficients of $F^j(j=0,-1,-2)$ to zero, we obtain

$$k_1A_0+k_3A_0^3=0, \tag{40}$$
$$A_1F'''+k_1A_1F'+3k_3A_0^2A_1F'=0, \tag{41}$$
$$-3A_1F'F''+3k_3A_0A_1^2F'^2=0, \tag{42}$$
$$2A_1F'^3+k_3A_1^3F'^3=0. \tag{43}$$

Eqs. (40) and (43) directly imply following solutions:

$$A_0=\pm\sqrt{-\frac{k_1}{k_3}},\quad A_1=\pm\sqrt{\frac{-2}{k_3}},\quad k_1>0,\ k_3<0.$$

Thus, Eqs. (41) and (42) become

$$F''' - 2k_1 F' = 0, \quad (44)$$

$$-F'' + \sqrt{2k_1} F' = 0. \quad (45)$$

By substituting Eq. (45) into Eq. (44) we get

$$-\sqrt{2k_1} F'' + F''' = 0. \quad (46)$$

The general solution of Eq. (46) is

$$F(\xi) = a_0 + a_1 \xi + a_2 \exp(\sqrt{2k_1}\xi)$$

where $a_i (i = 0, 1, 2)$ are arbitrary constants.
Thus, we have

$$\phi(\xi) = \pm\sqrt{-\frac{k_1}{k_3}} \pm \sqrt{-\frac{2}{k_3}} \left(\frac{a_1 + \sqrt{2k_1} a_2 \exp(\sqrt{2k_1}\xi)}{a_0 + a_1\xi + a_2 \exp(\sqrt{2k_1}\xi)}\right)$$

Now, the exact solution of Eq.(19) have the form

$$u(x,t) = \pm\sqrt{-\frac{k_1}{k_3}} \pm \sqrt{-\frac{2}{k_3}} \left(\frac{a_1 + \sqrt{2k_1} a_2 \exp(\sqrt{2k_1}(x-ct))}{a_0 + a_1(x-ct) + a_2 \exp(\sqrt{2k_1}(x-ct))}\right) \times \exp(i(kx - \Omega t))$$

4 Conclusion

In this work, we apply the modified simplest equation method to obtain exact solutions of the nonlinear (2+1)-dimensional nonlinear Schrödinger equation, the Schrödinger-Hirota equation and the perturbed nonlinear Schrödinger equation (NLSE) with Kerr law nonlinearity. The results have proven that modified simplest equation method is reliable and efficient in handling nonlinear problems. Considering the utility of these equations in semiconductor materials, optical fiber communications, plasma, fluid and solid mechanics and other branches of physics, these solutions may find practical applications. This method can be applied to solve other nonlinear partial differential equations.

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