

# Fractional Polynomial Spline for Solving Differential Equations of Fractional Order

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Received: 7 Nov. 2014, Revised: 3 Apr. 2015, Accepted: 23 Apr. 2015

Published online: 1 Sep. 2015

**Abstract:** In this paper, we implement new approximate techniques, the fractional lacunary interpolation by spline function for solving differential equations of fractional order. Convergence analysis, existence and uniqueness are shown by several theorems in the classes of  $C^3$  and  $C^4$  depend on the degree of spline polynomials. The numerical results demonstrates the errors bounds is quite validity and applicability of this method.

**Keywords:** Caputo Fractional Derivatives , lacunary Interpolation Series, Spline Approximation, Convergence Analysis.

## 1 Introduction

The applications of the fractional calculus that are allowed the related problems to be more development, are extended in almost all fields of mathematics and the other sciences. Also the interpolation by spline function is more the flexibility required for modern data analysis, curve fitting and easily in some cases. For examples, derivatives and integrals are hard to compute. In recent years, several new spline methods have been proposed for solving initial and boundary value problems, for instance, [7,9,10,13,20] introduces several degree spline methods, which are solving the linear and nonlinear ordinary differential equations.

Fractional calculus have been the focus of many studies due to their frequent appearance in various applications in engineering, biology, physics and fluid mechanics. Consequently, considerable attention has been given to the solutions of fractional ordinary differential equations, fractional integral equations of physics interest. Recently, fractional differential equations has developed some applications in assorted fields, such as engineering , physics and bioengineering [3,8,16] etc. and there has been significant interest in developing numerical schemes for their solution [14,15,19].

In this paper we begin by considering the fractional spline interpolations generated on the interval. Such splines share most of the properties as those of the lacunary

interpolations in polynomial space which defined by several works as [17,18] , the main part of this paper constructed the fractional spline method. We then prove that the fractional spline method have existence, uniqueness and convergence analysis, and there has been significant interest in developing numerical schemes for their derivation [11,20,21] . As applications, we represent the several examples of fractional differential equations solved numerically by this method.

## 2 Preliminaries of Method

The fractional differential equations are solving by lacunary interpolation by spline fractional method on the interval  $[a, b]$ . Since the interpolating polynomial coefficients are dependent on the uniform nodes  $a = x_1 < x_2 < x_3 < \dots < x_n = b$  such that  $x_{i+1} - x_i = h, i = 1, 2, \dots, n - 1$ , on the real line axis and normal derivatives as [4, 17, 18]. The Riemann-Liouville derivatives has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall use the fractional derivatives definition and Taylor's series from [1,3] has been written a formula of the Taylor series as:

$$f(x+h) = \sum_{m=-\infty}^{\infty} \frac{h^{m+r}}{\Gamma(m+r+1)} * (J_S^{m+r} f)(x) \quad (1)$$

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where  $J_S^{m+r} f$  is the Reimann-Liouville fractional integral of order  $m+r$ . Also from [3] the Caputo derivatives of order  $\alpha > 0$  of  $f \in C_{-1}^\infty$ ,  $n \in \mathbb{N}$  is defined as:

$$D_s^\alpha f(x) = \begin{cases} J_s^{n-\alpha} f^{(n)}(x), & x > s \geq 0, \quad n-1 < \alpha < n, \\ \frac{d^n f(x)}{dx^n}, & \alpha = n. \end{cases} \quad (2)$$

For more details on fractional derivatives definitions and lacunary interpolations (see [2, 14, 6, 9]). The main goal in this article is concerned with the applications of fractional spline function method to obtain the numerical solution of fractional differential equations.

### 3 Contraction Fractional Lacunary Interpolations

In this section, we will construct spline function and theorems related with the fractional order. Looking at the existence and uniqueness of this spline model, the new results to the convergence and error bounds are presented. We follow the methods introduced by T. Fawzy as [4] and present a spline solution for the normal order derivatives lacunary interpolation. We are going to construct fractional spline interpolate  $S_\Delta(x)$  for which

$$S_\Delta^{(q)}(x_k) = f_k^{(q)}, k = 0, 1, 2, \dots, n \quad \text{and} \quad q = 0, 1, 2, 3 \quad (3)$$

this construct has the following theorems:

**Theorem 1.** Let  $S(x)$  be the spline interpolate function, then there exist a unique  $S_\Delta(x) \in S_{n,3}^3$  such that:

$$\begin{aligned} S_\Delta(x) = S_k(x) = & y_k + \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} a_k + (x-x_k) D y_k \\ & + \frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}} b_k + \frac{(x-x_k)^2}{2} D^2 y_k \\ & + \frac{8(x-x_k)^{5/2}}{15\sqrt{\pi}} c_k + \frac{(x-x_k)^3}{6} D^3 y_k \end{aligned} \quad (4)$$

where  $x_k \leq x \leq x_{k+1}$ ,  $k = 1, 2, \dots, n-1$ ,  $a_k, b_k$  and  $c_k$  are lacunary interpolation of fractional order.

**Proof** Since  $S_\Delta(x) \in S_{n,3}^3[0, 1]$ , then it is easy to prove that the formula (4) with satisfies the condition of (3) assures that this spline polynomial exists and is unique, as the following

$$\begin{aligned} D^{(1/2)} S_k(x) = & a_k + \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} D y_k + (x-x_k) b_k \\ & + \frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}} D^{3/2} y_k + \frac{(x-x_k)^2}{2} c_k \\ & + \frac{8(x-x_k)^{5/2}}{15\sqrt{\pi}} D^{5/2} y_k \end{aligned} \quad (5)$$

Let

$$A_k = y_{k+1} - y_k - (x-x_k) D y_k - \frac{(x-x_k)^2}{2} D^2 y_k - \frac{(x-x_k)^3}{6} D^3 y_k$$

where

$$A_k = \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} a_k + \frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}} b_k + \frac{8(x-x_k)^{5/2}}{15\sqrt{\pi}} c_k$$

$$\text{and } DS_k(x) = D y_k + \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} b_k + (x-x_k) D^2 y_k + \frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}} c_k + \frac{(x-x_k)^2}{2} D^3 y_k$$

Also

$$D^2 S_k(x) = D^2 y_k + \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} c_k + (x-x_k) D^3 y_k \quad \text{and}$$

$$B_k = D y_{k+1} - D y_k - (x-x_k) D^2 y_k - \frac{(x-x_k)^2}{2} D^3 y_k,$$

$$\text{where } B_k = \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} b_k + \frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}} c_k,$$

Also  $C_k = D^2 y_{k+1} - D^2 y_k - (x-x_k) D^3 y_k$ , where

$$C_k = \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} c_k$$

Solving these three equations  $A_k, B_k$  and  $C_k$  we obtain the following:

$$a_k = \frac{\sqrt{\pi}}{90(x-x_k)^{1/2}} (45A_k - 30(x-x_k)B_k + 8(x-x_k)^2 C_k), \quad (6)$$

$$b_k = \frac{\sqrt{\pi}}{6(x-x_k)^{1/2}} (3B_k - 2(x-x_k)C_k), \quad (7)$$

$$c_k = \frac{\sqrt{\pi}}{2(x-x_k)^{1/2}} C_k. \quad (8)$$

Which is the proof of theorem 1.

**Theorem 2.** Let  $S_k(x)$  be the spline polynomial of degree three in case fractional lacunary, and  $y = f(x) \in C^3[a, b]$  then for all  $x \in [a, b]$ , we have

$$|(D^\alpha)^m S_k(x) - (D^\alpha)^m f(x)| \leq q_{m\alpha} h^{3-m\alpha} \omega(f, h)$$

where  $m = 0, 1, 2, \dots, 6$ ,  $\alpha = 1/2$ , and

$$q_0 = \frac{7}{9}, \quad q_{1/2} = \frac{91\sqrt{\pi^3} + 120\sqrt{\pi}}{180\sqrt{\pi}}, \quad q_1 = \frac{5}{3}, \quad q_{3/2} = \frac{4}{3\sqrt{\pi}} + \frac{3\sqrt{\pi}}{4}, \quad q_2 = 2, \quad q_{5/2} = \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\pi}} \quad \text{and} \quad q_3 = 1.$$

To prove this theorem we shall need the following lemma.

**Lemma 1** If the function  $f(x)$  in  $C^{m\alpha}[a, b]$ , where  $\alpha = \frac{1}{2}$  and  $m = 0, 1, \dots, 6$ . Then for  $k = 1, 2, \dots, n-1$ , the following are hold

$$|a_k - \frac{d^{1/2}}{dx^{1/2}} f(x_k)| \leq \frac{\sqrt{\pi}}{180} h^{5/2} \omega(f, h) \quad (9)$$

$$|b_k - \frac{d^{3/2}}{dx^{3/2}} f(x_k)| \leq \frac{\sqrt{\pi}}{4} h^{3/2} \omega(f, h) \quad (10)$$

$$|c_k - \frac{d^{5/2}}{dx^{5/2}}f(x_k)| \leq \frac{\sqrt{\pi}}{2}h^{1/2}\omega(f, h). \quad (11)$$

**Proof:** As we identified previously, we need only discuss the case fractional order  $m\alpha = 1/2, 3/2$  and  $5/2$ . In this situation, equation (6) become

$$|a_k - \frac{d^{1/2}}{dx^{1/2}}f(x_k)| = |\frac{\sqrt{\pi}}{90h^{1/2}}[45(f_{k+1} - f_k - \frac{2h^{1/2}}{\sqrt{\pi}}\frac{df_k}{dx_k} - h\frac{df_k}{dx_k} - \frac{h^2}{2}\frac{d^2f_k}{dx_k^2} - \frac{h^3}{6}\frac{d^3f_k}{dx_k^3}) - 30h(\frac{df_{k+1}}{dx_{k+1}} - \frac{df_k}{dx_k} - h\frac{d^2f_k}{dx_k^2} - \frac{h^2}{2}\frac{d^3f_k}{dx_k^3}) + 8h^2(\frac{d^2f_{k+1}}{dx_{k+1}^2} - \frac{d^2f_k}{dx_k^2} - h\frac{d^3f_k}{dx_k^3})]|$$

After expanding the function  $f(x)$  with derivatives by using equations(1) and (2), the above equation become:

$$a_k - \frac{d^{1/2}}{dx^{1/2}}f(x_k) \leq \frac{\sqrt{\pi}}{90}h^{5/2}|\frac{15}{2}\omega(f_{k_1}, h) - \frac{15}{2}\omega(f_{k_2}, h) + 8\omega(f_{k_3}, h)|$$

$$\leq \frac{\sqrt{\pi}}{90}h^{5/2}\omega(f, h), \text{ where } a \leq k_1, k_2 \text{ and } k_3 \leq b.$$

form equation (7), by the same manner using Taylor's fractional expansion in equation (1) and (2), we obtain

$$|b_k - \frac{d^{3/2}}{dx^{3/2}}f(x_k)| = \frac{\sqrt{\pi}}{6h^{1/2}}|3(\frac{df_{k+1}}{dx_{k+1}} - \frac{df_k}{dx_k} - \frac{2h^{1/2}}{\sqrt{\pi}}\frac{d^{3/2}f_k}{dx_k^{3/2}} - h\frac{d^2f_k}{dx_k^2} - \frac{h^2}{2}\frac{d^3f_k}{dx_k^3}) - 2h(\frac{d^2f_{k+1}}{dx_{k+1}^2} - \frac{d^2f_k}{dx_k^2} - h\frac{d^3f_k}{dx_k^3})|$$

$$\leq \frac{\sqrt{\pi}}{6h^{1/2}}|\frac{h^2}{2}\omega(f_{k_4}, h) + h^2\omega(f_{k_5}, h)| \rightarrow$$

$$|b_k - \frac{d^{3/2}}{dx^{3/2}}f(x_k)| \leq \frac{\sqrt{\pi}}{4}h^{3/2}\omega(f, h),$$

where  $a \leq k_4, k_5 \leq b$ .

Similarly can be find

$$|c_k - \frac{d^{5/2}}{dx^{5/2}}f(x_k)| = \frac{\sqrt{\pi}}{2}h^{1/2}|\frac{d^2f_{k+1}}{dx_{k+1}^2} - \frac{d^2f_k}{dx_k^2} - h\frac{d^3f_k}{dx_k^3} - \frac{d^{5/2}f_k}{dx_k^{5/2}}| \leq \frac{\sqrt{\pi}}{2}h^{1/2}\omega(f, h).$$

**Proof Theorem 2** For  $\xi_k \in [x_k, x_{k+1}]$  and  $k = 1, 2, \dots, n-1$ ,  $\frac{df}{dx} = Df(x)$ , using Lemma 1 and Taylor series in equation(1), we obtained the following:

$$|S_k(x) - f(x)| = |y_k + \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}}a_k + (x-x_k)Dy_k + \frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}}b_k + \frac{(x-x_k)^2}{2}D^2y_k + \frac{8(x-x_k)^{5/2}}{15\sqrt{\pi}}c_k + \frac{(x-x_k)^3}{6}D^3y_k - y_k + \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}}Dy_k + (x-x_k)Dy_k + \frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}}D^{3/2}y_k + \frac{(x-x_k)^2}{2}D^2y_k + \frac{8(x-x_k)^{5/2}}{15\sqrt{\pi}}D^{5/2}y_k + \frac{(x-x_k)^3}{6}D^3y_k|$$

$$\leq \frac{2(x-x_k)^{1/2}}{\sqrt{\pi}}|a_k - D^{1/2}y_k| + \frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}}|b_k - D^{3/2}y_k| + \frac{8(x-x_k)^{5/2}}{15\sqrt{\pi}}|c_k - D^{5/2}y_k| + \frac{h^3}{6}\omega(f_{\xi_1}, h). \quad \text{Where}$$

$h = (x - x_k)$  and from equation (9), (10) and (11), we obtain

$$|S_k(x) - f(x)| \leq \frac{h^3}{90}\omega(f_{\xi_1}, h) + \frac{h^3}{3}\omega(f_{\xi_1}, h) + \frac{4h^3}{15}\omega(f_{\xi_1}, h) + \frac{h^3}{6}\omega(f_{\xi_1}, h),$$

$$|S_k(x) - f(x)| \leq \frac{7h^3}{9}\omega(f, h)$$

and

$$|S_k^{(1/2)}(x) - f^{(1/2)}(x)| \leq |a_k - D^{(1/2)}y_k| + h|b_k - D^{(3/2)}y_k| + \frac{h^2}{2}|c_k - D^{(5/2)}y_k| + \frac{8h^{5/2}}{15\sqrt{\pi}}\omega(f, h),$$

substituting equations(9), (10) and (11), after some simplifications, we obtain

$$|S_k^{(1/2)}(x) - f^{(1/2)}(x)| \leq \frac{91\sqrt{\pi^3} + 120\sqrt{\pi}}{180\sqrt{\pi}}\omega(f, h),$$

$$|S_k'(x) - f'(x)| \leq \frac{2h^{1/2}}{\sqrt{\pi}}|b_k - D^{(3/2)}y_k| + \frac{4h^{3/2}}{3\sqrt{\pi}}|c_k - D^{(5/2)}y_k| + \frac{h^2}{2}\omega(f, h)$$

$$\leq \frac{5h^2}{3}\omega(f, h),$$

Similarly we can find the following:

$$|S_k^{(3/2)}(x) - f^{(3/2)}(x)| \leq \frac{4}{3\sqrt{\pi}} + \frac{3\sqrt{\pi}}{4}h^{3/2}\omega(f, h),$$

$$|S_k''(x) - f''(x)| \leq 2h\omega(f, h), \quad |S_k^{(5/2)}(x) - f^{(5/2)}(x)| \leq \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\pi}},$$

and  $|S_k'''(x) - f'''(x)| \leq \omega(f, h)$ .

**Theorem 3.** Let  $y = f(x) \in C^4[a, b]$ ,  $S(x)$  be the spline interpolate function, then there exist a unique  $S_{\Delta}(x) \in S_{n,4}^4$  such that:

$$S_{\Delta}(x) = S_k(x) = y_k + \frac{2h^{1/2}}{\sqrt{\pi}}a_k + hDy_k + \frac{4h^{3/2}}{3\sqrt{\pi}}b_k + \frac{h^2}{2}D^{(2)}y_k + \frac{8h^{5/2}}{15\sqrt{\pi}}c_k + \frac{h^3}{6}D^{(3)}y_k + \frac{16h^{7/2}}{150\sqrt{\pi}}D^{(7/2)}y_k + \frac{h^4}{24}D^{(4)}y_k(x) \quad (12)$$

where  $x_k \leq x \leq x_{k+1}$ ,  $k = 1, 2, \dots, n-1$ ,  $a_k, b_k$  and  $c_k$  are lacunary derivatives of fractional order.

**Proof** Now if  $S_{\Delta}(x) \in S_{n,4}^4[0, 1]$ , then the existence and uniqueness of  $S_{\Delta}(x)$  is easy to proved, since  $a_k, b_k$  and  $c_k$  are uniquely determined by using formula (12)

with satisfies the condition of (3), as the following

$$A_k = y_{k+1} - y_k - hDy_k - \frac{h^2}{2}D^{(2)}y_k - \frac{h^3}{6}D^{(3)}y_k - \frac{16h^{7/2}}{150\sqrt{\pi}}D^{(7/2)}y_k - \frac{h^4}{24}D^{(4)}y_k(x), \tag{13}$$

where  $A_k = \frac{2h^{1/2}}{\sqrt{\pi}}a_k + \frac{4h^{3/2}}{3\sqrt{\pi}}b_k + \frac{8h^{5/2}}{15\sqrt{\pi}}c_k$

$$B_k = Dy_{k+1} - Dy_k - hD^{(2)}y_k - \frac{h^2}{2}D^{(3)}y_k - \frac{56h^{5/2}}{150\sqrt{\pi}}D^{(7/2)}y_k - \frac{h^3}{6}D^{(4)}y_k, \tag{14}$$

where  $B_k = \frac{2h^{1/2}}{\sqrt{\pi}}b_k + \frac{4h^{3/2}}{3\sqrt{\pi}}c_k$ , and

$$C_k = D^{(2)}y_{k+1} - D^{(2)}y_k - hD^{(3)}y_k - \frac{28h^{3/2}}{30\sqrt{\pi}}D^{(7/2)}y_k - \frac{h^2}{4}D^{(4)}y_k, \tag{15}$$

where  $C_k = \frac{2h^{1/2}}{\sqrt{\pi}}c_k$ .

Solving these three equation  $A_k, B_k$  and  $C_k$  we obtain the following:

$$a_k = \frac{\sqrt{\pi}}{90h^{1/2}}(45A_k - 30hB_k + 8h^2C_k), \tag{16}$$

$$b_k = \frac{\sqrt{\pi}}{6h^{1/2}}(3B_k - 2hC_k), \tag{17}$$

$$c_k = \frac{\sqrt{\pi}}{2h^{1/2}}C_k. \tag{18}$$

**Theorem 4.** Let  $S_k(x)$  be the spline polynomial of degree four in case fractional lacunary from equation (12), and  $y = f(x) \in C^4[a, b]$  then for all  $x \in [a, b]$ , we have

$$|(D^\alpha)^m S_k(x) - (D^\alpha)^m f(x)| \leq r_{m\alpha} h^{4-m\alpha} \omega(f, h)$$

where  $m = 0, 1, 2, \dots, 8, \alpha = 1/2$ , and

$$r_0 = \frac{3}{4}, \quad r_{1/2} = \frac{1920 + 8925\pi}{18000\sqrt{\pi}}, \quad r_1 = 1, \quad r_{3/2} = \frac{16 + 15\pi}{30\sqrt{\pi}}, \quad r_2 = 1,$$

$$r_{5/2} = \frac{4}{3\sqrt{\pi}}, \quad r_3 = 1, \quad r_{7/2} = \frac{2}{\sqrt{\pi}} \text{ and } r_4 = 1.$$

**Proof** From equations (13), (16), (17) and (18), we obtain

$$|a_k - \frac{d^{1/2}}{dx^{1/2}}f(x_k)| \leq \frac{\sqrt{29\pi}}{240}h^{7/2}\omega(f, h) \tag{19}$$

and from (14), (16), (17) and (18), we obtain

$$|b_k - \frac{d^{3/2}}{dx^{3/2}}f(x_k)| \leq \frac{\sqrt{\pi}}{4}h^{5/2}\omega(f, h) \tag{20}$$

Also from (15), (16), (17) and (18), we obtain

$$|c_k - \frac{d^{5/2}}{dx^{5/2}}f(x_k)| \leq \frac{\sqrt{\pi}}{4}h^{3/2}\omega(f, h). \tag{21}$$

Now form equation (12) and using (19), (20) and (21), can be written thus:

$$\begin{aligned} |S_k(x) - f(x)| &= |y_k + \frac{2h^{1/2}}{\sqrt{\pi}}a_k + hDy_k + \frac{4h^{3/2}}{3\sqrt{\pi}}b_k + \frac{h^2}{2}D^2y_k + \frac{8h^{5/2}}{15\sqrt{\pi}}c_k + \frac{h^3}{6}D^3y_k - |y_k + \frac{2h^{1/2}}{\sqrt{\pi}}Dy_k + hDy_k + \frac{4h^{3/2}}{3\sqrt{\pi}}D^{3/2}y_k + \frac{h^2}{2}D^2y_k + \frac{8h^{5/2}}{15\sqrt{\pi}}D^{5/2}y_k + \frac{h^3}{6}D^3y_k| + \frac{16h^{7/2}}{150\sqrt{\pi}}D^{(7/2)}y_k + \frac{h^4}{24}D^{(4)}y_k(x) \\ &\leq \frac{2h^{1/2}}{\sqrt{\pi}}|a_k - D^{1/2}y_k| + \frac{4h^{3/2}}{3\sqrt{\pi}}|b_k - D^{3/2}y_k| + \frac{8h^{5/2}}{15\sqrt{\pi}}|c_k - D^{5/2}y_k| + \frac{h^4}{24}\omega(f, h) \\ &\rightarrow |S_k(x) - f(x)| \leq \frac{3}{4}h^4\omega(f, h), \end{aligned}$$

and take half fraction derivative of equation (12) with using equations (19), (20) and (21), can be obtain

$$\begin{aligned} |S_k^{(1/2)}(x) - f^{(1/2)}(x)| &\leq |a_k - D^{(1/2)}y_k| + h|b_k - D^{(3/2)}y_k| + \frac{h^2}{2}|c_k - D^{(5/2)}y_k| + \frac{16h^{7/2}}{150\sqrt{\pi}}\omega(f, h) \leq \frac{1920 + 8925\pi}{18000\sqrt{\pi}}h^{7/2}\omega(f, h) \\ &\rightarrow |S_k^{(1/2)}(x) - f^{(1/2)}(x)| \leq \frac{1920 + 8925\pi}{18000\sqrt{\pi}}h^{7/2}\omega(f, h) \end{aligned}$$

Similarly, it is easy to complete the proof of theorem 4 for  $|(D^\alpha)^m S_k(x) - (D^\alpha)^m f(x)|$  where  $m = 2, 3, \dots, 8$ .

### 4 Numerical Results

In this section, we applied (theorems 2 and 4) on previous sections to support our theoretical discussion by difference problems to compare the maximum absolute errors with the analytical solutions for the step sizes of  $h$ . The absolute errors in the function value and all derivatives can be find easily, especially the lacunary fractional order were seen be small, the results of the present work are compared with exact solution of all the problems and with that lacunary spline method.

*Example 1.* Consider the fractional differential equation as [5]

$$D^2y(t) - t^2D^{(3/2)}y(t) - \sqrt{t}D^{(1/2)}y(t) - t^{1/3}y(t) = 6\sqrt{\pi}t - \frac{16}{5}t^3 - t^{10/3}\sqrt{\pi},$$

$$y(0) = y'(0) = 0, \quad 0 \leq t \leq 1,$$

The exact solution is given by  $y(t) = \sqrt{\pi}t^3$ , the results of maximum absolute errors  $\omega(f, h) = \max_{1 \leq k \leq n} |s^{(m\alpha)}(x_k) - y^{(m\alpha)}(x_k)|$ ,  $\alpha = 1/2$  and  $m = 0, 1, \dots, 6$  such that (see [11, 20]), for this problem use theorem 2 and 4 are tabulated in Table 1 and 2 respectively.

**Table 1:** Comparison present method with exact solution by absolute errors of lacunary fractional order derivatives of example 1.

h values	$ s^{(1/2)} - y^{(1/2)} $	$ s^{(3/2)} - y^{(3/2)} $	$ s^{(5/2)} - y^{(5/2)} $
0.1	$4.0254 \times 10^{-3}$	$7 \times 10^{-2}$	$6.7751 \times 10^{-1}$
0.05	$3.558 \times 10^{-4}$	$1.2375 \times 10^{-2}$	$5.9884 \times 10^{-1}$
0.01	$1.2729 \times 10^{-6}$	$2.2137 \times 10^{-4}$	$2.1424 \times 10^{-2}$
0.001	$4.0252 \times 10^{-10}$	$7.3003 \times 10^{-7}$	$6.551 \times 10^{-4}$

**Table 2:** Comparison present method with exact solution by absolute errors of lacunary fractional order derivatives of example 1.

h values	$ s^{(1/2)} - y^{(1/2)} $	$ s^{(3/2)} - y^{(3/2)} $	$ s^{(5/2)} - y^{(5/2)} $
0.1	$3.1579 \times 10^{-4}$	$4 \times 10^{-3}$	$2.531 \times 10^{-2}$
0.05	$1.3956 \times 10^{-5}$	$3.5287 \times 10^{-4}$	$4.5 \times 10^{-5}$
0.01	$9.9862 \times 10^{-9}$	$1.2625 \times 10^{-6}$	$8 \times 10^{-5}$
0.001	$3.1579 \times 10^{-13}$	$3.9923 \times 10^{-10}$	$2.5298 \times 10^{-7}$

$ s^{(7/2)} - y^{(7/2)} $
$3.7951 \times 10^{-1}$
$1.342 \times 10^{-1}$
$1.2 \times 10^{-2}$
$3.7947 \times 10^{-4}$

*Example 2.* Consider the fractional differential equation:

$$D^{3/2}y(t) + y^2(t) = f(t), y(0) = y'(0) = 0, \quad 0 \leq t \leq 1,$$

$$\text{and } f(t) = \frac{\Gamma(6)}{\Gamma(6-\alpha)}t^{5-\alpha} - \frac{3\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{2\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + (t^5 - 3t^4 + 2t^3)^2,$$

The exact solution is given by  $y(t) = t^5 - 3t^4 + 2t^3$  for  $\alpha = 1.5$ , for this problem use theorem 2 and 4 are tabulated in Table 3 and 4 respectively .

**Table 3:** The maximum absolute errors of lacunary fractional order derivatives in theorem 1 applying of example 2.

h values	$ s^{(1/2)} - y^{(1/2)} $	$ s^{(3/2)} - y^{(3/2)} $	$ s^{(5/2)} - y^{(5/2)} $
0.1	$2.1953 \times 10^{-2}$	$3.8178 \times 10^{-1}$	$3.6950 \times 10^0$
0.05	$2.1412 \times 10^{-3}$	$7.1447 \times 10^{-2}$	$1.4415 \times 10^0$
0.01	$8.2351 \times 10^{-6}$	$1.4321 \times 10^{-3}$	$1.3860 \times 10^{-1}$
0.001	$2.645 \times 10^{-11}$	$4.5999 \times 10^{-6}$	$4.5180 \times 10^{-3}$

**Table 4:** The maximum absolute errors of present method to lacunary fractional order derivatives in theorem 2 of example 2.

h values	$ s^{(1/2)} - y^{(1/2)} $	$ s^{(3/2)} - y^{(3/2)} $	$ s^{(5/2)} - y^{(5/2)} $
0.1	$1.7 \times 10^{-3}$	$2.18 \times 10^{-3}$	$1.38 \times 10^{-2}$
0.05	$8.7 \times 10^{-5}$	$2.1 \times 10^{-3}$	$2.69 \times 10^{-2}$
0.01	$6.4604 \times 10^{-8}$	$8.1674 \times 10^{-6}$	$5.1755 \times 10^{-4}$
0.001	$2.075 \times 10^{-12}$	$2.6535 \times 10^{-9}$	$1.6623 \times 10^{-6}$

$ s^{(7/2)} - y^{(7/2)} $
$2.0634 \times 10^0$
$8.074 \times 10^{-1}$
$7.76 \times 10^{-2}$
$2.5 \times 10^{-3}$

## 5 Conclusions

In the present paper, it has been described and demonstrated the applicability and efficiency of the lacunary spline polynomial method with fractional order for solving fractional differential equations. For the error estimates have small accuracy than the errors bound obtained in [4, 17, 18, 20]. The lacunary spline polynomial method is tested on different problems and the approximate solution have been found, the errors bounds from all the tables clearly indicate that our numerical solution converges to the exact solution and its an applicable technique depend on the degree of spline polynomial and the class of derivatives.

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