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Numerical Simulation of Acoustic Problems with High Wavenumbers

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Abstract: The boundary knot method (BKM) is a kind of boundary-type meshless method, only boundary points are needed in the solution process. Like the boundary element method and method of fundamental solutions, the BKM needs nonsingular general solutions of governing equations. Therefore, collocation points and source points can be located on the physical boundary simultaneously which is superior in dealing with Helmholtz problems. In this paper, the BKM is extended to investigate acoustic problems with high wavenumbers. By comparing with the finite element method, numerical results show that the BKM has excellent advantages in dealing with high wave number acoustic problems.

Keywords: Meshless method, boundary knot method, finite element method, Helmholtz equation

1 Introduction

Acoustic problems have very important applications in real world, such as noise control, nondestructive testing, medical imaging, sonar. Investigation on the computational methods of acoustic problems is a hot topic for engineers and mathematicians. The challenging conundrum is the simulation of acoustic problems with high wavenumbers. Several numerical methods have been applied to this problem [1,2,3].

The finite element method (FEM) can provide solutions of the whole physical domain, but the whole domain should be discretized which makes the computational very large. Furthermore, enough FEM cells should be used in each wavelength to ensure the approximation accuracy for Helmholtz equations modeled from acoustic problems. The mesh-refinement should be implemented with the increasing wavenumbers. The FEM simulation depends on the size of minimal cells, so the mesh-refinement enlarges the computational time and increases the accumulating errors [4,5,6]. An alternative to the FEM is the boundary element method (BEM) which is restricted to the boundary of the considered domain [7]. Apart from this, the BEM possesses advantages when dealing with problems involving infinite or semi-infinite domains. However, it still has several drawbacks, such as the boundary layer effect and singularity or hyper-singularity [8,9,10]. To the best of our knowledge, the BEM for acoustic problems with high wavenumbers is rare in literatures.

Corresponding to the BEM, boundary-type meshless methods behave eminent in analyzing many difficult problems. Among which the method of fundamental solutions (MFS) [11,12], also called the T-Trefftz method [13,14,15], and the boundary knot method (BKM) are typical examples. The BKM, proposed by Kang et al [16], employs the nonsingular general solutions instead of fundamental solutions used in the MFS. The fictitious boundary appeared in the MFS is eliminated while the other advantages are maintained at the same time. Previous numerical results show that the BKM is suitable for many problems associated with Helmholtz equations[17,18,19,20]

Based on the above analysis, this paper uses the BKM to study acoustic problems with high wavenumbers. Numerical results are compared with the FEM to show the superiority. Section 2 briefly introduces the BKM. Followed by Section 3, numerical examples are presented by comparing with the FEM. Some conclusions are made in Section 4.

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2 The boundary knot method

Most acoustic sources encountered in real applications always have definite characters. For convenience, we consider harmonic waves of point acoustic source, which can be modeled to the following Helmholtz problem, to illustrate the BKM

$$\nabla^2 u(\mathbf{x}) + \lambda^2 u(\mathbf{x}) = 0, \qquad \text{in } \Omega \tag{1}$$

$$u(\mathbf{x}) = \bar{u}(\mathbf{x}), \qquad \text{on } \Gamma_D \tag{2}$$

$$\frac{\partial u(\mathbf{x})}{\partial n} = q(\mathbf{x}) = \bar{q}(\mathbf{x}), \qquad \text{on } \Gamma_N \tag{3}$$

$\bar{u}(\mathbf{x})$ and $\bar{q}(\mathbf{x})$ are pre-defined boundary conditions on Dirichlet boundary Γ_D and Neumann boundary Γ_N, respectively. Ω means the physical domain in $\mathbf{R}^d$, d is the dimensionality, $\partial\Omega (= \Gamma_D \cup \Gamma_N)$ represents the physical boundary, λ and n are wavenumber and unit normal vector, respectively.

The the BKM has similar basic theory with the other boundary-type meshless methods. In the BKM, the nonsingular general solutions has no singularity, the source points and collocation points can be located on the physical boundary simultaneously. The approximate solution $u_N(\mathbf{x})$ can be expressed in terms of nonsingular general solutions

$$u_N(\mathbf{x}) = \sum_{j=1}^{N} c_j Q(\mathbf{x}, \mathbf{y}_j), \qquad \mathbf{y}_j \in \partial\Omega \tag{4}$$

where $\mathbf{y}_j$ are source points on the boundary, c_j are coefficient to be determined, N the total number of source points,

$$Q(\mathbf{x}, \mathbf{y}) = \left(\frac{\lambda}{2\pi r}\right)^{(d/2)-1} J_{(d/2)-1}(\lambda r), \quad d \geq 2 \tag{5}$$

is the nonsingular general solutions for the Helmholtz equation with J denoting the Bessel function of the first kind and $r = \| \mathbf{x} - \mathbf{y} \|$ the Euclidean distance.

Differentiation of Eq. (4) is

$$\frac{\partial u_N(\mathbf{x})}{\partial n} = \sum_{j=1}^{N} c_j \frac{\partial Q(\mathbf{x}, \mathbf{y}_j)}{\partial n}. \qquad \mathbf{x} \in \partial\Omega \tag{6}$$

Substitute Eqs. (4) and (5) into boundary conditions Eqs. (2) and (3), we have the following equations on N collocation points

$$\sum_{j=1}^{N} c_j Q(\mathbf{x}_i, \mathbf{y}_j) = \bar{u}(\mathbf{x}_i) \quad i = 1, \ldots, N_1 \tag{7}$$

$$\sum_{j=1}^{N} c_j \frac{\partial Q(\mathbf{x}_k, \mathbf{y}_j)}{\partial n} = \bar{q}(\mathbf{x}_k) \quad k = 1, \ldots, N_2 \tag{8}$$

where N_1 and N_2 are number of collocation points on Γ_D and Γ_N, respectively. $N_1 + N_2 = N$.

The matrix form of Eq. (7) and (8) are

$$Q\alpha = b \tag{9}$$

where

$$Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} & \cdots & Q_{1,N} \\ \cdots & \cdots & \cdots & \cdots \\ Q_{N_1,1} & Q_{N_1,2} & \cdots & Q_{N_1,N} \\ \frac{\partial Q_{1,1}}{\partial n} & \frac{\partial Q_{1,2}}{\partial n} & \cdots & \frac{\partial Q_{1,N}}{\partial n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial Q_{N_2,1}}{\partial n} & \frac{\partial Q_{N_2,2}}{\partial n} & \cdots & \frac{\partial Q_{N_2,N}}{\partial n} \end{bmatrix} \tag{10}$$

are $N \times N$ coefficient matrix, $b = (\bar{u}_1, \ldots, \bar{u}_{N_1}, \bar{q}_1, \ldots, \bar{q}_{N_2})^T$ is $N \times 1$ vector composed by boundary conditions and $\alpha = (c_1, c_2, \ldots, c_N)^T$ is $N \times 1$ coefficient vector. ^{T} stands for the transpose of vector.

For the solvability of equation (9), we have the following theorem.

Theorem 1. The radial basis function given by Eq. (5) will give nonsingular interpolation matrix, thus, Eq. (9) is solvable and has unique solution.

Proof. The non-singularity of the interpolation matrix in Eq. (9) can be similarly proved as in Ref. [21]. After which we can get the conclusion of this theorem by Cramer's rule.

After the coefficient a is solved, we can calculate values from Eq. (4) for arbitrary point on the whole physical domain.

3 Numerical example and discussions

We use the above-given BKM to test the Helmholtz problems with high wavenumbers. Unless otherwise specified, we use the following relative average error (root mean-square relative error: RMSE) [22]:

$$\mathbf{RMSE} = \sqrt{\frac{1}{N_t} \sum_{j=1}^{N_t} \left| \frac{u(\mathbf{x}_j) - \tilde{u}(\mathbf{x}_j)}{u(\mathbf{x}_j)} \right|^2}, \tag{11}$$

for $|u(\mathbf{x}_j)| \geq 10^{-3}$,

$$\mathbf{RMSE} = \sqrt{\frac{1}{N_t} \sum_{j=1}^{N_t} \left| u(\mathbf{x}_j) - \tilde{u}(\mathbf{x}_j) \right|^2}, \tag{12}$$

for $|u(\mathbf{x}_j)| < 10^{-3}$, where j is the index of test points, $u(\mathbf{x}_j)$ and $\tilde{u}(\mathbf{x}_j)$ are the exact and numerical solutions on the test point $\mathbf{x}_j$, respectively. N_t means the total number of test points.

The unit square domain $\Omega = \{(x,y) | 0 \leq x, y \leq 1\}$ is considered. In order to show the variation of the BKM versus the wavenumber λ, we choose the following exact solution

$$u(x,y) = \sin(\lambda x) + \cos(\lambda y). \tag{13}$$

For wavenumber $\lambda = 150$, Fig. 1 shows the convergence curve of the RMSE versus the number of unknowns per line. We can see that the convergence curve is very smooth with the increasing number of boundary

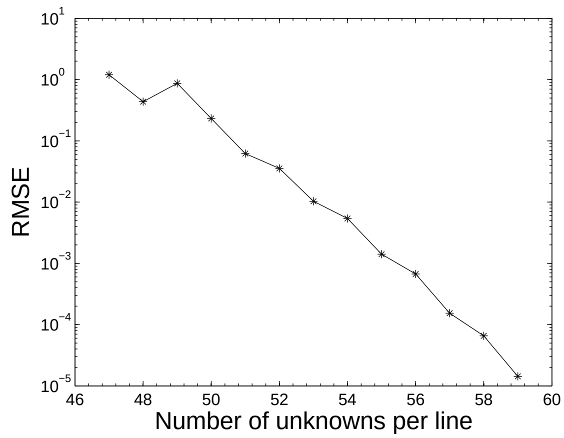

Fig. 1: Numerical results of the BKM for wavenumber $\lambda = 150$.

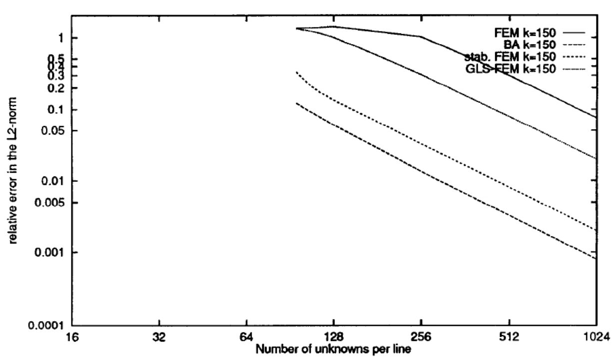

Fig. 2: The FEM results in reference [2] for wavenumber $\lambda = 150$.

points. Only 55 boundary points per line can achieve the relative average error RMSE $= 10^{-3}$.

In order to illustrate the superiority of the BKM, we use the FEM numerical results of reference [2] for comparison. In Fig. 2, the 1024 boundary points per line merely reaches the average relative error RMSE $= 10^{-1}$ for the traditional FEM. Despite the generalized least square finite element method (GLS-FEM) has the best results in Fig. 2, we find that 1024 points per line leads to the average relative error RMSE $= 10^{-3}$ while the same results can be obtained by using 55 points per line for the BKM.

Note: If one uses the MATLAB toolbox to generate mesh or an unit square domain, seven times mesh-generation corresponds to 704 points per line and totally 1116546 points number. For eight times mesh-generation, the computer doesn't work. Simultaneously, 55 points per line corresponds to totally 216 number for the BKM.

From the above analysis, we can conclude that the FEM is not suitable for Helmholtz problems with high wavenumbers. On the contrary, the BKM has obvious advantage in dealing with such high wavenumber problems.

Table 1: Relationship among wavenumber λ, boundary point number N, relative average error RMSE and condition number Cond

λ	N	RMSE	Cond
100	41	4.9635×10^{-4}	1.0906×10^{12}
110	44	7.0030×10^{-4}	3.7007×10^{11}
120	47	6.8327×10^{-4}	1.3554×10^{11}
130	49	3.1715×10^{-4}	1.2850×10^{9}
140	54	4.1059×10^{-4}	8.8116×10^{11}
150	56	6.7026×10^{-4}	8.9634×10^{9}
200	73	5.5960×10^{-4}	2.3042×10^{16}
300	107	3.9554×10^{-4}	1.6903×10^{14}
400	147	1.1242×10^{-4}	1.7559×10^{16}
500	188	8.9665×10^{-4}	1.8549×10^{17}
600	214	5.2239×10^{-4}	3.9693×10^{17}
700	253	6.5808×10^{-4}	3.9613×10^{15}
800	280	5.1530×10^{-4}	4.7599×10^{16}
900	312	8.4212×10^{-4}	5.9307×10^{15}
1000	345	9.0818×10^{-4}	1.1787×10^{16}

With the increasing number of wavenumbers, Table 1 gives the relationship among boundary point number, relative average error and condition number. It is found that the increment of 10 wavenumbers corresponds with addition of $2 \sim 3$ boundary points to maintain the solution accuracy. For the increment of 100 wavenumbers, about adding 35 boundary points keeps the original solution accuracy. When the wavenumber $\lambda = 1000$, only 345 boundary points can calculate very good result RMSE $= 9.0818 \times 10^{-4}$. This phenomenon support the conclusion given in [23], i.e., to approximate Helmholtz equation with acceptable accuracy the resolution of the mesh should be adjusted to the wave number according to the rule of thumb. Compared with the FEM, this phenomenon shows that the boundary point number of the BKM is rather less sensitive to the increasing wavenumbers.

Furthermore, Table 1 shows that the condition numbers of coefficient matrix increase with the increasing numbers of boundary points or wavenumbers. We also note that the increasing condition number may lead to the instability of a method [18,24].

4 Conclusions

In this paper, we use the BKM, which eliminates the mesh-generation, to simulate acoustic problems with high wavenumbers in terms of stability and convergence. A theorem is given to prove the feasibility of the BKM. Numerical results show that the BKM has better accuracy than the FEM. Only minor boundary points can achieve very good results for acoustic problems with high wavenumbers which proves the applicability of the BKM.

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