Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090245

A Note on Certain Qualitative Properties of a Second Order Linear Differential System

Cemil TUNÇ and Osman TUNÇ*

Department of Mathematics, Faculty of Sciences, Yüzüncü Yil University, 65080 Van, Turkey

Received: 16 Jun. 2014, Revised: 14 Sep. 2014, Accepted: 16 Sep. 2014
Published online: 1 Mar. 2015

Abstract: In this note, two theorems are presented concerning the well known second order linear differential system $\ddot{X}+a(t)X=P(t)$. While the results are not new, the proofs presented simplify previous works since the Gronwall inequality is avoided which is the usual case. The technique of proof involves the integral test and examples are included to illustrate the results.

Keywords: Qualitative property, linear differential system, second order.

1 Introduction

In 2013, Kroopnick [3] considered the second order scalar linear differential equation of the form

$$x'' + a(t)x = 0. \tag{1}$$

In his brief note, Kroopnick [3] gave two new and elementary proofs proving the stability of solutions and the boundedness of solutions as $t \to \infty$ for the well known linear differential equation, equation (1) given various constraints on $a(.)$. While the results of Kroopnick [3] are not new (see Sanchez [5, pp. 111-117] for some classical results), the proofs in [3] are less complex and quite general.

Kroopnick [3] first proved the following theorem.

Theorem A(Kroopnick [3, Theorem I]). Given equation (1) where $a(.)$ is in $C^1[0,\infty)$ such that

$$a(t) > 0 \textit{ and } a^{'}(t) \geq 0,$$

then all solutions of equation (1) are bounded as $t \to \infty$ and the absolute values of the amplitudes form a non-increasing sequence.
Kroopnick [3] second proved the following theorem.
Theorem B(Kroopnick [3, Theorem II]). Given equation (1) where $a(.)$ is a continuous function on $[0,\infty)$ such that $a(t) > 0$, $a^{'}(t) \geq 0$ and $K > a(t) > k > 0$ for some positive constants K and k, then all solutions of equation (1) are bounded as $t \to \infty$ and stable. Furthermore, the absolute values of the amplitudes form a non-increasing sequence.
It should be noted that Kroopnick [3] proved both of Theorem A and Theorem B by the integral test.
In this note, instead of equation (1), we consider the more general vector linear differential equation of the second order

$$\ddot{X} + a(t)X = P(t), \tag{2}$$

where $X \in \Re^n$, $t \in \Re^+$, $\Re^+ = [0,\infty)$; $a(.) : \Re^+ \to \Re$ and $P(.) : \Re^+ \to \Re^n$ are continuous functions. It is worth mentioning that equation (2) represents the vector version for the system of real second order linear differential equations of the form

$$\ddot{x}_i + a(t)x_i = p_i(t), (i = 1,2,...,n).$$

This shows that equation (1) is a special case of equation (2).
It should be noted that equation (1) is known as Hill equation in the literature. Hill equation used to describe many phenomena of physical interest. For example, it is significant in investigation of stability and instability of geodesic on Riemannian manifolds where Jacobi fields can be expressed in form of Hill equation system (Gallot et al [2]). This fact has been used by some physicists to study dynamics in Hamiltonian systems (Pettini and Valdettaro [4]). Besides, equation (1) is frequently

* Corresponding author e-mail: cemtunc@yahoo.com

encountered as mathematical models of most dynamics process in electromechanical system of physics and engineering (Ahmad and Rama Mohana Rao [1]). Here, we would not like to give more details of applications concerning equations (1) and (2). Therefore, it is worth to work on the qualitative properties of equation (2). Furthermore, the motivation of this note has been inspired by the results established in Kroopnick [3] and those in ([1], [2], [4], [5], [6], [7]). The aim of paper is to give the proofs of stability/boundedness solutions which are less complex and quite general than those in the literature by the integral test. The paper has a new contribution to the topic in the literature. This case shows the novelty of this work. The results to be established here may be useful for researchers working on the qualitative theory of solutions.

2 Main problems

The first main problem of this paper is the following theorem.

Theorem 1. In addition to the basic assumptions imposed on the functions $a(.)$ and $P(.)$, we assume that the following assumptions hold: Given equation (2) where $a(.)$ is in $C^1[0,\infty)$ such that

$$a(t) > 0, a^{'}(t) \geq 0,$$

$$\|P(t)\| \leq e(t), q(t) = \frac{e(t)}{a(t)}$$

and

$$q(.) \in L^1[0,\infty) \ for \ all \ t \in \Re^+.$$

Then, all solutions of equation (2) are bounded as $t \to \infty$ Furthermore, if $P(t) \equiv 0$ in equation (2), then all solutions of equation (2) are bounded as $t \to \infty$ and the absolute values of the amplitudes form a non-increasing sequence.

Proof. When we multiply equation (2) by $\frac{2}{a(t)}\dot{X}(t)$ it follows that

$$\frac{2}{a(t)}\langle \dot{X}(t), \ddot{X}(t)\rangle + 2\langle X(t), \dot{X}(t)\rangle = \frac{2}{a(t)}\langle \dot{X}(t), P(t)\rangle. \quad (3)$$

Integrating estimate (3) from 0 to t and then applying integration by parts to the first term on the left hand side of (3), we have

$$2\int_0^t \frac{1}{a(s)}\langle \dot{X}(s), \ddot{X}(s)\rangle ds + 2\int_0^t \langle X(s), \dot{X}(s)\rangle ds =$$

$$2\int_0^t \frac{1}{a(s)}\langle \dot{X}(s), P(s)\rangle ds,$$

and then

$$\frac{1}{a(t)} \parallel \dot{X}(t) \parallel^2 + \int_0^t \frac{a^{'}(s)}{a^2(s)} \parallel \dot{X}(t) \parallel^2 ds + \parallel X(t) \parallel^2$$

$$= \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2 + 2\int_0^t \frac{1}{a(s)}\langle \dot{X}(s), P(s)\rangle ds \quad (4)$$

$$\leq \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2 + 2\int_0^t \frac{\parallel P(s) \parallel}{a(s)} \parallel \dot{X}(s) \parallel ds$$

$$\leq \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2 + 2\int_0^t \frac{e(s)}{a(s)} \parallel \dot{X}(s) \parallel ds$$

$$= \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2 + 2\int_0^t q(s) \parallel \dot{X}(s) \parallel ds.$$

We now apply the mean value theorem for integrals to the term

$$2\int_0^t q(s) \parallel \dot{X}(s) \parallel ds.$$

Then

$$\frac{1}{a(t)} \parallel \dot{X}(t) \parallel^2 + \int_0^t \frac{a^{'}(s)}{a^2(s)} \parallel \dot{X}(t) \parallel^2 ds + \parallel X(t) \parallel^2$$

$$\leq \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2 + 2 \parallel \dot{X}(t^*) \parallel \int_0^\infty q(s) ds,$$

where $0 < t^* < t$.

Hence, it is clear that

$$\frac{1}{a(t)} \parallel \dot{X}(t) \parallel^2 + \parallel X(t) \parallel^2$$

$$\leq \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2 + 2 \parallel \dot{X}(t^*) \parallel \int_0^\infty q(s) ds. \quad (5)$$

Since all terms in estimate (5) are positive, boundedness follows. Otherwise, the left hand side of (5) would become infinite as $t \to \infty$ while the right hand side of (5) remained fixed which is impossible. Further, when $P(t) \equiv 0$ in equation (2), by the Sturm comparison theorem ([5, pp. 114-115)]), all solutions oscillate when we compare equation (2) to the equation $\ddot{X} + a_0 X = 0$ on the interval $[0,\infty)$ and $a(0) = a_0$.

When $P(t) \equiv 0$ in equation (2), consider now two successive critical points t_1 and t_2, where $\dot{X}(t_1) = \dot{X}(t_2) = 0$, and integrate this time estimate (3) from t_1 to t_2 rather than 0 to t. In that case estimate (4) becomes

$$\int_0^t \frac{a^{'}(s)}{a^2(s)} \parallel \dot{X}(t) \parallel^2 ds + \parallel X(t_2) \parallel^2 = \parallel X(t_1) \parallel^2 . \quad (6)$$

It now follows from estimate (6) that $\parallel X(t_2) \parallel \leq \parallel X(t_1) \parallel$ which proves that the absolute values of the amplitudes are non-increasing. This completes the proof of Theorem 1.

The second main and final problem of this paper is the following theorem concerns the boundedness of solutions when $P(t) \neq 0$ and $t \to \infty$ and the stability of the solutions when $P(t) \equiv 0$ in equation (2).

Theorem 2. In addition to the basic assumptions imposed on the functions $a(.)$ and $P(.)$, we assume that the following assumptions hold: Given equation (2) where $a(.)$ and $P(.)$ continuous functions on $[0,\infty)$ such that

$$a(t) > 0, a^{'}(t) \geq 0, K > a(t) > k > 0$$

for some positive constants K and k, and

$$\parallel P(t) \parallel \leq e(t), q(t) = \frac{e(t)}{a(t)}, q(.) \in L^1[0,\infty).$$

Then, all solutions of equation (2) are bounded as $t \to \infty$. Further, if $P(t) \equiv 0$ in equation (2), then all solutions of equation (2) are stable. Furthermore, if $P(t) \equiv 0$ in equation (2), then the absolute values of the amplitudes form a non-increasing sequence.

Proof. The boundedness of solutions as $t \to \infty$ can be proved by following the way shown in Theorem 1. We need only to prove stability. Since $P(t) \equiv 0$, in view of the assumptions of Theorem 2, it follows from estimate (4) that

$$\frac{1}{a(t)} \parallel \dot{X}(t) \parallel^2 + \int_0^t \frac{a^{'}(s)}{a^2(s)} \parallel \dot{X}(t) \parallel^2 ds + \parallel X(t) \parallel^2$$
$$= \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2$$

so that

$$\parallel X(t) \parallel^2 \leq \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2$$
$$\leq \parallel X(0) \parallel^2 + \frac{1}{k} \parallel \dot{X}(0) \parallel^2 \quad (7)$$

and

$$\frac{1}{a(t)} \parallel \dot{X}(t) \parallel^2$$
$$\leq \parallel X(0) \parallel^2 + \frac{1}{a(0)} \parallel \dot{X}(0) \parallel^2$$

which implies that

$$\parallel \dot{X}(t) \parallel^2 \leq a(t) \parallel \dot{X}(0) \parallel^2 + \frac{a(t)}{a(0)} \parallel \dot{X}(0) \parallel^2$$
$$K \parallel \dot{X}(0) \parallel^2 + \frac{K}{k} \parallel \dot{X}(0) \parallel^2 . \quad (8)$$

Hence, relations (7) and (8) show that given small initial conditions, both $\parallel X(.) \parallel$ and $\parallel \dot{X}(.) \parallel$ remain small so the solutions are indeed stable. The proof of Theorem 2 is now completed.

Example 1. As a special case of equation (2) for $n = 2$, we consider the second order linear differential system

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + (t+1) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t(1+t^2)^{-1} \\ cost(1+t^2)^{-1} \end{bmatrix}, t \geq 0. \quad (9)$$

Then, it follows that

$$a(t) = t+1, a^{'}(t) = 1 > 0,\ t \geq 0,$$

$$P(t) = \begin{bmatrix} \frac{t}{1+t^2} \\ \frac{cost}{1+t^2} \end{bmatrix}, ||P(t)|| = \left|\left| \begin{bmatrix} \frac{t}{1+t^2} \\ \frac{cost}{1+t^2} \end{bmatrix} \right|\right| \leq \frac{t+1}{1+t^2} = e(t),$$

$$\int_0^\infty \frac{e(s)}{a(s)} ds = \int_0^\infty q(s) ds = \int_0^\infty \frac{1}{1+s^2} ds = \frac{\pi}{2},$$

that is, $q(.) \in L^1(0,\infty)$. Hence, all the conditions of Theorem 1 hold. Therefore, all solutions of equation (9) are bounded as $t \to \infty$ by Theorem 1.

Example 2. As a special case of equation (2) for $n = 2$, we consider the second order linear differential system

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + 2(t+e^{-t}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t(1+t^2)^{-1} \\ e^{-t}(1+t^2)^{-1} \end{bmatrix}, t \geq 0. \quad (10)$$

Hence, it follows that

$$a(t) = 2t + 2e^{-t},\ a^{'}(t) = 2 - 2e^{-t} \geq 0,\ t \geq 0,$$

$$P(t) = \begin{bmatrix} \frac{t}{1+t^2} \\ \frac{e^{-t}}{1+t^2} \end{bmatrix}, ||P(t)|| = \left|\left| \begin{bmatrix} \frac{t}{1+t^2} \\ \frac{e^{-t}}{1+t^2} \end{bmatrix} \right|\right| \leq \frac{t+e^{-t}}{1+t^2} = e(t),$$

$$\int_0^\infty \frac{e(s)}{a(s)} ds = \int_0^\infty q(s) ds = \int_0^\infty \frac{1}{1+s^2} ds = \frac{\pi}{4},$$

that is, $q(.) \in L^1(0,\infty)$.

Thus, all the conditions of Theorem 2 hold. Hence, all solutions of equation (10) are bounded as $t \to \infty$, If $P(t) \equiv 0$ in equation (10), then all solutions of equation (2) are stable.

Remark. Kroopnick [3] proved Theorem A and Theorem B by the integral test for a scalar homogenous linear differential equation of second order, $x^{''} + a(t)x = 0$. In spite of the results of Kroopnick [3] are not new, the proofs presented in [3] are new and simplify previous works since the Gronwall inequality is avoided which is the usual case. It also is worth mentioning that the equation discussed in [3] is special case of our equation, $\ddot{X} + a(t)X = P(t)$. When we take $n = 1$ and $P(t) = 0$, then our equation and assumptions, the assumptions of Theorem 1 and Theorem 2, reduce to those of Kroopnick [3, Theorem 1, Theorem 2]. Since the Gronwall inequality is avoided here, which is the usual case, the proofs of this paper are new and the results of this paper simplify previous works in the literature (see [1], [5]). Furthermore, our results extend and improve the results of Kroopnick [3, Theorem 1, Theorem 2] and those in the literature.

3 Conclusion

A well known linear differential system of second order has been considered. Certain sufficient conditions have been constructed which guarantee that all solutions are bounded as $t \to \infty$ and stable, and the absolute values of the amplitudes form a non-increasing sequence. The technique of proof involves the integral test. Two examples are included to illustrate the results. Our results extend and improve some recent results in the literature.

References

[1] Ahmad, S.; Rama Mohana Rao, M., Theory of Ordinary differential Equations. With Applications in Biology and engineering. Affiliated East-West Press Pvt. Ltd., New Delhi, 1999.
[2] Gallot, S., Hulin, D., Lafontaine, J., Riemannian geometry, Universitext, Springer-Verlag, Berlin, 2004.
[3] Kroopnick, Allan J., Two new proofs for the boundedness of solutions to Missouri J. Math. Sci. **25**, 103-105 (2013).
[4] Pettini, M., Valdettaro, R., On the Riemannian description of chaotic instability in Hamiltonian dynamics, Chaos **5**, (1995), 646-652.
[5] Sanchez, David A., Ordinary differential equations and stability theory. An introduction. Republ. of 1968 orig. publ. by Freeman and Company. Dover Books on Advanced Mathematics. New York: Dover Publications, Inc. VI, (1979).
[6] D Becherer, Bounded solutions to backward SDEs with jumps for utility optimization and indifference hedging, The Annals of Applied Probability, **16**, 2027-2054 (2006)
[7] VK Chandrasekar, M Senthilvelan, M Lakshmanan, Unusual Linard-type nonlinear oscillator, Physical Review E, **72**, 066203 (2005).

Cemil Tunç was born in Yeşilöz Köyü (Kalbulas), Horasan-Erzurum, Turkey, in 1958. He received the Ph. D. degree in Applied Mathematics from Erciyes University, Kayseri, in 1993. His research interests include qualitative behaviors to differential and integral equations. At present he is Professor of Mathematics at Yüzüncü Yıl University, Van-Turkey.

Osman Tunç was born in Talas, Kayseri-Turkey, in 1989. He received the MS degree in Mathematics from Institute of Science, Fatih University, Istanbul-Turkey, in 2014.