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On Some Subclasses of Analytic Functions Defined by Fractional Derivative in the Conic Regions

*Khalida Inayat Noor**, *Qazi Zahoor Ahmad and Muhammad Aslam Noor*

Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.

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Abstract: In this paper, we use fractional derivative operator to define certain classes of analytic functions related to conic domains. Using differential subordination and convolution techniques, we prove coefficient and inclusion results. It is also shown that these classes are closed under convolution with convex functions. Some interesting applications of this result are also highlighted.

Keywords: Conic domain, bounded boundary rotation, subordination, convolution, fractional derivative, convex, uniformly starlike, univalent.
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1 Introduction

Let $\mathscr{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let $S^*(\gamma), C(\gamma)$ be the subclasses of $\mathscr{A}$ of all the functions which are starlike and convex of order $\gamma(0 \leq \gamma < 1)$, respectively. A functions $f \in \mathscr{A}$ is said to be in the class $UST(k,\gamma)$ of uniformly starlike function of order γ, if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \quad k \geq 0, \quad \gamma \in [0,1).$$

It is obvious that, for $k = 0$, we get the class $S^*(\gamma)$. The class $UST(1,0) \equiv UST$ was defined and studied by Ronning [14] and $UST(1,\gamma) \equiv UST(\gamma)$ was investigated in [7], $UST(k,0) \equiv k - UST$ is defined in [10].

Following Acu [1], we define the class $UK(k,\gamma,\beta)$ as follows.

An analytic function $f \in \mathscr{A}$ is in the class $UK(k,\gamma,\beta)$ if

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} > k\left|\frac{zf'(z)}{g(z)} - 1\right| + \beta, \quad \text{for some } g \in UST(k,\gamma),$$
$$k \geq 0, \beta \in [0,1).$$

We now define the following.

Definition 1.1. Let $p(z)$ be analytic in E with $p(0) = 1$. Then $p(z)$ is said to belong to the class $P(p_{k,\gamma})$, if and only if, $p(z)$ takes all the values in the conic domain $\Omega_{k,\gamma}$, $k \geq 0$, $\gamma \in [0,1)$, such that

$$\Omega_{k,\gamma} = (1-\gamma)\Omega_k + \gamma, \quad 1 \in \Omega_{k,\gamma},$$

where

$$\Omega_k = \{u + iv : u > k\sqrt{(u-1)^2 + v^2} \quad \}. \tag{2}$$

Remark 1.1. The domain Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$ and right half plane when $k = 0$.

The functions $p_{k,\gamma}(z)$, given below, play the role of extremal functions for the class $P(p_{k,\gamma})$.

$$p_{k,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z} & (k=0), \\ 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k=1) \\ \frac{1-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}(\arccos k)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2-\gamma}{1-k^2}, & (0<k<1) \\ 1 + \frac{1-\gamma}{k^2-1}\sin\left(\frac{\pi}{2K(x)}\int_0^{\frac{u(z)}{\sqrt{x}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dt\right) + \frac{k^2-\gamma}{k^2-1}, & (k>1), \end{cases} \tag{3}$$

where $u(z) = \frac{z-\sqrt{K}}{1-\sqrt{K}z}$, $t \in (0,1)$, $z \in E$ and K is such that $k = \cosh\left(\frac{\pi K'(x)}{4K(x)}\right)$, and $K'(x)$ is complementary

* Corresponding author e-mail: khalidanoor@hotmail.com

integral of Legendere's complete elliptic integral $K(x)$ of the first kind. See also [5,8].

For $k=0$, it is clear that

$$p_{0,\gamma}(z)=1+2(1-\gamma)z+2(1-\gamma)z^2+\dots,$$

For $k=1$,

$$p_{1,\gamma}(z)=1+\frac{8}{\pi^2}(1-\gamma)z+\frac{16}{3\pi^2}(1-\gamma)z^2+\dots.$$

By comparing Taylor series expansion, we have, for $0<k<1$,

$$p_{k,\gamma}(z)=1+\frac{8(1-\gamma)}{\pi^2(1-k^2)}(\arccos k)^2 z+\dots,$$

where $B=\frac{2}{\pi}\arccos k$, and for $k>1$,

$$\begin{aligned}&p_{k,\gamma}(z)\\&=1+\frac{\pi^2(1-\gamma)}{4\sqrt{x}(k^2-1)K^2(x)(1+x)}\\&\times\left\{z+\frac{4K^2(x)(x^2+6x+1)-\pi^2}{24\sqrt{x}K^2(x)(1+x)}z^2+\dots.\right\}\end{aligned}$$

Remark 1.2. If $p\in P(p_{k,\gamma})$, then $p\prec p_{k,\gamma}$, ($\prec$ denotes subordinate to) and using the properties of the domain Ω_k,we have

$$Re\{p(z)\}>Re\left(p_{k,\gamma}(z)\right)>\frac{k+\gamma}{k+1}.$$

From the definition of $UST(k,\gamma)$, it follows that $f\in UST(k,\gamma)$ implies that $\frac{zf'}{f}$ takes all the values in the conic domain $\Omega_{k,\gamma}$.

Definition 1.3. Let $h(z)$ be analytic in E with $h(0)=1$. Then $h(z)$ is said to belong to the class $P_m(p_{k,\gamma})$, $m\geq 2$, if and only if there exist $h_1,h_2\in P(p_{k,\gamma})$ such that

$$h(z)=\left(\frac{m}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)h_2(z),\ \text{for } z\in E. \quad (4)$$

We note the following special cases.

(i) $P_2(p_{k,\gamma})=P(p_{k,\gamma})$
(ii) $P_m(p_{0,\gamma})=P_m(\gamma)$. For this class, we refer to [11].
(iii) For $\gamma=0,k=0$, $P_m(p_{0,0})=P_m$, which is a well known class introduced and studied in [12].
(iv) With $\gamma=k=0$ and $m=2$, we obtain the class P of Caratheodory functions with positive real part.

Let $\phi(a,c;z)$ be the incomplete beta function defined as follows:

$$\phi(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1},\quad z\in E,\quad c\neq 0,-1,-2,\dots,$$

where $(x)_n$ is Pochhamer symbol defined in terms of Gamma function Γ by

$$\begin{aligned}(x)_n&=\frac{\Gamma(x+n)}{\Gamma(x)}\\&=\begin{cases}1, & n=0,\\ x(x+1)(x+2)\dots(x+n-1), & n\in N=\{1,2,\dots\}.\end{cases}\end{aligned}$$

The fractional derivative of order α is defined in [3,10] for a function $f(z)$ by

$$D_z^{\alpha}f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z\frac{f(\xi)}{(z-\xi)^{\alpha}}d\xi,\quad 0\leq\alpha<1,$$

where $f(z)$ is an analytic function in a simply connected domain of the z-plane containing the origin and the multiplicity of $(z-\xi)^{-\alpha})$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi)>0$. Using $D_z^{\alpha}f(z)$, Owa and Srivastava [10] introduced the operator $L_{\alpha}:\mathscr{A}\longrightarrow\mathscr{A}$, known as the extension of fractional derivative and fractional integral, as follows.

$$\begin{aligned}L_{\alpha}f(z)&=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}f(z) &(5)\\&=z+\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_nz^n\\&=\phi(2,2-\alpha;z)\star f(z),\quad \alpha\neq 2,3,4,\dots, &(6)\end{aligned}$$

where $L_0f(z)=f(z)$.

We now have

Definition 1.4. Let $f\in\mathscr{A}$. Then $f(z)$ is said to be in the class $UK_m(k,\gamma,\beta)$,
$k\geq 0$, $\gamma,\beta\in[0,1), m\geq 2$, if and only if there exists $g\in UST(k,\gamma)$ such that
$\frac{zf'}{g}\in P_m(p_{k,\beta})$, $z\in E$.

Also, $f\in UK_{\lambda}^{\alpha}(m,k,\gamma,beta)$, if and only if

$\{(1-\lambda)L_{\alpha}f(z)+\lambda z(L_{\alpha}f(z))'\}\in UK_m(k,\gamma,\beta)$, for $z\in E$ and $\lambda\geq 0$.

It can easily be seen that $f\in UK_{\lambda}^{\alpha}(m,k,\gamma,\beta)$ implies that

$$\left\{(1-\lambda)\frac{z(L_{\alpha}f(z))'}{L_{\alpha}g(z)}+\lambda\frac{z\left(z(L_{\alpha}f(z))'\right)'}{L_{\alpha}g(z)}\right\}\in P_m)p_{k,\beta}),(7)$$

where

$g\in UST(k,\gamma), k\geq 0,\beta,\gamma\in[0,1), m\geq 2,\lambda\geq 0$, and $z\in E$.

We now discuss some special cases.

(i) For $\alpha=k=\beta=\gamma=0$, $m=2$, the class $UK_1^0(2,0,0,0)$ coincides with the class C^* of quasi-convex functions which was first introduced and studied in [6]. Also see [4].

(ii) $UK_0^0(2,0,0,0)\equiv K$, the well-known class of close-to-convex univalent functions, see [2].

For other different choices of parameters α,λ, k,m, β and γ, we obtain several known and new subclasses of analytic functions.

Remark 1.3. Let f be given (1). Then, from (4) and (5), we can write

$$D_{\lambda}^{\alpha}f(z)=(1-\lambda)L_{\alpha}f(z)+\lambda z(L_{\alpha}f(z))'=z+\sum_{n=2}^{\infty}A_nz^n,$$

where

$$A_n=\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}(1+\lambda(n-1))a_n \quad (8)$$

Also, from (4) and (5), we can write $D^{\alpha}_{\lambda} f(z)$ as
$D^{\alpha}_{\lambda} f(z) = \phi(2,2-\alpha;z) \star \Psi_{\lambda}(z) \star f(z), \quad (\alpha \neq 2,3,\ldots),$
$\Psi_{\lambda}(z) = \dfrac{z-(1-\lambda)z^2}{(1-z)^2}.$
Throughout this paper, we shall assume $k \geq 0, \lambda \geq 0, m \geq 2, \quad \alpha,\beta,\gamma \in [0,1)$ and $z \in E$. unless otherwise stated.

2 Preliminary Results

We need the following results in our investigation.
Lemma 2.1[15]. Let $f(z)$ and $g(z)$ be convex and starlike respectively. Then, for every function $F(z)$ analytic in E with $F(0)=1$, we have

$$\frac{f(z)\star g(z)F(z)}{f(z)\star g(z)} \subseteq \overline{co}F(E), \quad z \in E,$$

where $\overline{co}$ denotes the closed convex hull.
Lemma 2.2 [7]. Let $f \in UST(k,\gamma)$. Then $L_{\alpha} f \in S^*(\frac{1}{2})$ for $k \geq 1$.
Lemma 2.3 [13]. Let

$$h(z) = 1+\sum_{n=1}^{\infty} c_n z^n \text{ be subordinate to } H(z) = 1+\sum_{n=1}^{\infty} C_n z^n \text{ in } E.$$

If $H(z)$ is univalent in E and $H(E)$ is convex, then $|c_n| \leq |C_1|, \quad n \geq 1$.
Lemma 2.4. Let N, D be analytic in E with $N(0) = D(0) = 0$ and $D \in UST(k,\gamma) \subset S^*$ in E. Then

$$\left\{(1-\lambda)\frac{N(z)}{D(z)} + \lambda\frac{N'(z)}{D'(z)}\right\} \in P_m(p_{k,\beta})$$

$$\text{implies } \frac{N}{D} \in P_m(p_{k,\beta}) \quad \text{in} \quad E.$$

Proof. Let

$$\frac{N(z)}{D(z)} = p(z) = \left(\frac{m}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)p_2(z), \tag{9}$$

where $p(z)$ is analytic in E with $p(0)=1$. Now

$$\frac{N'(z)}{D'(z)} = p(z) + \frac{zp'(z)}{\frac{zD'(z)}{D(z)}} = p(z) + \frac{zp'(z)}{p_0(z)},$$

where

$$p_0(z) = \frac{zD'(z)}{D(z)} \in P(p_{k,\beta}).$$

So

$$\begin{aligned}(1-\lambda)\frac{N(z)}{D(z)} + \lambda\frac{N'(z)}{D'(z)} &= p(z) + \lambda\frac{zp'(z)}{p_0(z)} \\ &= \left\{p(z) + B(z)\lambda zp'(z)\right\} \in P_m(p_{k,\beta}),\end{aligned}$$

where $B(z) = \frac{1}{p_0(z)} \in P$ in E.
Using (2.1), it follows that
$\left\{p_i(z) + B(z)\lambda zp'(z)\right\} \in P(p_{k,\beta}) \quad \text{in} \quad E.$
We now use a result [1, Theorem 2.2] to have $p_i \in P(p_{k,\beta}), \quad i = 1,2$ in E which implies that $p \in P_m(p_{k,\beta})$ in E. □

3 Main Results

Theorem 3.1. $UK^{\alpha}_{\lambda}(m,k,\gamma,\beta) \subset UK^{0}_{\lambda}(m,k,\gamma,\beta)$.
Proof. Let $f \in UK^{\alpha}_{\lambda}(m,k,\gamma,\beta)$. Then

$$\left\{(1-\lambda\frac{z(L_{\alpha} f(z))'}{L_{\alpha} g(z)} + \lambda\frac{z\left(z(L_{\alpha} f(z))'\right)'}{L_{\alpha}(z)}\right\} \in P_m(p_{k,\beta}),$$

where

$$\frac{z(L_{\alpha} g(z))'}{L_{\alpha} g(z)} \in P(p_{k,\gamma}) \quad \text{in} \quad E.$$

It is known [7] that, if $L_{\alpha} g \in UST(k,\gamma)$, then $g \in UST(k,\gamma)$ in E. Now

$$\begin{aligned}&(1-\lambda)\left[\frac{zf'(z)}{g(z)} + \lambda\frac{(zf'(z))'}{g(z)}\right] \\ &= (1-\lambda)\left[\frac{\phi(2-\alpha,2;z)\star\phi(2,2-\alpha;z)\star zf'(z)}{\phi(2-\alpha,2;z)\star\phi(2,2-\alpha;z)\star g(z)}\right] \\ &+\lambda\left[\frac{\phi(2-\alpha,2;z)\star\phi(2,2-\alpha;z)\star z(zf'(z))'}{\phi(2-\alpha,2;z)\star\phi(2,2-\alpha;z)\star g(z)}\right] \\ &= (1-\lambda)\left[\frac{\phi(2-\alpha,2;z)\star z(L_{\alpha} f(z))'}{\phi(2-\alpha,2;z)\star L_{\alpha} g(z)}\right] \\ &+\lambda\left[\frac{\phi(2-\alpha,2;z)\star z(z(L_{\alpha} f(z))')'}{\phi(2-\alpha,2;z)\star L_{\alpha} g(z)}\right] \\ &= \frac{\phi(2-\alpha,2;z)\star\left[(1-\lambda)\frac{z(L_{\alpha} f)'}{L_{\alpha} g} + \lambda\frac{z(z(L_{\alpha} f)')'}{L_{\alpha} g}\right](L_{\alpha} g)}{\phi(2-\alpha,2;z)\star L_{\alpha} g} \\ &= \frac{\phi(2-\alpha,2;z)\star F(L_{\alpha} g)}{\phi(2-\alpha,2;z)\star(L_{\alpha} g)} \\ &= \left(\frac{m}{4}+\frac{1}{2}\right)\left[\frac{\phi(2-\alpha,2;z)\star F_1(L_{\alpha} g)}{\phi(2-\alpha,2;z)\star(L_{\alpha} g)}\right] \\ &-\left(\frac{m}{4}-\frac{1}{2}\right)\left[\frac{\phi(2-\alpha,2;z)\star F_2(L_{\alpha} g)}{\phi(2-\alpha,2;z)\star(L_{\alpha} g)}\right],\end{aligned}$$

where $F_i \in P(p_{k,\beta}), \quad i = 1,2$ and $L_{\alpha} g \in S^*$.
Now $\phi(2-\alpha,2;z)$ is a convex function, since

$$z\phi'(2-\alpha,2;z) = \Phi(2-\alpha,1;z) = \frac{z}{(1-z)^{2-\alpha}}$$

belongs to $S^*(\dfrac{\alpha}{2}) \subseteq S^*$.

Using Lemma 2.1, we have for $i = 1,2$

$$\frac{\phi(2-\alpha,2;z)\star F_i(L_{\alpha} g)}{\phi(2-\alpha,2;z)\star L_{\alpha} g} \subseteq \overline{co}\left(p_{k,\beta}(E)\right).$$

This implies

$$\left\{(1-\lambda)\frac{zf'(z)}{g(z)} + \lambda\frac{z(zf'(z))'}{g(z)}\right\} \in P_m(p_{k,\beta}),$$

where $g \in UST(k,\gamma), \quad z \in E$. This proves that $f \in UK^{0}_{\lambda}(m,k,\gamma,\beta)$ in E. □

With similar argument, we can easily prove the following.

Theorem 3.2. Let $0 \le \alpha_1 \le \alpha - 2 < 1$. Then, for $k \ge 1$, $UK_\lambda^{\alpha_2}(m,k,\gamma,\beta) \subset UK_\lambda^{\alpha_1}(m,k,\gamma,\beta)$.

Theorem 3.3. Let

$$\frac{\left(z(L_\alpha f)'\right)'}{(L_\alpha g)'} \in P_m(p_{k,\beta}), \quad (L_\alpha g) \in UST(k,\gamma).$$

Then $f \in UK_0^\alpha(m,k,\gamma,\beta)$ for $z \in E$.

Proof. Let

$$\frac{z(L_\alpha f)'}{L_\alpha g} = p(z).$$

Then

$$\left(z(L_\alpha f)'\right)' = (L_\alpha g)' p(z) + (L_\alpha g) p'(z),$$

and this give us

$$\frac{\left(z(L_\alpha f)'\right)'}{(L_\alpha g)'} = \left\{ p(z) + \frac{zp'(z)}{p_0(z)} \right\} \in P_m(p_{k,\beta}),$$

where

$$p_0(z) = \frac{z(L_\alpha g)'}{L_\alpha g} \in P(p_{\alpha,\gamma}).$$

We use Lemma 2.4 with $N(z) = z(L_\alpha f)'$, $D(z) = L_\alpha g$ and $\lambda = 1$. This gives us that $p \in P_m(p_{k,\beta})$ in E which leads to the desired results that $f \in UK_0^\alpha(m,k,\gamma,\beta)$ in E. □

Next we prove that the class $UK_\lambda^\alpha(m,k,\gamma,\beta)$ is invariant under convolution with convex univalent functions.

Theorem 3.4. Let $f \in UK_\lambda^\alpha(m,k,\gamma,\beta)$ and let $h(z)$ be a convex univalent function. Then

$$(f \star h)(z) \in UK_\lambda^\alpha(m,k,\gamma,\beta), \quad z \in E.$$

Proof. Let $f \in UK_\lambda^\alpha(m,k,\gamma,\beta)$. Then

$$\left\{ (1-\lambda)\frac{z(L_\alpha f)'}{L_\alpha g} + \lambda \frac{z\left(z(L_\alpha f)'\right)'}{L_\alpha g} \right\} \in P_m(p_{k,\beta}),$$

where

$$L_\alpha g \in UST(k,\gamma) \subset S^*\left(\frac{k+\gamma}{k+1}\right) \subset S^*.$$

Now

$$(1-\lambda)\frac{z(L_\alpha(f\star h))'}{L_\alpha(g\star h)} + \lambda \frac{z\left(z(L_\alpha(f\star h))'\right)'}{L_\alpha(g\star h)}$$

$$= \frac{h \star \left\{ \frac{(1-\lambda)z(L_\alpha f)' + \lambda z\left(z(L_\alpha f)'\right)'}{(L_\alpha g)} \right\}(L_\alpha g)}{h \star (L_\alpha g)}$$

$$= \frac{h \star F(L_\alpha g)}{h \star (L_\alpha g)}.$$

The desired result follows by applying Lemma 2.1. □

Let $p_{k,\beta}(z)$, given by (2) be given by

$$p_{k,\beta}(z) = 1 + \delta_1 z + \delta_2 z^2 + \ldots.$$

Then

$$\delta_1 = \delta(k,\beta) = \begin{cases} 8(1-\beta)(\arccos k)^2, & 0 \le k < 1, \\ \frac{8(1-\beta)}{\pi^2}, & k = 1, \\ \frac{\pi^2(1-\beta)}{4\sqrt{x}(k^2-1)K^2(x)(1+x)}, & k > 1. \end{cases} \tag{10}$$

Theorem 3.5. Let $f \in UK_\lambda^\alpha(m,k,\gamma,\beta)$ and be given by (1). Then

$$|a_n| \le \frac{\Gamma(n+1-\alpha)}{(1+\lambda(n-1))\Gamma(n+1)\Gamma(2-\alpha)} \times \left\{ \frac{(\delta(k,\gamma))_{n-1}}{n!} + \frac{m}{2n}|\delta(k,\beta)| \sum_{j=1}^{n-1} \frac{|\delta(k,\gamma)|_{j-1}}{(j-1)!} \right\}.$$

Proof. Let $G(z) = L_\alpha g(z) \in UST(k,\gamma)$ and write

$$G(z) = z + \sum_{n=2}^\infty B_n z^n, \quad g(z) = z + \sum_{n=2}^\infty b_n z^n.$$

Then

$$B_n = \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} b_n, \quad n \ge 2. \tag{11}$$

For $p \in P_m(p_{k,\beta})$, and $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$, let

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4}, \frac{1}{2}\right) p_2(z), \quad p_i \prec p_{k,\beta}, \quad i = 1,2.$$

Writing

$$p_i(z) = 1 + d_1 z + d_2 z^2 + \ldots, \quad n \ge 1,$$

we have

$$|d_n| \le |\delta(k,\beta)|,$$

where $\delta(k,\beta)$ is given by (10) and we have used Lemma 2.3. Combining these facts, we have

$$|c_n| \le \frac{m}{2}|\delta(k,\beta)|. \tag{12}$$

Now, using (7) and (10), we have

$$nA_n = B_n + \sum_{j=1}^{n-1} c_{n-j} B_j, \quad n \ge 2. \tag{13}$$

From (11), (12) and (13), it follows that

$$|A_n| \le \frac{(|\delta(k,\gamma)|)_{n-1}}{n!} + \frac{m|\delta(k,\beta)|}{2n} \sum_{j=1}^{n-1} \frac{(|\delta(k,\gamma)|)_{j-1}}{(j-1)!}. \tag{14}$$

We obtain the required result form (7) and (14). □

We now consider the integral operator $I_a : \mathscr{A} \longrightarrow \mathscr{A}$ defined as

$$I_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) t^{a-1} dt, \quad a \in \mathscr{C}, Re\{a\} > 0. \tag{15}$$

For $a = 1$, we obtain Libera integral operator, $a = 0$ gives us Alexander integral operator and in the cases $a = 1,2,3,\ldots$, we obtain Bernardi operator.

Theorem 3.6. Let $F \in UK_{\lambda}^{\alpha}(m,k,\gamma,\beta)$, and let $f(z) = I_a F(z)$, where I_a is the integral operator defined by (3.6). Then $f \in UK_{\lambda}^{\alpha}(m,k,\gamma,\beta)$ for $z \in E$.

Proof. Since $I_a F(z) = \phi(z) \star F(z)$, where

$$\phi_a(z) = \sum_{n=1}^{\infty} \frac{1+a}{n+a} z^n, \quad Re\{a\} > 0$$

is convex in E, see [15]. Proof follows immediately by applying Theorem 3.4, and
hence $f \in UK_{\lambda}^{\alpha}(m,k,\gamma,\beta)$ for z in E. □

Define

$$\Psi_a(z) = \frac{a}{a+1}\frac{z}{1-z} + \frac{1}{a+1}\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} \frac{a+n}{a+1} z^n, \quad (a > -1).$$

Then Ψ_a is convex for $|z| < r_a = \frac{a+1}{2+\sqrt{3+a^2}}$.

Since $F(z) = I_a F(z) \star \Psi_a(z)$, where $I_a F$ is defined by (15). We use Theorem 3.3 to have:

Theorem 3.7. Let, for $a > -1$, $I_a F \in UK_{\lambda}^{\alpha}(m,k,\gamma,\beta)$. Then $F \in UK_{\lambda}^{\alpha}(m,k,\gamma,\beta)$ for $|z| < r_a$, where

$$r_a = \frac{a+1}{2+\sqrt{3+a^2}}.$$

By choosing suitable values for different parapmeters, we obtain several known and new results as special cases.

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Khalida Inayat Noor is a leading world-known figure in mathematics. She obtained her PhD from University of Wales (Swansea)(UK). She has a vast experience of teaching and research at university levels in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. She was awarded HEC best research paper award in 2009 and CIIT Medal for innovation in 2009. She has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2010 for her outstanding contributions in mathematical sciences and other fields. Her field of interest and specialization is Complex analysis, Geometric function theory, Functional and Convex analysis. She introduced a new technique, now called as Noor Integral Operator which proved to be an innovation in the field of geometric function theory and has brought new dimensions in the realm of research in this area. She has been personally instrumental in establishing PhD/MS programs at COMSATS Institute of Information Technology, Pakistan. Dr. Khalida Inayat Noor has supervised successfully several Ph.D and MS/M.Phil students. She has been an invited speaker of number of conferences and has published more than 400 (Four hundred) research articles in reputed international journals of mathematical and engineering sciences. She is member of editorial boards of several international journals of mathematical and engineering sciences.

M. Aslam Noor earned his PhD degree from Brunel University, London, UK (1975) in the field of Applied Mathematics(Numerical Analysis and Optimization). He has vast experience of teaching and research at university levels in various countries including Pakistan, Iran, Canada, Saudi Arabia and United Arab Emirates. His field of interest and specialization is versatile in nature. It covers many areas of Mathematical and Engineering sciences such as Variational Inequalities, Operations Research and Numerical Analysis. He has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2008, in recognition of his contributions in the field of Mathematical Sciences. He was awarded HEC Best Research paper award in 2009. He has supervised successfully several Ph.D and MS/M.Phil students. He is currently member of the Editorial Board of several reputed international journals of Mathematics and Engineering sciences. He has more than 780 research papers to his credit which were published in leading world class journals.

Qazi Zahoor Ahmad is Ph.D scholar at COMSATS Institute of Information Technology, Pakistan. He is working under the supervision of Prof. Dr. Khalida Inayat Noor. His field of interest is Geometric function theory and related areas.