

Kumaraswamy Modified Inverse Weibull Distribution: Theory and Application

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Abstract: In this article, we propose a generalization of the modified inverse Weibull distribution. The generalization are motivated by the recent work of Cordeiro *et al.* [2] and are based on the Kumaraswamy distribution. The generalized distribution is called the Kumaraswamy modified inverse Weibull(KMIW) distribution. We provide a comprehensive description of the structural properties of the subject distribution and explore some of its special cases. It will be shown that the analytical results are applicable to model real world data.

Keywords: Kumaraswamy distribution; Hazard function; Modified inverse Weibull distribution; Maximum likelihood; Order statistic

1 Introduction

The inverse Weibull distribution is the life time probability distribution which has been used in reliability analysis. It can be used to model a variety of failure characteristics such as infant mortality, useful life and wear-out periods. Reliability and failure data both from life testing and in service records are often modeled by the life time distributions such as the inverse exponential, inverse Rayleigh, inverse Weibull distributions. Keller *et al.* [9] introduced the use of the inverse Weibull distribution to describe the degeneration phenomena of mechanical components such as the dynamic components of diesel engines. The inverse Weibull distribution also provides a good fit to several data such as the times to breakdown of an insulating fluid, subject to the action of constant tension [18]. A random variable X is said to have inverse Weibull (IW) distribution with parameters $\theta > 0$ and $\alpha > 0$ if its cumulative distribution function (cdf) is given by

$$F_{IW}(x) = \exp(-\theta x^{-\alpha}), \quad x > 0 \quad (1)$$

Note that in equation (1) θ is the scale parameter and α is the shape parameter. The corresponding probability density function (pdf) of inverse Weibull distribution is

$$f_{IW}(x) = \alpha \theta x^{-(\alpha+1)} \exp(-\theta x^{-\alpha}) \quad (2)$$

Observe that the cdf given in (1) becomes identical with the cdf of inverse Rayleigh distribution if $\alpha = 2$ and it coincides with the inverse exponential distribution if $\alpha = 1$. Also note that when $\theta = 1$ we have the Fréchet distribution function. In the probability theory of statistics the Weibull and inverse Weibull distributions are the family of continuous probability distributions which have the capability to develop many other life time distributions such as exponential, negative exponential, Rayleigh, inverse Rayleigh distributions and Weibull families also known as type *I, II* and *III* extreme value distributions. Many generalizations of the Weibull distribution and their successful applications have been studied by several authors, see Almalki *et al.* [1]. Some work also has been done on inverse Weibull distribution by Khan *et al.* [10, 11, 12]. Recently, Khan *et al.* [13] introduced the generalized version of four parameter modified inverse Weibull distribution and also provided comprehensive description of the mathematical properties of the modified inverse Weibull distribution. The cumulative distribution function (cdf) of the modified inverse Weibull distribution is given by

$$G_{MIW}(x) = \exp\{-(\lambda x^{-1} + \theta x^{-\alpha})\}, \quad \lambda, \theta, \alpha > 0 \quad (3)$$

where α is the shape parameter representing the different patterns of the modified inverse Weibull probability

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distribution. Here θ is a scale parameter representing the characteristic life and is also positive and λ is a scale parameter. The probability density function of the modified inverse Weibull distribution is given as

$$g_{MIW}(x) = \left[\lambda x^{-2} + \alpha \theta x^{-(\alpha+1)} \right] \exp \left\{ -(\lambda x^{-1} + \theta x^{-\alpha}) \right\}, \quad (4)$$

The purpose of the present study is to generalize the modified inverse Weibull distribution by using a bounded continuous distribution.

The beta distribution with density function

$$f_B(x) = \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1}, \quad 0 < x < 1,$$

with $a > 0$ and $b > 0$, is perhaps one of the most popular bounded continuous probability distribution. Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. This distribution has attracted lot of attention within the area of theoretical and applied statistics. The distribution has found extensive application in areas including; but not limited to engineering, computer sciences and hydrology. The beta distribution can be used to generalize any parent cumulative distribution function $G(x)$ using

$$F_{BG}(x) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1} (1-w)^{b-1} dw \quad (5)$$

The generalization given in (5) has been used by number of authors to propose new distributions. Some notable references include Nadarajah *et al.* [15, 16, 17], Famoye *et al.* [6], Eugene *et al.* [5], Hanook *et al.* [7] and many others. Although the generalization of distribution functions given in (5) has attracted number of researchers, it still involves the complexity of incomplete beta function ratio. Some researchers have suggested to use other bounded distribution, with support (0,1), to obtain the generalization of any parent cumulative distribution function. One such distribution is the Kumaraswamy [14] distribution having density and distribution function as

$$f_K(x) = abx^{a-1}(1-x^a)^{b-1} \quad (6)$$

and

$$F_K(x) = 1 - [1 - x^a]^b \quad (7)$$

where $a > 0$ and $b > 0$ are parameters whose role is to introduce asymmetry and produce distribution with heavier tails. It should be noted that the Kumaraswamy pdf can be unimodal, uniantimodal, increasing, decreasing or constant depending on the choice of the parameters a and b . In particular $a = b = 1$ yields a Uniform (0,1) distribution. Jones [8] explored the background and genesis of the Kumaraswamy distribution and made clear some similarities and differences between the beta and Kumaraswamy distributions. He highlighted several advantages of the

Kumaraswamy distribution over the beta distribution. For a detailed survey of the Kumaraswamy distribution, the readers are referred to [8]. In this note, we combine the works of Kumaraswamy [14] and Cordeiro *et al.* [2] to derive some mathematical properties of a new model, the so-called Kumaraswamy modified inverse Weibull (KMIW) distribution, which stems from the following general construction: if G denotes the baseline cumulative function of a random variable, then a generalized class of distributions can be defined by

$$F_{KG}(x) = 1 - [1 - G(x)^a]^b \quad (8)$$

where $a > 0$ and $b > 0$ are two additional shape parameters. Note that since the cdf is quite tractable the KG- distribution can be used quite effectively even if the data are censored. Correspondingly, the probability density function of KG-distribution is given by

$$f_{KG}(x) = abg(x)G(x)^{a-1} [1 - G(x)^a]^{b-1} \quad (9)$$

The density family in (9) has many of the same properties of the class of Beta- G distributions see [5] for details. It should be noted that the KG-distribution has some advantages over BG-distribution in terms of tractability, since it does not involve any special function. Equivalently, as occurs with the BG-family of distributions, special KG-distributions can be generated by choosing the $G(x)$ in (8) the cdf of the specified distribution of choice. For extensive discussion and some examples of the proposed KG- distribution including the Kumaraswamy Exponentiated Pareto distribution readers are refer to Elbatal [4] and the references therein. A physical interpretation of the KG-distribution for a and b positive integers, as described in Corderio *et al.* [3], is as below:

Suppose a system is made of b independent components and that each component is made up of a independent subcomponents. Suppose the system fails if any of the b components fails and that each component fails if all of the a subcomponents fail. Let $X_{j1}, X_{j2}, \dots, X_{ja}$ denote the life times of the subcomponents within the j^{th} component, $j = 1, 2, \dots, b$ with common cdf $G(x)$. Let X_j denote the lifetime of the j^{th} component, $j = 1, \dots, b$ and let X denote the lifetime of the entire system. Then the cdf of X is given by

$$P(X \leq x) = 1 - \{1 - G^a(x)\}^b.$$

So, it follows that the KG distribution given by (8) is precisely the time to failure distribution of the entire system.

The rest of the article is organized as follows: In section 2, we define the Kumaraswamy modified inverse Weibull (KMIW) distribution and present some of its special cases. We also provide the series representation of pdf and cdf of the KMIW distribution which will be quite useful for characterizing the distribution. Some statistical

properties including the moments, quantiles and moment generating function are discussed in section 3. In section 4 we will discuss the reliability behavior of the subject distribution. We will discuss distribution of the order statistics in section 5. Parameter estimations are discussed in section 6. Finally, in section 7 we present a real world data analysis to illustrate the usefulness of the proposed distribution.

2 Kumaraswamy Modified Inverse Weibull Distribution

In this section, we introduce the five-parameter Kumaraswamy modified inverse Weibull (KMIW) distribution. By taking $G(x)$ in equation (8) to be the cdf of the modified inverse Weibull (MIW) distribution (3) the cdf of the KMIW distribution can be written as

$$F_{KMIW}(x) = 1 - \left[1 - \exp \left\{ -a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right\} \right]^b \tag{10}$$

The corresponding pdf of the KMIW distribution is

$$f_{KMIW}(x) = ab \left[\frac{\lambda}{x^2} + \frac{\alpha\theta}{x^{\alpha+1}} \right] \exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right] \times \left[1 - \exp \left\{ -a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right\} \right]^{b-1} \tag{11}$$

Figure 1 and figure 2 illustrate the graphical behavior of the pdf of Kumaraswamy modified inverse Weibull distribution for selected values of the parameters.

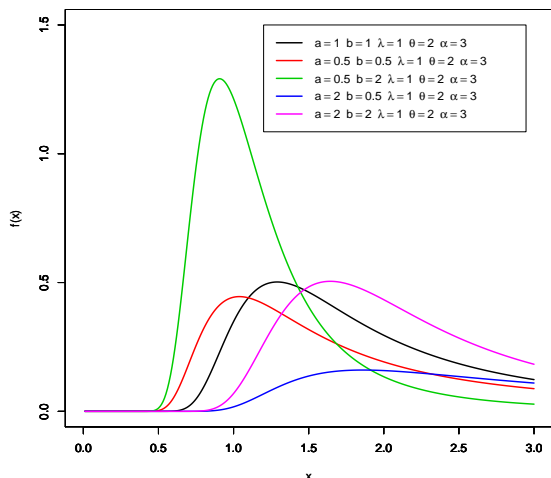


Fig. 1: pdf of KMIW distribution for different values of a and b with $\lambda = 1, \theta = 2, \& \alpha = 3$

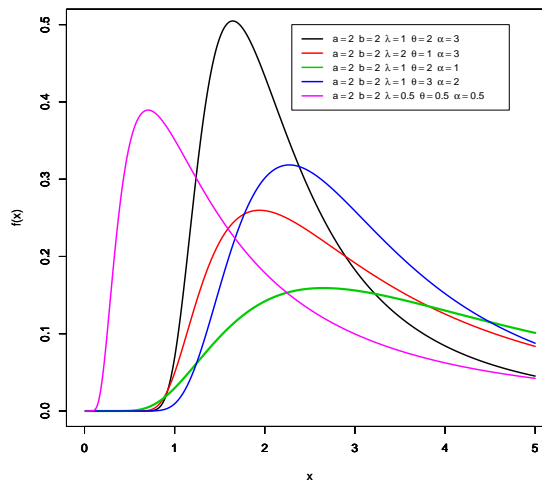


Fig. 2: pdf of KMIW distribution for different values of λ, α & θ with $a = 2 \& b = 2$

2.1 Special cases of the KMIW distribution

The Kumaraswamy modified inverse Weibull distribution is very flexible model that approaches to different distributions when its parameters are changed. In addition to some standard distribution the KMIW distribution includes the following well-known distributions as special models.

- (a) If $b = 1$, then the KMIW distribution reduces to the exponentiated modified inverse Weibull (EMIW) distribution.
- (b) If $\alpha = 2$, then KMIW distribution reduces to Kumaraswamy modified inverse Rayleigh (KMIR) distribution.
- (c) If $\alpha = 1$, then KMIW distribution reduces to the Kumaraswamy modified inverse exponential (KMIE) distribution which is in fact the Kumaraswamy inverse exponential distribution with exponentiated parameters $\lambda + \theta$.
- (d) If $\lambda = 0$, then KMIW distribution reduces to the Kumaraswamy inverse Weibull (KIW) distribution.
- (e) If $a = b = 1$, then the KMIW distribution reduces to the modified inverse Weibull (MIW) distribution.
- (f) If $a = b = 1$ and $\lambda = 0$, then KMIW distribution reduces to the inverse Weibull (IW) distribution.

Table 1 summarizes the special cases of KMIW distribution for selected values of the parameters.

Table 1: Kumaraswamy modified inverse Weibull distribution and some of its sub-models

Model	a	b	α	θ	λ	CDF
KMIW	-	-	-	-	-	$1 - \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right] \right\}^b$
EMIW	-	1	-	-	-	$\exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right]$
KMIR	-	-	2	-	-	$1 - \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^2} \right) \right] \right\}^b$
KMIE	-	-	1	-	-	$1 - \left\{ 1 - \exp \left[-a \left(\frac{\lambda + \theta}{x} \right) \right] \right\}^b$
KIW	-	-	-	-	0	$1 - \left\{ 1 - \exp \left[-a \frac{\theta}{x^\alpha} \right] \right\}^b$
MIW	1	1	-	-	-	$\exp \left[- \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right]$
IW	1	1	-	-	0	$\exp \left[- \left(\frac{\theta}{x^\alpha} \right) \right]$

2.2 Series Representation of CDF and PDF

In this subsection we present series representations of cdf and pdf Kumaraswamy modified inverse Weibull distribution which will be useful to study the mathematical properties of the subject distribution. Using following series representation of Prudnikov *et al.* [19]

$$(1+x)^\alpha = \sum_{k=0}^\infty \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} \frac{x^k}{k!}$$

We have the cdf of Kumaraswamy modified inverse Weibull distribution is

$$F(x) = 1 - \Gamma(b+1) \sum_{k=0}^\infty (-1)^k \frac{\exp \{ -ak(\lambda x^{-1} + \theta x^{-\alpha}) \}}{k! \Gamma(b-k+1)}, \tag{12}$$

Similarly the pdf of Kumaraswamy modified inverse Weibull distribution becomes

$$f(x) = a\Gamma(b+1) \left(\lambda x^{-2} + \alpha \theta x^{-(\alpha+1)} \right) \times \sum_{k=0}^\infty (-1)^k \frac{\exp \{ -a(k+1)(\lambda x^{-1} + \theta x^{-\alpha}) \}}{k! \Gamma(b-k)}$$

Note that in particular, for $b = 1$, the pdf and cdf are respectively given by

$$F(x) = \exp \{ -a(\lambda x^{-1} + \theta x^{-\alpha}) \} = \sum_{n=0}^\infty (-1)^n \frac{a^n}{n!} (\lambda x^{-1} + \theta x^{-\alpha})^n,$$

and

$$f(x) = a \left(\lambda x^{-2} + \alpha \theta x^{-(\alpha+1)} \right) \exp \{ -a(\lambda x^{-1} + \theta x^{-\alpha}) \} = a \left(\lambda x^{-2} + \alpha \theta x^{-(\alpha+1)} \right) \sum_{n=0}^\infty (-1)^n \frac{a^n}{n!} (\lambda x^{-1} + \theta x^{-\alpha})^n$$

It is worth noting that the particular case described above is simply the expressions of the exponentiated modified inverse Weibull distribution. Also note that if X has Kumaraswamy($a, 1$) distribution then $(1 - X)$ has Kumaraswamy($1, a$) and $-\ln(X)$ has $Exp(a)$ distribution. Therefore, the special case described above can be related to the generalization using these special distribution.

3 Statistical Properties

In this section we will present some the statistical properties of the KMIW distribution.

3.1 Quantile Function

The KMIW quantile function, say $x = Q(u)$, can be obtained by inverting (10). We have

$$\frac{\lambda}{x} + \theta \left(\frac{1}{x} \right)^\alpha + \ln \left\{ 1 - (1-u)^{\frac{1}{b}} \right\}^{\frac{1}{a}} = 0.$$

We can easily generate X by taking u as a uniform random variable in $(0, 1)$.

3.2 Moments

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). The r^{th} moments of the KMIW distribution can be expressed as

$$E(X^r) = \int_0^\infty x^r f(x) dx = \int_0^\infty x^r \left[a\Gamma(b+1) \left(\lambda x^{-2} + \alpha \theta x^{-(\alpha+1)} \right) \times \sum_{k=0}^\infty (-1)^k \frac{\exp \{ -a(k+1)(\lambda x^{-1} + \theta x^{-\alpha}) \}}{k! \Gamma(b-k)} \right] dx = a\Gamma(b+1) \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{k+j} (a(k+1)\theta)^j}{k! \Gamma(b-k) j!} [\lambda I_1 + \alpha \theta I_2]$$

where,

$$I_1 = \int_0^\infty x^{r-\alpha j-2} \exp \{ -a(k+1)\lambda x^{-1} \} dx = (a\lambda(k+1))^{r-\alpha j-1} \Gamma(r(j-1)+1)$$

and

$$I_2 = \int_0^\infty x^{r-\alpha j-\alpha-1} \exp \{ -a(k+1)(\lambda x^{-1}) \} dx = (a(k+1)\lambda)^{r-\alpha j-\alpha} \Gamma((j+1)\alpha-r)$$

Hence, the r^{th} moment of KMIW distribution is given by

$$E(X^r) = a\Gamma(b+1) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+j} (a(k+1)\theta)^j}{k!\Gamma(b-k)j!} \times \left\{ \frac{\lambda\Gamma(\alpha j - r + 1)}{[a(k+1)\lambda]^{\alpha j - r + 1}} + \frac{\alpha\theta\Gamma(\alpha j + \alpha - r)}{[a(k+1)\lambda]^{\alpha j + \alpha - r}} \right\} \quad (13)$$

The mean, variance, skewness and kurtosis of the KMIW distribution can be calculated from (13) using the relations:

$$\begin{aligned} \text{Mean}(X) &= E(x) \\ \text{Variance}(X) &= E(x^2) - [E(x)]^2 \\ \text{Skewness}(X) &= \frac{E(x^3) - 3E(x)E(x^2) + 2E^3(x)}{\text{Var}^{3/2}(x)} \\ \text{Kurtosis}(X) &= \frac{E(x^4) - 4E(x)E(x^3) + 6E(x^2)E^2(x) - 3E^4(x)}{\text{Var}^2(x)} \end{aligned}$$

3.3 Moment Generating Function

In this subsection we derived the moment generating function of KMIW distribution. The well-known definition of the moment generating function is given by $M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f(x) dx$. Since $\sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x)$ converges and each term is integrable for all t close to 0, then we can rewrite the moment generating function as $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$ by replacing $E(X^r)$. Hence using (13) the MGF of KMIW distribution is given by

$$\begin{aligned} M_X(t) &= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \\ &= a\Gamma(b+1) \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+j} t^r [a(k+1)\theta]^j}{r!k!\Gamma(b-k)j!} \\ &\quad \times \left\{ \frac{\lambda\Gamma(\alpha j - r + 1)}{(a(k+1)\lambda)^{\alpha j - r + 1}} + \frac{\alpha\theta\Gamma(\alpha(j+1) - r)}{(a(k+1)\lambda)^{\alpha(j+1) - r}} \right\}. \end{aligned}$$

Similarly, the characteristic function of the KMIW distribution becomes

$$\varphi_X(t) = M_X(it)$$

where $i = \sqrt{-1}$ is the unit imaginary number.

4 Reliability Analysis

Because of the analytical structure of the Kumaraswamy modified inverse Weibull distribution, it can be a useful model to characterize failure time of a system. The reliability function also known as survival function of the KMIW distribution is denoted by $R_{KMIW}(t)$ and is given as

$$R_{KMIW}(t) = \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \right\}^b \quad (14)$$

and the hazard rate function which is an important quantity characterizing life phenomenon of a system is given by

$$\begin{aligned} h(t, \lambda, \mu, \theta, a, b) &= \frac{f(t, \lambda, \mu, \theta, a, b)}{R(t, \lambda, \mu, \theta, a, b)} \\ &= \frac{ab \exp \left[-a \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \left[\frac{\lambda}{t^2} + \frac{\alpha\theta}{t^{\alpha+1}} \right]}{\left\{ 1 - \exp \left[-a \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \right\}^b} \quad (15) \end{aligned}$$

In table 2 we summarize few special cases of the hazard rate function of the Kumaraswamy modified inverse Weibull distribution(15).

Table 2: Hazard rate function of some of the sub-models of the Kumaraswamy modified inverse Weibull distribution

Model	a	b	α	θ	λ	Hazard Rate Function
KMIWD	-	-	-	-	-	$\frac{ab \exp \left[-a \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \left[\frac{\lambda}{t^2} + \frac{\alpha\theta}{t^{\alpha+1}} \right]}{\left\{ 1 - \exp \left[-a \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \right\}^b}$
EMIWD	-	1	-	-	-	$\frac{a \exp \left[-a \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \left[\frac{\lambda}{t^2} + \frac{\alpha\theta}{t^{\alpha+1}} \right]}{\left\{ 1 - \exp \left[-a \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \right\}^b}$
KMIRD	-	-	2	-	-	$\frac{ab}{t^3} \frac{(\lambda t + 2\theta)}{\left\{ \exp \left(\frac{a(\lambda t + \theta)}{t^2} \right) - 1 \right\}}$
KMIED	-	-	1	-	-	$\frac{ab}{t^2} \frac{(\lambda + \theta)}{\left\{ \exp \left(\frac{a(\lambda + \theta)}{t} \right) - 1 \right\}}$
KIWD	-	-	-	-	0	$\frac{ab}{t^{\alpha+1}} \frac{\alpha\theta}{\left\{ \exp \left(\frac{a\theta}{t^\alpha} \right) - 1 \right\}}$
MIWD	1	1	-	-	-	$\frac{\exp \left[- \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \left[\frac{\lambda}{t^2} + \frac{\alpha\theta}{t^{\alpha+1}} \right]}{\left\{ 1 - \exp \left[- \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \right\}^b}$
IWD	1	1	-	-	0	$\frac{1}{t^{\alpha+1}} \frac{\alpha\theta}{\left\{ \exp \left(\frac{\alpha\theta}{t^\alpha} \right) - 1 \right\}}$

Figure 3 and figure 4 illustrate the graphical behavior of the hazard rate of Kumaraswamy modified inverse Weibull distribution for selected values of the parameters.

Many generalized probability models have been proposed in reliability literature through the fundamental relationship between the reliability function $R(t)$ and its cumulative hazard function(CHF) $H(t)$ given by $H(t) = -\ln R(t)$. The CHF describes how the risk of a particular outcome changes with time. The cumulative hazard rate function of a Kumaraswamy modified inverse Weibull distribution is given by

$$\begin{aligned} H(t) &= -\ln \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right] \right\}^b \\ &= b \sum_{n=1}^{\infty} \frac{\exp \left\{ -na \left(\frac{\lambda}{t} + \frac{\theta}{t^\alpha} \right) \right\}}{n}. \end{aligned}$$

5 Order Statistics

The order statistics are among the most fundamental tools in non-parametric statistics and statistical inference. In

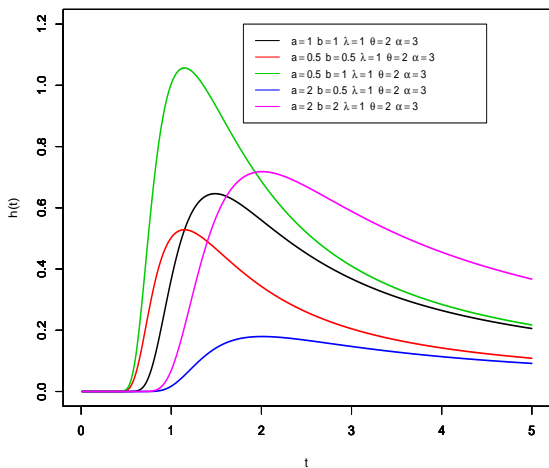


Fig. 3: Hazard rate function of KMIW distribution for different values of a and b with $\lambda = 1, \theta = 2, \& \alpha = 3$

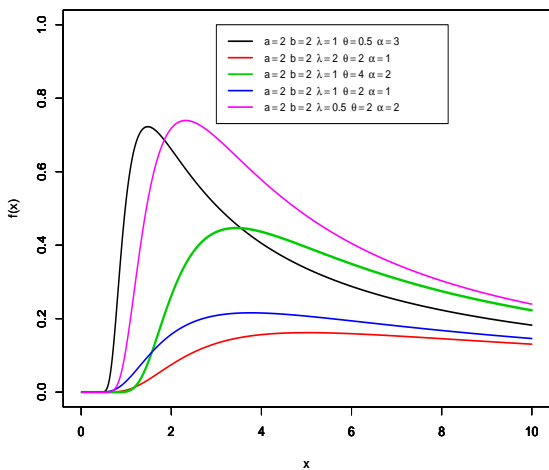


Fig. 4: Hazard rate function of KMIW distribution for different values $\alpha, \theta \& \lambda$ with $a = 2 \& b = 2$

fact, the order statistics have many applications in reliability and life testing. Some distributional properties of the maximum and minimum of random variables have been extensively studied in the literature. In addition, the literature on order statistics contains a huge work about the maximum.

Let X_1, X_2, \dots, X_n be a simple random sample from KMIW distribution with cumulative distribution function and probability density function as in (10) and (11),

respectively. Let $X_{(1:n)} \leq X_{(2:n)} \leq \dots \leq X_{(n:n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(i:n)}$ denote the lifetime of an $(n - i + 1)$ -out-of- n system which consists of n independent and identically components. When $i = 1$, and when $i = n$, such systems are better known as series, and parallel systems, respectively. Considerable attention has been given to establish several reliability properties of such systems. It is well known that the cdf and pdf of $X_{(i:n)}$ for $1 \leq i \leq n$ are, respectively, given by

$$F_{i:n}(x) = \sum_{k=i}^n \binom{n}{k} [F(x)]^k [1 - F(x)]^{n-k} = \int_0^{F(x)} \frac{n!}{(i-1)!(n-i)!} t^{i-1} (1-t)^{n-i} dt \quad (16)$$

and

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x) \quad (17)$$

We define the smallest order statistic $X_{(1)} = \text{Min}(X_1, X_2, \dots, X_n)$, the largest order statistic as $X_{(n)} = \text{Max}(X_1, X_2, \dots, X_n)$ and median order statistic X_{m+1} where $m = \lfloor \frac{n}{2} \rfloor$.

5.1 Distribution of Minimum, Maximum and Median

Let X_1, X_2, \dots, X_n be a random sample from a Kumaraswamy modified inverse Weibull distribution then the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{1:n}(x) = n[1 - F(x)]^{n-1} f(x) = nab \left[\frac{\lambda}{x^2} + \frac{\alpha\theta}{x^{\alpha+1}} \right] \exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right] \times \left[1 - \exp \left\{ -a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right\} \right]^{nb-1}$$

and the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x) = nab \left[\frac{\lambda}{x^2} + \frac{\alpha\theta}{x^{\alpha+1}} \right] \left[1 - \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right] \right\}^b \right]^{n-1} \times \exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right] \left[1 - \exp \left\{ -a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right\} \right]^{b-1}$$

and the pdf of median order statistic is

$$f_{m+1:n}(\tilde{x}) = \frac{(2m+1)!}{m!m!} (F(\tilde{x}))^m (1 - F(\tilde{x}))^m f(\tilde{x}) = \frac{(2m+1)!}{m!m!} ab \left[\frac{\lambda}{x^2} + \frac{\alpha\theta}{x^{\alpha+1}} \right] \exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right] \times \left[1 - \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right] \right\}^b \right]^m \times \left[1 - \exp \left\{ -a \left(\frac{\lambda}{x} + \frac{\theta}{x^\alpha} \right) \right\} \right]^{bm+b-1}$$

Also note that the minimum, maximum and median order statistics of the five parameter Kumaraswamy modified inverse Weibull distribution converges to the order statistics of several life time distributions when its parameters are changed.

6 Parameter Estimation

In this section we will discuss about the method of parameter estimation of the KMIW distribution. The Maximum Likelihood Estimation is one of the most widely used estimation method for finding the unknown parameters. Here we find the estimators for the *KMIW*. Let X_1, X_2, \dots, X_n be a random sample from $X \sim KMIW(\alpha, \theta, \lambda, a, b)$ with observed values x_1, x_2, \dots, x_n then the likelihood function $L \equiv L(\alpha, \theta, \lambda, a, b : x_i)$ can be written as

$$L = \prod_{i=1}^n ab \left[\frac{\lambda}{x_i^2} + \frac{\alpha\theta}{x_i^{\alpha+1}} \right] \exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right] \times \left[1 - \exp \left\{ -a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right\} \right]^{b-1} \tag{18}$$

Hence, the log-likelihood function $\ell = \ln L$ becomes

$$\ell = n \ln a + n \ln b + \sum_{i=1}^n \ln \left[\frac{\lambda}{x_i^2} + \frac{\alpha\theta}{x_i^{\alpha+1}} \right] - a \sum_{i=1}^n \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) + (b-1) \sum_{i=1}^n \ln \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right] \right\} \tag{19}$$

The components of the score vector are obtained by differentiating (19) with respect to the parameters

$\alpha, \theta, \lambda, a$ and b given as as below.

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \sum_{i=1}^n \frac{\theta(1 - \alpha \ln x_i)}{\alpha\theta + \lambda x_i^{\alpha-1}} + a\theta \sum_{i=1}^n \frac{\ln x_i}{x_i^\alpha} \\ &\quad - a\theta(b-1) \sum_{i=1}^n \frac{\ln x_i}{x_i^\alpha} \frac{\exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right]}{\left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right] \right\}} \\ \frac{\partial \ell}{\partial \theta} &= \sum_{i=1}^n \frac{\alpha}{\alpha\theta + \lambda x_i^{\alpha-1}} - \sum_{i=1}^n \frac{a}{x_i^\alpha} \\ &\quad + a(b-1) \sum_{i=1}^n \frac{\exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right]}{x_i^\alpha \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right] \right\}} \\ \frac{\partial \ell}{\partial \lambda} &= \sum_{i=1}^n \frac{1}{\lambda + \alpha\theta x_i^{-\alpha+1}} - \sum_{i=1}^n \frac{a}{x_i} \\ &\quad + a(b-1) \sum_{i=1}^n \frac{\exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right]}{x_i \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right] \right\}} \\ \frac{\partial \ell}{\partial a} &= \frac{n}{a} - \sum_{i=1}^n \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \\ &\quad + (b-1) \sum_{i=1}^n \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \frac{\exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right]}{\left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right] \right\}} \\ \frac{\partial \ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \ln \left\{ 1 - \exp \left[-a \left(\frac{\lambda}{x_i} + \frac{\theta}{x_i^\alpha} \right) \right] \right\} \end{aligned}$$

The maximum likelihood estimators $\hat{\alpha}, \hat{\theta}, \hat{\lambda}, \hat{a}$ and \hat{b} of $\alpha, \theta, \lambda, a$ and b are obtained by setting the score vector to zero and solving the system of nonlinear equations. It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the log-likelihood function given in (19). For the five parameter Kumaraswamy modified inverse Weibull distribution all second order derivatives exist. Thus, we have

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \\ \hat{\lambda} \\ \hat{a} \\ \hat{b} \end{pmatrix} \sim \text{Normal} \left[\begin{pmatrix} \alpha \\ \theta \\ \lambda \\ a \\ b \end{pmatrix}, \Sigma \right] \tag{20}$$

with

$$\Sigma = -E \begin{bmatrix} V_{\alpha\alpha} & V_{\alpha\theta} & V_{\alpha\lambda} & V_{\alpha a} & V_{\alpha b} \\ V_{\theta\alpha} & V_{\theta\theta} & V_{\theta\lambda} & V_{\theta a} & V_{\theta b} \\ V_{\lambda\alpha} & V_{\lambda\theta} & V_{\lambda\lambda} & V_{\lambda a} & V_{\lambda b} \\ V_{a\alpha} & V_{a\theta} & V_{a\lambda} & V_{a a} & V_{a b} \\ V_{b\alpha} & V_{b\theta} & V_{b\lambda} & V_{b a} & V_{b b} \end{bmatrix}^{-1} \tag{21}$$

where

$$\begin{aligned}
 V_{\alpha\alpha} &= \frac{\partial^2 \ell}{\partial \alpha^2}, & V_{\theta\theta} &= \frac{\partial^2 \ell}{\partial \theta^2}, & V_{\lambda\lambda} &= \frac{\partial^2 \ell}{\partial \lambda^2} \\
 V_{aa} &= \frac{\partial^2 \ell}{\partial a^2}, & V_{bb} &= \frac{\partial^2 \ell}{\partial b^2}, & V_{\alpha\theta} &= \frac{\partial^2 \ell}{\partial \alpha \partial \theta} \\
 V_{\alpha\lambda} &= \frac{\partial^2 \ell}{\partial \alpha \partial \lambda}, & V_{\alpha a} &= \frac{\partial^2 \ell}{\partial \alpha \partial a}, & V_{\alpha b} &= \frac{\partial^2 \ell}{\partial \alpha \partial b} \\
 V_{\theta\lambda} &= \frac{\partial^2 \ell}{\partial \theta \partial \lambda}, & V_{\theta a} &= \frac{\partial^2 \ell}{\partial \theta \partial a}, & V_{\theta b} &= \frac{\partial^2 \ell}{\partial \theta \partial b} \\
 V_{\lambda a} &= \frac{\partial^2 \ell}{\partial \lambda \partial a}, & V_{\lambda b} &= \frac{\partial^2 \ell}{\partial \lambda \partial b}, & V_{ab} &= \frac{\partial^2 \ell}{\partial a \partial b}
 \end{aligned}$$

By calculating this inverse dispersion matrix will yield asymptotic variance and covariances of these MLEs. Approximate $100(1 - \phi)\%$ confidence intervals for $\alpha, \theta, \lambda, a$ and b can be determined, respectively, as

$$\begin{aligned}
 \hat{\alpha} \pm z_{\frac{\phi}{2}} \sqrt{\widehat{V}_{\alpha\alpha}}, & \quad \hat{\theta} \pm z_{\frac{\phi}{2}} \sqrt{\widehat{V}_{\theta\theta}}, & \quad \hat{\lambda} \pm z_{\frac{\phi}{2}} \sqrt{\widehat{V}_{\lambda\lambda}}, \\
 \hat{a} \pm z_{\frac{\phi}{2}} \sqrt{\widehat{V}_{aa}} & \quad \text{and} & \quad \hat{b} \pm z_{\frac{\phi}{2}} \sqrt{\widehat{V}_{bb}}
 \end{aligned}$$

where z_{ϕ} is the upper ϕ^{th} percentile of the standard normal distribution.

We can compute the maximum values of the unrestricted and restricted log-likelihood functions to obtain likelihood ratio (LR) statistics for testing the sub-model of the new distribution. For example, we can use the LR statistic to check whether the fitted Kumaraswamy modified inverse Weibull distribution is statistically “superior” to a fitted Kumaraswamy inverse Weibull distribution for a given data set. In this case we can compare these models by testing $H_0 : \lambda = 0$ (KIW model) versus $H_a : \lambda \neq 0$ (KMIW model). If we want to test whether the fitted Kumaraswamy modified inverse Weibull distribution is statistically “superior” to a fitted modified inverse Weibull distribution for a given data set we calculate the LR statistic by testing $H_0 : a = b = 1$ (MIW model) versus $H_a : \text{either } a \text{ or } b \text{ is not equal to } 1$ (KMIW model). Similarly, if we want to check whether the fitted Kumaraswamy modified inverse Weibull distribution is statistically “superior” to a fitted exponentiated modified inverse Weibull distribution for a given data we calculate the LR statistic by testing $H_0 : b = 1$ (EMIW model) versus $H_a : b \neq 1$ (KMIW model).

7 Applications

In this section, we provide a data analysis in order to assess the goodness-of-fit of the model. The data set refers to testing the tensile fatigue characteristics of a polyester/viscose yarn to study the problem of warp breakage during weaving. The study consists of 100 centimeter yarn sample at 2.3 percent strain level. This

data was studied by Quesenberry *et al.* [20] and discussed by several other authors to compare the reliability models. The descriptive summary of the data is provided in table 3.

Table 3: Descriptive statistics of the yarn data

<i>n</i>	Mean	Median	Variance	Minimum	Maximum
100	222	195.5	20914.38	15	829.0

We fitted the modified inverse Weibull(MIW), exponentiated modified inverse Weibull(EMIW) and Kumaraswamy modified inverse Weibull(KMIW) distribution to the subject data. The MLEs for MIW, EMIW, and KMIW distribution are given in table 4 and the corresponding measures of fit statistic using the AIC and BIC criteria are provided in table 5.

Table 4: Estimated Parameters of MIW , EMIW and KMIW distribution

Model	Parameter Estimates
MIW	$\hat{\alpha} = 1.103, \hat{\theta} = 150.491, \hat{\lambda} = 23.868$
EMIW	$\hat{\alpha} = 19.109, \hat{\theta} = 40.343, \hat{\lambda} = 1.535, \hat{a} = 78.578$
KMIW	$\hat{\alpha} = 0.230, \hat{\theta} = 37.582, \hat{\lambda} = 34.921, \hat{a} = 0.568, \hat{b} = 480.214$

Table 5: Measures of fit statistics of MIW, EMIW and KMIW distribution

Model	$-\ell(\cdot, x)$	AIC	BIC
MIW	652.499	1310.998	1318.814
EMIW	653.293	1314.586	1325.007
KMIW	627.024	1264.048	1277.074

One can use the likelihood ratio (LR) test to show that Kumaraswamy modified inverse Weibull distribution fits the subject data better than the 3-parameter modified inverse Weibull distribution and 4-parameter exponentiated modified inverse Weibull distribution. The LR test statistic for testing $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$ is $\omega = 2(\ell(\hat{\theta}; x) - \ell(\theta_0; x))$ where, $\hat{\theta}$ and θ_0 are MLEs under H_a and H_0 , respectively. The likelihood ratio test is based on the fact that the test statistic ω is asymptotically chi-square distributed with suitable degrees of freedom. The LR test statistic for testing the KMIW distribution versus the MIW distribution is $\omega = 50.9498$. The test statistic is larger than χ_2^2 for any typical level of significance so we conclude that the KMIW distribution provides a significantly better fit to the subject data. Similarly, we can compare the KMIW distribution with

EMIW distribution and arrive in the similar conclusion. Furthermore, the graphical comparison corresponding to these fits to confirm our claim is illustrated in figure 5.

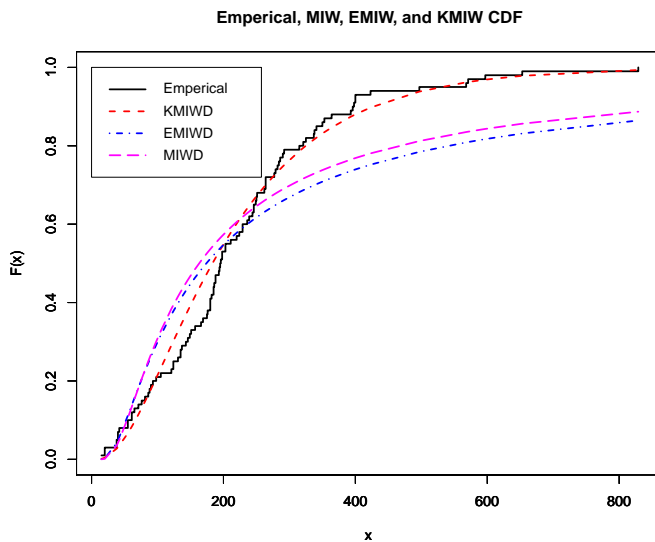


Fig. 5: Empirical, fitted MIW, EMIW and KMIW distribution of warp breakage data

8 Concluding Remarks

In the present study, we have introduced a new generalization of the modified inverse Weibull distribution the so-called Kumaraswamy modified inverse Weibull distribution. Many standard distribution including the inverse Weibull and modified inverse Weibull distribution are embedded in the proposed distribution. Some mathematical properties along with estimation issues are addressed. We have presented an example where the Kumaraswamy modified inverse Weibull distribution fits better than the modified inverse Weibull distribution. Since the proposed distribution appear to include several other distributions we believe that it can be used to model data arising from several different areas. We expect that this study will serve as a reference and help to advance future research in the subject area.

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