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An Axiomatic Characterization of Generalized Entropies under Analyticity Condition

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Abstract: We present axiomatic characterizations of the Nath, Rényi and Havrda-Charvát-Tsallis entropies under the assumption that they are analytic functions with respect to the distribution dimension, unlike the previous characterizations, which suppose that they are expandable and maximized for uniform distribution.

Keywords: Axiomatic characterization; Information measures; Nath entropy; Rényi entropy; Havrda-Charvát-Tsallis entropy

1 Introduction

Nath [1], Rényi [2] and Havrda-Charvát-Tsallis [3], [4] entropies are well-known generalizations of the Shannon entropy. All of them have a more general strong additivity property in comparison to the Shannon entropy. By the strong additivity property, the entropy of joint distribution can be represented as the sum of the entropy of the first one and the conditional entropy of the second one with respect to the first one. The conditional entropy is defined as the P expected value of the entropy of the conditional distribution Q conditioned on P. For the Nath entropy, and its normalized instance, the Rényi entropy, the definition of the expectation is generalized from the linear to the quasi-linear mean, while in the Havrda-Charvát-Tsallis case the linear expectation is used but the additivity is generalized to the γ-additivity [5].

Previous axiomatic systems [6], [7], [8], [9] take the additivity condition as an axiom, and another three axioms - continuity, expandability (which means that adding the zero probability event in the sample space does not affect the entropy distribution) and maximality (which means that the entropy is maximized for uniform distribution). In this letter we provide alternative axiomatic systems, which replace the expendability and maximality axioms with the axiom that states the uniform distribution entropy can be analytically continued if it is taken as the function of the distribution dimension, the property that has an important role in the asymptotic analysis of entropy [10]. The presented results generalize the theorem of Nambiar et al. [11], which characterizes the Shannon entropy.

The letter is organized as follows. In section 2 we review the Shannon entropy uniqueness theorem given by Nambiar et al. [11]. The theorem is generalized to the Nath and Rényi entropies in section 3 and to the Havrda-Charvát-Tsallis entropy in section 4.

2 Shannon entropy

Let the set of all n-dimensional distributions be denoted with

$$\Delta_n \equiv \left\{ (p_1,\dots,p_n) \middle| \, p_i \ge 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n > 1 \qquad (1)$$

and the n-dimensional uniform distribution be denoted with

$$U_n = \left(\frac{1}{n},\dots,\frac{1}{n}\right) \in \Delta_n. \qquad (2)$$

The Shannon entropy [12] of n-dimensional distribution is a function $\mathscr{H}_n : \Delta_n \to \mathbb{R}_{>0}$ given with the family parameterized by $\tau \in \mathbb{R}$:

$$\mathscr{S}_n(P) = \tau \cdot \sum_{k=1}^{n} p_k \log p_k; \qquad \tau < 0, \qquad (3)$$

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where log stands for the logarithm to the base 2. The following theorem characterizes the Shannon entropy.

Theorem 1. Let the function $\mathscr{H}_n : \Delta_n \to \mathbb{R}_{>0}$ satisfy the following axioms, for all $n \in \mathbb{N}$, $n > 1$:

SA1:$\mathscr{H}_n(U_n) = s(1/n)$, where $s : \mathbb{C} \to \mathbb{C}$ is an analytic function.
SA2:Let $P = (p_1, \dots, p_n) \in \Delta_n$, $PQ = (r_{11}, r_{12}, \dots, r_{nm}) \in \Delta_{nm}$, $n, m \in \mathbb{N}$ such that $p_i = \sum_{j=1}^{n} r_{ij}$ and $Q_{|k} = (q_{1|k}, \dots, q_{m|k}) \in \Delta_m$, where $q_{i|k} = r_{ik}/p_k$. Then,

$$\mathscr{H}_{nm}(PQ) = \mathscr{H}_n(P) + \mathscr{H}_m(Q|P), \tag{4}$$

where

$$\mathscr{H}_m(Q|P) = \sum_k p_k \mathscr{H}_m(Q_{|k}) \tag{5}$$

Thus, $\mathscr{H}_n$ is the Shannon entropy.

The previous theorem slightly differs from the one presented in [11], but it can be proven by straightforward repetition of the steps from [11]. First, we do not assume the normalization condition $s(1/2) = 1$, which fixes the constant to $\tau = -1$. Second, the statement of the theorem from [11] assumes that the entropy is a complex analytic function with respect to the distribution. We assume the equivalent statement that the entropy is a continuous real function (note that the assumption about the analyticity with respect to the distribution dimension is kept).

3 Nath entropy

For a distribution $P \in \Delta_n$, we define the α-escort distribution $P^{(\alpha)} = (p_1^{(\alpha)}, \dots, p_n^{(\alpha)})$, where

$$p_k^{(\alpha)} = \frac{p_k^\alpha}{\sum_{i=1}^n p_i^\alpha}; \; k = 1, \dots, n. \tag{6}$$

If $P = (p_1, \dots, p_n) \in \Delta_n$ and $Q = (q_1, \dots, q_m) \in \Delta_m$, the *direct product*, $P \star Q \in \Delta_{nm}$, is defined as

$$P \star Q = (p_1 q_1, p_1 q_2, \dots, p_n q_m). \tag{7}$$

The proof of the following theorem can be found in [7], [13].

Theorem 2. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous invertible function and $\mathscr{H}_n : \Delta_n \to \mathbb{R}_{>0}$ a continuous function

$$\mathscr{H}_n(P) = g^{-1}\left(\sum_{k=1}^n p_k^{(\alpha)}(\alpha) g(\tau \cdot \log p_k)\right); \quad \tau < 0, \tag{8}$$

for all $P = (p_1, \dots, p_n) \in \Delta_n$, and let $\mathscr{H}_n$ be additive for all $n \in \mathbb{N}$, i.e. $\mathscr{H}_{nm}(P \star Q) = \mathscr{H}_n(P) + \mathscr{H}_m(Q)$ for all $P = (p_1, \dots, p_n) \in \Delta_n$, $Q = (q_1, \dots, q_m) \in \Delta_m$ $n, m \in \mathbb{N}$. Then, g is the function from the class parameterized by $\lambda \in \mathbb{R}$:

$$g(x) = \begin{cases} -cx, & \text{for } \lambda = 0 \\ \dfrac{2^{-\lambda x} - 1}{\gamma}, & \text{for } \lambda \neq 0, \end{cases} \tag{9}$$

with the inverse function

$$g^{-1}(x) = \begin{cases} -\dfrac{1}{c}\, x, & \text{for } \lambda = 0 \\ -\dfrac{1}{\lambda} \log(\gamma x + 1), & \text{for } \lambda \neq 0, \end{cases} \tag{10}$$

where $c, \gamma \neq 0$, and the entropy is uniquely determined with

$$\mathscr{H}_n(P) = \begin{cases} \tau \cdot \sum_{k=1}^n p_k^{(\alpha)} \log p_k & \text{for } \lambda = 0 \\ -\dfrac{1}{\lambda} \log\left(\dfrac{\sum_{k=1}^n p_k^{\alpha - \tau\lambda}}{\sum_{k=1}^n p_k^\alpha}\right) & \text{for } \lambda \neq 0, \end{cases} \tag{11}$$

where $\tau < 0$ and $\alpha - \tau\lambda > 0$.

The Nath entropy of n-dimensional distribution is a function $\mathscr{H}_n : \Delta_n \to \mathbb{R}_{>0}$ from the family parameterized by $\alpha, \lambda, \tau \in \mathbb{R}$:

$$\mathscr{H}_n(P) = \begin{cases} \tau \cdot \sum_{k=1}^n p_k \log p_k; \quad \tau < 0 & \text{for } \alpha = 1 \\ \dfrac{1}{\lambda} \log\left(\sum_{k=1}^n p_k^\alpha\right); \; \dfrac{1-\alpha}{\lambda} > 0 \,,\; \lambda \neq 0 & \text{for } \alpha \neq 1. \end{cases} \tag{12}$$

The Rényi entropy is a function $\mathscr{R}_n : \Delta_n \to \mathbb{R}_{>0}$ from the family (12) with $\lambda = 1 - \alpha$ and $\tau = -1$. The following theorem characterizes the Nath and Rényi entropies.

Theorem 3. Let the function $\mathscr{H}_n : \Delta_n \to \mathbb{R}_{>0}$ satisfy the following axioms, for all $n \in \mathbb{N}$, $n > 1$:

NA1:$\mathscr{H}_n(U_n) = v(1/n)$, where $v : \mathbb{C} \to \mathbb{C}$ is an analytic function.
NA2:Let $P = (p_1, \dots, p_n) \in \Delta_n$, $PQ = (r_{11}, r_{12}, \dots, r_{nm}) \in \Delta_{nm}$, $n, m \in \mathbb{N}$ such that $p_i = \sum_{j=1}^n r_{ij}$ and $Q_{|k} = (q_{1|k}, \dots, q_{m|k}) \in \Delta_m$, where $q_{i|k} = r_{ik}/p_k$ and $\alpha \in (0, \infty]$ is some fixed parameter. Then,

$$\mathscr{H}_{nm}(PQ) = \mathscr{H}_n(P) + \mathscr{H}_m(Q|P), \tag{13}$$

where

$$\mathscr{H}_m(Q|P) = f^{-1}\left(\sum_k p_k^{(\alpha)} f(\mathscr{H}_m(Q_{|k}))\right), \tag{14}$$

where f is an invertible continuous function.

Thus, $\mathscr{H}_n$ is the Nath entropy. In addition, if the normalization axiom $v(1/2) = 1$ is satisfied, $\mathscr{H}_n$ is the Rényi entropy.

Proof. Let $A=(1/2,1.2)$. By successive use of formula (13) we get

$$v\left(\frac{1}{2^n}\right)=\mathscr{H}_{2^n}(\underbrace{A\star A\star\ldots\star A}_{n\text{ times}})=\sum_{k=1}^{n}\mathscr{H}_2(A)=$$
$$=n\cdot\mathscr{H}_2(A)=-\log\frac{1}{2^n}\cdot\mathscr{H}_2\left(\frac{1}{2},\frac{1}{2}\right)=\tau\cdot\log\frac{1}{2^n},\quad (15)$$

where $\tau=-\mathscr{H}_2(1/2,1/2)$ is a negative real value, since $\mathscr{H}_2(1/2,1/2)$ is positive by assumption of the theorem. Accordingly, the values of functions $v(z)$ and $\tau\cdot\log z$ coincide at an infinite number of points converging to zero, $\{z=1/2^n\}_{n\in\mathbb{N}}$. Since both $f(z)$ and $\tau\cdot\log z$ are analytic functions, they must be the same:

$$v(z)=\tau\cdot\log z;\quad \tau<0.\quad (16)$$

Let us determine the entropy form for the distribution $P=(p_1,\ldots,p_n)\in\mathbb{C}^n$ when p_i are rational numbers and the case for irrational numbers follows from the continuity of the entropy. Let $P=(p_1,\ldots,p_n)\in\mathbb{C}^n$, $Q_{|k}=(q_{1|k},\ldots,q_{m_k|k})\in\mathbb{C}^{m_k};k=1,\ldots,n$ and $PQ=(r_{11},r_{12},\ldots,r_{nm})\in\mathbb{C}^{nm}$, for $n,m\in\mathbb{N}$, and $p_i=m_i/m$; $r_{ij}=1/m$, $q_{j|i}=1/m_i$, where $m=\sum_{i=1}^{n}m_i$ and $m_i\in\mathbb{N}$ for any $i=1,\ldots,n$ and $j=1,\ldots,m_i$. Then we have $\mathscr{H}_m(PQ)=\mathscr{H}_m(U_m)=v(1/m)=-\tau\cdot\log m$ and $\mathscr{H}_{m_k}(Q_{|k})=\mathscr{H}_{m_k}(U_{m_k})=v(1/m_k)=-\tau\cdot\log m_k$. Since $p_i=\sum_{j=1}^{n}r_{ij}$ and $q_{i|k}=r_{ik}/p_k$, we can apply the axiom [**NA2**] and get

$$\mathscr{H}_n(P)=-\tau\cdot\log m-f^{-1}\left(\sum_{k=1}^{n}p_k^{(\alpha)}f\left(-\tau\cdot\log m_k\right)\right)=$$
$$-\tau\cdot\log m-f^{-1}\left(\sum_{k=1}^{n}p_k^{(\alpha)}f\left(-\tau\cdot\log p_k-\tau\cdot\log m\right)\right).\quad (17)$$

Let us define $f_y(x)=f(-x-y)$, and $f_y^{-1}(z)=-y-f^{-1}(z)$. If we set $y=\tau\cdot\log m$, the equality (17) becomes

$$\mathscr{H}(P)=f_y^{-1}\left(\sum_{k=1}^{n}p_k^{(\alpha)}f_y\left(\tau\cdot\log p_k\right)\right);\qquad \tau<0.\quad (18)$$

Since f is continuous, both f_y and f_y^{-1} are continuous, as well as the entropy, and we may extend the result (18) from rational p_k's to any real valued p_k's defined in [0,1]. Now, if the axiom [**NA2**] is used with $PQ=P\star Q$, the conditions from the Theorem 2 are satisfied so that $f_y(x)=-cx$, for $\lambda=0$ and $f_y(x)=(2^{-\lambda x}-1)/\gamma$, for $\lambda\neq 0$. Accordingly, the entropy is uniquely determined by the class (11). The relationship between the parameters α, τ and λ is determined by use of the axiom [**NA2**].

For $\lambda=0$, since $f_y(x)=f(z)$, where $z=-x-y$, we get $f(z)=f_y(x)=f_y(-z-y)=cz+cy$. If the equality (11) is substituted in [**NA2**] with $f(z)=cz+cy$, we get

$$\sum_{k=1}^{n}\sum_{l=1}^{m}r_{kl}^{(\alpha)}\log\left(r_{kl}\right)=$$
$$=\sum_{k=1}^{n}p_k^{(\alpha)}\log p_k+\sum_{k=1}^{n}\sum_{l=1}^{m}p_k^{(\alpha)}q_{l|k}^{(\alpha)}\log\left(q_{l|k}\right)=$$
$$=\sum_{k=1}^{n}\sum_{l=1}^{m}p_k^{(\alpha)}q_{l|k}^{(\alpha)}\log\left(r_{kl}\right),\quad (19)$$

which can be transformed to

$$\sum_{k=1}^{n}\rho_k^{(\alpha)}\sum_{j=1}^{m}q_{j|k}^{\alpha}=\sum_{k=1}^{n}p_k^{(\alpha)}\sum_{l=1}^{m}q_{l|k}^{\alpha}\quad (20)$$

where

$$\rho_k^{(\alpha)}=\frac{p_k^{(\alpha)}\sum_{l=1}^{m}q_{l|k}^{(\alpha)}\log r_{kl}}{\sum_{i=1}^{n}p_i^{(\alpha)}\sum_{j=1}^{m}q_{j|i}^{(\alpha)}\log r_{ij}}.\quad (21)$$

The equality (20) holds for all distributions and we may consider the case $n=m=2$ and the distributions $P=(1/2,1/2)$, $Q_{|1}=(1,0)$, $Q_{|2}=(1/2,1/2)$. If we set $x_1=\sum_{l=1}^{2}q_{l|1}^{\alpha}=1$, $x_2=\sum_{l=1}^{2}q_{l|2}^{\alpha}=2^{1-\alpha}$ and $u_k=\rho_k^{(\alpha)}$, $v_k=p_k^{(\alpha)}$, the equality (20) can be transformed as follows:

$$\begin{aligned}&u_1x_1+u_2x_2=v_1x_1+v_2x_2\quad\Leftrightarrow\\ \Leftrightarrow\quad&u_1x_1+(1-u_1)x_2=v_1x_1+(1-v_1)x_2\quad\Leftrightarrow\\ \Leftrightarrow\quad&(u_1-v_1)x_1=(u_1-v_1)x_2.\end{aligned}\quad (22)$$

By using $u_1=\frac{1}{3}\neq v_1=\frac{1}{2}$, we get $x_1=x_2$, i.e. $1=2^{1-\alpha}$, which implies $\alpha=1$. Accordingly, the case $\lambda=0$ from (11) reduces to the case $\alpha=1$ from (12). Positivity of the entropy implies that $\tau<0$.

If $\lambda\neq 0$ and the equality (11) is substituted in [**NA2**], we get

$$-\frac{1}{\lambda}\log\left(\frac{\sum_{k=1}^{n}\sum_{l=1}^{m}r_{kl}^{\alpha-\tau\lambda}}{\sum_{k=1}^{n}\sum_{l=1}^{m}r_{kl}^{\alpha}}\right)=-\frac{1}{\lambda}\log\left(\frac{\sum_{k=1}^{n}p_k^{\alpha-\tau\lambda}}{\sum_{k=1}^{n}p_k^{\alpha}}\right)+$$
$$+f^{-1}\left(\sum_k p_k^{(\alpha)}f\left(-\frac{1}{\lambda}\log\left(\frac{\sum_{l=1}^{m}q_{l|k}^{\alpha-\tau\lambda}}{\sum_{l=1}^{m}q_{l|k}^{\alpha}}\right)\right)\right),\quad (23)$$

where $f(z)=(2^{\lambda(z+y)}-1)/\gamma$ (since for $z=-x-y$, we have $f(z)=f_y(-z-y)=(2^{\lambda(z+y)}-1)/\gamma$) or, equivalently,

$$\sum_{k=1}^{n}\sum_{l=1}^{m}r_{kl}^{(\alpha)}r_{kl}^{-\tau\lambda}=\frac{\sum_{k=1}^{n}p_k^{(\alpha)}\ p_k^{-\tau\lambda}}{\sum_{k=1}^{n}p_k^{(\alpha)}\cdot\dfrac{1}{\sum_{l=1}^{m}q_{l|k}^{(\alpha)}q_{l|k}^{-\tau\lambda}}},\quad (24)$$

which can further be transformed into

$$\sum_{k=1}^{n}\sum_{l=1}^{m}r_{kl}^{(\alpha)}\cdot\frac{1}{\sum_{j=1}^{m}q_{j|k}^{\beta}}=\sum_{k=1}^{n}\sum_{l=1}^{m}r_{kl}^{(\beta)}\cdot\frac{1}{\sum_{j=1}^{m}q_{j|k}^{\beta}},\quad (25)$$

where $\beta=\alpha-\tau\lambda$. Similarly to the case $\lambda=0$, we note that the equality (20) holds for all distributions and we consider the case $n=m=2$ and the distributions $P=(1/2,1/2)$, $Q_{|1}=(1,0)$, $Q_{|2}=(1/2,1/2)$. If we set $x_1=\frac{1}{\sum_{l=1}^{2}q_{l|1}^{\beta}}=1$, $x_2=\frac{1}{\sum_{l=1}^{2}q_{l|2}^{\beta}}=2^{\beta-1}$ and $u_k=\sum_{l=1}^{m}r_{kl}^{(\alpha)}=\frac{\sum_{l=1}^{2}q_{l|k}^{\alpha}}{\sum_{i=1}^{2}\sum_{l=1}^{2}q_{l|k}^{\alpha}}$, $v_k=\sum_{l=1}^{m}r_{kl}^{(\beta)}=\frac{\sum_{l=1}^{2}q_{l|k}^{\beta}}{\sum_{i=1}^{2}\sum_{l=1}^{2}q_{l|k}^{\beta}}$, the equality (24) can be transformed into the form (22). By using $u_1=\frac{1}{1+2^{1-\alpha}}\neq v_1=\frac{1}{1+2^{1-\beta}}$, we get $x_1=x_2$, i.e. $1=2^{\beta-1}$, which implies $\beta=1$, i.e. $\alpha-\tau\lambda=1$ with $\tau<0$, and the case $\lambda\neq 0$ from (11) reduces to the case $\alpha\neq 1$ from (12). Positivity of the entropy implies $(1-\alpha)/\lambda>0$.

Finally, if $v(1/2)=\mathscr{H}_2(1/2)=1$, the equation (12) implies $\tau=-1$, $\lambda=1-\alpha$, which proves the theorem.

4 Havrda-Charvát-Tsallis entropy

The Havrda-Charvát-Tsallis entropy of n-dimensional distribution is a function from the family parameterized by $\tau, \lambda, \alpha \in \mathbb{R}$:

$$\mathscr{T}(P) = \begin{cases} \tau \cdot \sum_{k=1}^{n} p_k \log\, p_k; & \tau < 0, \quad \text{for} \quad \lambda = 0 \\ \frac{1}{\lambda} \cdot \left(\sum_k p_k^{\alpha} - 1 \right); & \alpha \neq 1, \quad \text{for} \quad \lambda \neq 0. \end{cases} \tag{26}$$

For $\lambda = 2^{1-\alpha} - 1$, the entropy reduces to the Havrda-Charvát entropy [3], while in the case of $\lambda = 1 - \alpha$ it reduces to the Tsallis entropy [4]. Let us define $\oplus_\lambda$-addition, defined as [5],

$$x \oplus_\lambda y = x + y + \lambda xy; \quad a \in \mathbb{R}, \tag{27}$$

for all $x, y \in \mathbb{R}$. For the case $\lambda = 0$, $\oplus_\lambda$-addition reduces to ordinary addition. The pair $(\mathbb{R}, \oplus_\lambda)$ forms a commutative group where the inverse operation and the $\ominus_\lambda$-difference are defined as

$$\ominus_\lambda x = \frac{-x}{1+\lambda x} \qquad x \ominus_\lambda y = \frac{x-y}{1+\lambda y}. \tag{28}$$

It is easy to see that the structure $(\mathbb{R}, \oplus_\lambda)$ is a topological group isomorphic to $(\mathbb{R}, +)$, with an isomorphism

$$h(x) = \begin{cases} x, & \text{for } \lambda = 0 \\ \frac{2^{\lambda \cdot x} - 1}{\lambda}, & \text{for } \lambda \neq 0 \end{cases} \tag{29}$$

so that

$$h(x+y) = h(x) \oplus_\lambda h(y). \tag{30}$$

Equivalently, for the inverse

$$h^{-1}(y) = \begin{cases} y, & \text{for } \lambda = 0 \\ \frac{\log(\lambda \cdot y + 1)}{\lambda}, & \text{for } \lambda \neq 0, \end{cases} \tag{31}$$

we have

$$h^{-1}(x \oplus_a y) = h^{-1}(x) + h^{-1}(y). \tag{32}$$

The following theorem characterizes the Havrda-Charvát-Tsallis entropy.

Theorem 4. Let the function $\mathscr{T}_n : \Delta_n \to \mathbb{R}_{>0}$ satisfy the following axioms, for all $n \in \mathbb{N}$, $n > 1$:

TA1: $\mathscr{T}_n(U_n) = t(1/n)$, where $t : \mathbb{C} \to \mathbb{C}$ is an analytic function.
TA2: Let $P = (p_1, \ldots, p_n) \in \Delta_n$, $PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{nm}$, $n, m \in \mathbb{N}$ such that $p_i = \sum_{j=1}^n r_{ij}$ and $Q_{|k} = (q_{1|k}, \ldots, q_{m|k}) \in \Delta_m$, where $q_{i|k} = r_{ik}/p_k$. Then,

$$\mathscr{T}_{nm}(PQ) = \mathscr{T}_n(P) \oplus_\lambda \mathscr{T}_m(Q|P), \tag{33}$$

where

$$\mathscr{T}_m(Q|P) = \sum_k p_k^{(\alpha)} \mathscr{T}_m(Q_{|k}). \tag{34}$$

Thus, $\mathscr{T}_n$ is the Havrda-Charvát-Tsallis entropy.

Proof. If $\lambda = 0$, the Theorem 4 reduces to the Theorem 1, so we prove the theorem for $\lambda \neq 0$. Similarly as in the proof of the Theorem 3, we set $A = (1/2, 1.2)$. By successive usage of the formula (33), we get

$$\begin{aligned} t\left(\frac{1}{2^n}\right) &= \mathscr{T}_{2^n}(\underbrace{A \star A \star \ldots \star A}_{n \text{ times}}) = \bigoplus_{k=1}^{n} \mathscr{T}_2(A) = \\ h\left(\sum_{k=1}^{n} \mathscr{T}_2(A)\right) &= h\Big(n \cdot \mathscr{T}_2(A)\Big) = h\left(\tau \cdot \log \frac{1}{2^n}\right), \end{aligned} \tag{35}$$

where $\tau = -\mathscr{T}_2(1/2, 1/2)$ is a negative real value, since $\mathscr{T}_2(1/2, 1/2)$ is positive, by assumption of the theorem and $\operatorname{sgn}(h(x)) = \operatorname{sgn}(x)$. Accordingly, the values of functions $t(z)$ and $h(\tau \cdot \log z)$ coincide at an infinite number of points converging to zero, $\{z = 1/2^n\}_{n \in \mathbb{N}}$. Since both $t(z)$ and $h(\tau \cdot \log z)$ are analytic functions [1], they must be the same,

$$t(z) = h(\tau \cdot \log z) = \frac{1}{\lambda}\left(z^{\tau\lambda} - 1\right); \quad \tau < 0. \tag{36}$$

Similarly as in the proof of the Theorem 3, we determine the entropy form for the distribution $P = (p_1, \ldots, p_n) \in \mathbb{C}^n$, when p_i are rational numbers and the case for irrational numbers follows from the continuity of the entropy. Let $P = (p_1, \ldots, p_n) \in \mathbb{C}^n$, $Q_{|k} = (q_{1|k}, \ldots, q_{m_k|k}) \in \mathbb{C}^{m_k}; k = 1, \ldots, n$, and $PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \mathbb{C}^{nm}$, for $n, m \in \mathbb{N}$, where $p_i = m_i/m$; $r_{ij} = 1/m$, and $q_{j|i} = 1/m_i$, where $m = \sum_{i=1}^n m_i$ and $m_i \in \mathbb{N}$, for any $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$. Then, $p_i = \sum_{j=1}^n r_{ij}$ and $q_{i|k} = r_{ik}/p_k$ and we can apply the axiom [**TA2**], which yields

$$\begin{aligned} \mathscr{T}(P) &= t\left(\frac{1}{m}\right) \ominus_\lambda \sum_{k=1}^{n} p_k^{\alpha}\, t\left(\frac{1}{m_k}\right) = \\ &= \frac{t\left(\frac{1}{m}\right) - \sum_{k=1}^n p_k^{(\alpha)}\, t\left(\frac{1}{m_k}\right)}{1 + \lambda \cdot \sum_{k=1}^n p_k^{(\alpha)}\, t\left(\frac{1}{m_k}\right)} = \frac{1}{\lambda} \cdot \left(\frac{\sum_{k=1}^n p_k^{\alpha}}{\sum_{k=1}^n p_k^{\alpha - \tau\lambda}} - 1 \right). \end{aligned} \tag{37}$$

The relationship between τ and λ is determined by use of the axiom [**TA2**]. If we apply the map h on both sides of the equality (33), using the equality (32), by which $h^{-1}(x \oplus_a y) = h^{-1}(x) + h^{-1}(y)$, we get $h^{-1}(\mathscr{T}(PQ)) = h^{-1}(\mathscr{T}(P)) + h^{-1}(\sum_k p_k^{(\alpha)} \mathscr{T}(Q_{|k}))$. If we now use the equalities (29) and (31) for the mappings h and h^{-1} and the entropy form (37), we get

$$\begin{aligned} \frac{1}{\lambda} \cdot \log\left(\frac{\sum_{k=1}^n r_{kl}^{\alpha}}{\sum_{k=1}^n r_{kl}^{\alpha-\tau\lambda}} \right) &= \frac{1}{\lambda} \cdot \log\left(\frac{\sum_{k=1}^n p_k^{\alpha}}{\sum_{k=1}^n p_k^{\alpha-\tau\lambda}} \right) + \\ &+ h^{-1}\left(\sum_{k=1}^{n} p_k^{(\alpha)}\, h\left(\frac{1}{\lambda} \cdot \log\left(\frac{\sum_{k=1}^n q_{l|k}^{\alpha}}{\sum_{k=1}^n q_{l|k}^{\alpha-\tau\lambda}} \right) \right) \right). \end{aligned} \tag{38}$$

The function $h(z) = (2^{\lambda \cdot z} - 1)/\lambda$ is the linear function of $f(z) = (2^{\lambda(z+y)} - 1)/\gamma$. It is a well-known fact from the mean theory that, if h is a linear function of f, they generate the same quasi-linear mean [14], and the function h in the equality (38)

[1] Note that in this context, h is an analytic continuation of the real function defined by the formula (29).

can be substituted with f. Accordingly, the equation (38) can be rewritten in the form of the equation (23). As shown in the proof of the Theorem 3, the equation (23) is satisfied iff $\alpha - \tau\lambda = 1$ and the equation (37) reduces to the equation (26), which proves the theorem.

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