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Further Results on the Instability of Solutions of Certain Nonlinear Vector Differential Equations of Fifth Order

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By using Lyapunov's second method [13], some new results are established, which insure that the zero solution of non-linear vector differential equations of the form

$$X^{(5)} + \Psi(\dot{X}, \ddot{X})\dddot{X} + \Phi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}) + \Theta(\dot{X}) + F(X) = 0$$

is unstable.

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1 Introduction

So far, instability problems for solutions of various linear and non-linear differential equations of higher order, third-, fourth-, fifth-, sixth-, seventh and eighth orders, have been studied and still are being investigated by many authors. For some related contributors to the subject, we refer to the papers of Ezeilo ([1], [2], [3], [4], [5]), Krasovskii [7], Liao and Lu [8], Li and Yu [9], Li and Duan [10], Lu [11], Lu and Liao [12], Sadek ([16], [17]), Skrapek ([18], [19]), Sun and Hou [20], Tejumola [21], Tunç ([22], [23], [24], [25], [26], [27], [28], [29]), Tunç and Sevli [30], C. Tunç and E. Tunç ([31], [32], [33], [34]), E. Tunç [35] and the references listed in these papers. In all of the above mentioned papers, authors took into consideration of Krasovskii's criteria [7] and used the Lyapunov's second (or direct) method [13]. The reason is, perhaps, due to the effectiveness of Krasovskii's criterion [7] and Lyapunov's second method [13]. In [6], Iggidr and Sallet expressed that "The most efficient tool for the study of the stability of a given non-linear system is provided by Lyapunov's theory". Similarly, in [15], Qian stated that "So far, the most effective method to study the stability of non-linear differential equations is still the Lyapunov's direct method".

Now, it should be better to summarize some works, in particular, focused on the instability of nonlinear differential equations of fifth order. Namely, with respect to our observations in the literature, first, in 1978 and 1979 for the case $n = 1$, Ezeilo ([2], [3], [4]) investigated the instability of zero solution for the following nonlinear scalar differential equations, respectively,

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + a_3 \ddot{x} + a_4 \dot{x} + f(x) = 0,$$

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + h(\dot{x})\ddot{x} + g(x)\dot{x} + f(x) = 0,$$

$$x^{(5)} + \psi(\ddot{x})\dddot{x} + \phi(\ddot{x}) + \theta(\dot{x}) + f(x) = 0$$

and

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + g(\dot{x})\ddot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\dot{x} + f(x) = 0,$$

where a_1, a_2, a_3, a_4 are some constants and f, g, h, ψ, ϕ and θ are continuous functions depending only on the arguments shown such that $f(0) = \phi(0) = \theta(0) = 0$.

On the other hand, in 2003, Sadek [17] studied the instability behaviors of solutions of fifth order nonlinear vector differential equations described by

$$X^{(5)} + \Psi(\ddot{X})\dddot{X} + \Phi(\ddot{X}) + \Theta(\dot{X}) + F(X) = 0$$

and

$$X^{(5)} + AX^{(4)} + B\dddot{X} + H(\dot{X})\ddot{X} + G(X)\dot{X} + F(X) = 0.$$

More recently, Tunç ([26], [29]) and Tunç&Sevli [33], respectively, also gave sufficient conditions which guarantee that the zero solution of the vector differential equations of the form

$$X^{(5)} + AX^{(4)} + \Psi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dddot{X} + G(\dot{X})\ddot{X}$$
$$+ H(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dot{X} + F(X) = 0,$$

$$X^{(5)} + AX^{(4)} + B(t)\Psi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dddot{X} + C(t)G(\dot{X})\ddot{X}$$
$$+ D(t)H(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dot{X} + E(t)F(X) = 0$$

and

$$X^{(5)} + \Psi(\dot{X}, \ddot{X})\dddot{X} + \Phi(X, \dot{X}, \ddot{X}) + \Theta(\dot{X}) + F(X) = 0$$

is unstable.

Now, this paper is devoted to the investigation of instability of the zero solution of fifth-order non-linear vector differential equation

$$X^{(5)} + \Psi(\dot{X}, \ddot{X})\dddot{X} + \Phi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}) + \Theta(\dot{X}) + F(X) = 0 \qquad (1.1)$$

in the real Euclidean space $\Re^n$ (with the usual norm denoted in what follows by $\|.\|$), where $X \in \Re^n$; Ψ is an $n \times n$ -symmetric continuous matrix function depending, in each case, on the arguments shown; $\Phi : \Re^n \times \Re^n \times \Re^n \times \Re^n \times \Re^n \to \Re^n$, $\Theta : \Re^n \to \Re^n$, $F : \Re^n \to \Re^n$ and $\Phi(X, \dot{X}, 0, \dddot{X}, X^{(4)}) = \Theta(0) = F(0) = 0$. It is also supposed that the functions Φ, Θ and F are continuous. Throughout this paper, we consider, instead of equation (1.1), the equivalent differential system

$$\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = W, \quad \dot{W} = U,$$
$$\dot{U} = -\Psi(Y, Z)W - \Phi(X, Y, Z, W, U) - \Theta(Y) - F(X), \tag{1.2}$$

which was obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = W$, $X^{(4)} = U$ in (1.1). It is also assumed that the expressions $J\left(\Psi(Y, Z)Z\,|Y\right)$, $J\left(\Psi(Y, Z)\,|Z\right)$, $J_\Theta(Y)$ and $J_F(X)$, respectively, denote the Jacobian matrices as follows:

$$J\left(\Psi(Y, Z)Z\,|Y\right) = \left(\frac{\partial}{\partial y_j}\sum_{k=1}^{n}\psi_{ik}z_k\right) = \left(\sum_{k=1}^{n}\frac{\partial\psi_{ik}}{\partial y_j}z_k\right),$$
$$J\left(\Psi(Y, Z)\,|Z\right) = \left(\frac{\partial}{\partial z_j}\sum_{k=1}^{n}\psi_{ik}\right) = \left(\sum_{k=1}^{n}\frac{\partial\psi_{ik}}{\partial z_j}\right),$$
$$J_\Theta(Y) = \left(\frac{\partial\theta_i}{\partial y_j}\right), J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right), (i, j = 1, 2, ..., n),$$

where $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(z_1, z_2, \ldots, z_n)$, (ψ_{ik}), $(i,\ k = 1, 2, \ldots, n)$, $(\theta_1, \theta_2, \ldots, \theta_n)$ and $(f_1, f_2, \ldots, f_n)$ are the components of X, Y, Z, Ψ, Θ and F, respectively. In addition to these, it is assumed, as basic throughout the paper, that the Jacobian matrices $J\left(\Psi(Y, Z)Z\,|Y\right)$, $J\left(\Psi(Y, Z)\,|Z\right)$, $J_\Theta(Y)$ and $J_F(X)$ exist and are continuous and symmetric. The symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, and $\lambda_i(A)$, ($A = (a_{ij})$, $(i,\ j = 1,\ 2, \ldots,\ n)$), are the eigenvalues of the $n \times n$ - symmetric matrix A and the matrix $A = (a_{ij})$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \Re^n$ and X^T denotes the transpose of X .

Finally, it is also worth mentioning that the motivation for the present work has been inspired basically by the papers mentioned above. Next, equation (1.1) has never been the subject of systematic investigations in this direction.

2 Preliminaries

In order to prove our main results, we give some basic information which plays an essential role throughout the paper.

Lemma 2.1. *Let A be a real symmetric $n \times n$ -matrix and*

$$a' \geq \lambda_i(A) \geq a > 0 (i = 1, 2, \dots, n),$$

where a', a are constants. Then

$$a' \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle .$$

Proof. See Mirsky [14]. □

In order to prove our further coming results, it will suffice (see Krasovskii [7]) to show that there exists a continuous Lyapunov function $V_0 = V_0(X, Y, Z, W, U)$ which has the following Krasovskii properties:

(K_1) In every neighborhood of $(0, 0, 0, 0, 0)$ there exists a point $(\xi, \eta, \zeta, \mu, \rho)$ such that $V_0(\xi, \eta, \zeta, \mu, \rho) > 0$;

(K_2) the time derivative $\dot{V}_0 = \frac{d}{dt} V_0(X, Y, Z, W, U)$ along solution paths of the system (1.2) is positive semi-definite;

(K_3) the only solution $(X, Y, Z, W, U) = (X(t), Y(t), Z(t), W(t), U(t))$ of the system (1.2) which satisfies $\dot{V}_0 = 0$ $(t \geq 0)$ is the trivial solution $(0, 0, 0, 0, 0)$.

3 Main Results

Our main results are the following theorems concerned with the instability of zero solution of equation (1.1).

Theorem 3.1. *Beside the basic assumptions imposed on Ψ, Φ, Θ and F that appeared in equation (1.1), we assume the following conditions are satisfied:*

(i) $F(0) = 0$ *and* $F(X) \neq 0$ *if* $X \neq 0$*, and the Jacobian matrix* $J_F(X)$ *is symmetric and* $\lambda_i(J_F(X)) < 0$, $(i = 1, 2, \dots, n)$*, for all* $X \in \Re^n$,

(ii) $\Phi(X, Y, 0, W, U) = 0$, $\Phi(X, Y, Z, W, U) \neq 0$ *if* $Z \neq 0$, *and* $\sum_{i=1}^{n} z_i \phi_i(X, Y, Z, W, U) \geq 0$ *for all* X, Y, Z, W, $U \in \Re^n$, *where* $\Phi(X, Y, Z, W, U) = (\phi_1(X, Y, Z, W, U), \dots, \phi_n(X, Y, Z, W, U))$,

(iii) *The matrices* $\Psi(Y, Z)$ *and* $J(\Psi(Y, Z)Z \,|Y)$ *are symmetric and* $J(\Psi(Y, Z)Z \,|Y) \leq 0$ *for all* Y, $Z \in \Re^n$.

Then the zero solution $X = 0$ of equation (1) is unstable.

Proof. We define the Lyapunov function $V_0 = V_0(X, Y, Z, W, U)$ as follows:

$$V_0 = \frac{1}{2} \langle W, W \rangle - \langle Y, F(X) \rangle - \langle Z, U \rangle - \int_0^1 \langle \Theta(\sigma Y), Y \rangle \, d\sigma - \int_0^1 \langle \sigma \Psi(Y, \sigma Z) Z, Z \rangle \, d\sigma. \tag{3.1}$$

Taking notice of (3.1), we see that $V_0(0,0,0,0,0) = 0$. Next, in view of Lemma 2.1 and (3.1), we get that

$$V_0(0,0,0,\varepsilon,0) = \frac{1}{2}\langle\varepsilon,\varepsilon\rangle = \frac{1}{2}\|\varepsilon\|^2 > 0$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \Re^n$. These facts, clearly, show that the Lyapunov function $V_0 = V_0(X,Y,Z,W,U)$ satisfies the first property of Krasovskii, (K_1).

Now, let $(X,Y,Z,W,U) = (X(t),Y(t),Z(t),W(t),U(t))$ be an arbitrary solution of the system (1.2). Differentiating the Lyapunov function given by (3.1) and making use of the system (1.2), we find that

$$\begin{aligned}\dot{V}_0 &= \frac{d}{dt}V_0(X,Y,Z,W,U)\\ &= \langle Z,\Phi(X,Y,Z,W,U)\rangle - \langle Y,J_F(X)Y\rangle + \langle\Psi(Y,Z)W,Z\rangle\\ &\quad + \langle\Theta(Y),Z\rangle - \frac{d}{dt}\int_0^1\langle\Theta(\sigma Y),Y\rangle\,d\sigma - \frac{d}{dt}\int_0^1\langle\sigma\Psi(Y,\sigma Z)Z,Z\rangle\,d\sigma. \end{aligned} \tag{3.2}$$

Now, recall that

$$\begin{aligned}&\frac{d}{dt}\int_0^1\langle\sigma\Psi(Y,\sigma Z)Z,Z\rangle\,d\sigma\\ &= \int_0^1\langle\sigma\Psi(Y,\sigma Z)W,Z\rangle\,d\sigma + \int_0^1\sigma\frac{\partial}{\partial\sigma}\langle\sigma\Psi(Y,\sigma Z)W,Z\rangle\,d\sigma\\ &\quad + \int_0^1\langle\sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ ,Z\rangle\,d\sigma\\ &= \sigma^2\langle\Psi(Y,\sigma Z)W,Z)\rangle\big|_0^1 + \int_0^1\langle\sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ ,Z\rangle\,d\sigma\\ &= \langle\Psi(Y,Z)W,Z\rangle + \int_0^1\langle\sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ ,Z\rangle\,d\sigma. \end{aligned} \tag{3.3}$$

Similarly, we have that

$$\begin{aligned}\frac{d}{dt}\int_0^1\langle\Theta(\sigma Y),Y\rangle\,d\sigma &= \int_0^1\sigma\langle J_\Theta(\sigma Y)Z,Y\rangle\,d\sigma + \int_0^1\langle\Theta(\sigma Y),Z\rangle\,d\sigma\\ &= \int_0^1\sigma\frac{\partial}{\partial\sigma}\langle\Theta(\sigma Y),Z\rangle\,d\sigma + \int_0^1\langle\Theta(\sigma Y),Z\rangle\,d\sigma\\ &= \sigma\langle\Theta(\sigma Y),Z\rangle\big|_0^1\\ &= \langle\Theta(Y),Z\rangle\,. \end{aligned} \tag{3.4}$$

Substituting the estimations (3.3) and (3.4) into (3.2) we obtain

$$\dot{V}_0 = \langle Z, \Phi(X,Y,Z,W,U)\rangle - \langle Y, J_F(X)Y\rangle - \int_0^1 \langle \sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ , Z\rangle\, d\sigma. \quad (3.5)$$

Making use of the assumption $J\left(\Psi(Y,Z)Z\,|Y\right) \le 0$, we have from (3.5) that

$$\dot{V}_0 \ge \langle Z, \Phi(X,Y,Z,W,U)\rangle - \langle Y, J_F(X)Y\rangle\,. \quad (3.6)$$

Subject to the assumptions of Theorem 3.1, we deduce from (3.6) that $\dot{V}_0(t) \ge 0$ for all $t \ge 0$, that is, $\dot{V}_0$ is positive semi-definite. This shows that the property (K_2) of Krasovskii is satisfied. Finally, $\dot{V}_0 = 0$ $(t \ge 0)$ necessarily implies that $Y = 0$ for all $t \ge 0$, and hence also that $X = \xi$ (a constant vector), $Z = \dot{Y} = 0$, $W = \ddot{Y} = 0$, $U = \dddot{Y} = 0$, for all $t \ge 0$. By using the expressions

$$X = \xi, \quad Y = Z = W = U = 0$$

in the system (1.2), it can be seen easily that $F(\xi) = 0$, which necessarily leads that $\xi = 0$ because $F(0) = 0$ and $F(X) \ne 0$ if $X \ne 0$. In view of the above discussion, clearly, it follows that

$$X = Y = Z = W = U = 0 \quad \text{for all } t \ge 0.$$

That is, we now have the property of (K_3) Krasovskii. Therefore, subject to the assumptions of Theorem 1, the function V_0 has the entire the criteria of Krasovskii [7], (K_1), (K_2) and (K_3). Thus, the basic properties of the function $V_0(X,Y,Z,W,U)$, which were proved above, verify that the zero solution of system (1.2) is unstable. The system of equations (1.2) is equivalent to differential equation (1.1) and hence the proof of Theorem 1 is now complete. □

Theorem 3.2. *Beside the basic assumptions imposed on* Ψ, Φ, Θ *and* F *that appeared in equation (1.1), we assume the following conditions are satisfied:*

(i) $F(0) = 0$ *and* $F(X) \ne 0$ *if* $X \ne 0$ *and the Jacobian matrix* $J_F(X)$ *is symmetric and*

$$\lambda_i(J_F(X)) > 0, \quad (i = 1,2,\ldots,n), \quad \textit{for all} X \in \Re^n,$$

(ii) $\Phi(X,Y,0,W,U) = 0$, $\Phi(X,Y,Z,W,U) \ne 0$ *if* $Z \ne 0$, *and*

$$\sum_{i=1}^{n} z_i \phi_i(X,Y,Z,W,U) \le 0 \ \textit{ for all } X,Y,Z,W,U \in \Re^n,$$

where $\Phi(X,Y,Z,W,U) = (\phi_1(X,Y,Z,W,U),\ldots,\phi_n(X,Y,Z,W,U))$.

(iii) *The matrices* $\Psi(Y,Z)$ *and* $J\left(\Psi(Y,Z)Z\,|Y\right)$ *are symmetric,* $\lambda_i(\Psi(Y,Z)) > 0$, $(i = 1,2,\ldots,n)$, *and* $J\left(\Psi(Y,Z)Z\,|Y\right) \ge 0$ *for all* $Y,\ Z \in \Re^n$.

Then the zero solution $X = 0$ *of equation (1.1) is unstable.*

Proof. In a similar manner as in the proof of Theorem 3.1, we define for the proof of Theorem 3.2, the Lyapunov function $V_1 = V_1(X, Y, Z, W, U)$ such that V_1 = - V_0, where V_0 is defined as the same in (3.1), that is,

$$V_1 = -\frac{1}{2}\langle W, W\rangle + \langle Z, U\rangle + \langle Y, F(X)\rangle + \int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma + \int_0^1 \langle \sigma\Psi(Y, \sigma Z)Z, Z\rangle\, d\sigma.$$

Clearly, $V_1(0, 0, 0, 0, 0) = 0$ and in view of the assumption (iii), we have that

$$\begin{aligned} V_1(0, 0, \varepsilon, 0, \varepsilon) &= \langle \varepsilon, \varepsilon\rangle + \int_0^1 \langle \sigma\Psi(0, \sigma\varepsilon)\varepsilon, \varepsilon\rangle\, d\sigma \\ &\geq \|\varepsilon\|^2 + \int_0^1 \langle \sigma\Psi(0, \sigma\varepsilon)\varepsilon, \varepsilon\rangle\, d\sigma > 0, \end{aligned}$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \Re^n$. The rest of the proof is similar to that of Theorem 3.2, except for some minor modifications, hence it is omitted. □

Remark 3.3. It should be noted that, for the case $n = 1$, the result of Ezeilo [4; Theorem 3] is a special case of our first result. Next, the results constituted here give additional result to those of established by Sadek [17; Theorem 3] and Tunç&Şevli [30].

Example: As a special case of the system (1.2), if we take for $n = 5$,

$$\Psi(Z) = \begin{bmatrix} 1+z_1^2 & 0 & 0 & 0 & 0 \\ 0 & 1+z_2^2 & 0 & 0 & 0 \\ 0 & 0 & 1+z_3^2 & 0 & 0 \\ 0 & 0 & 0 & 1+z_4^2 & 0 \\ 0 & 0 & 0 & 0 & 1+z_5^2 \end{bmatrix}, \ \Phi(Z) = \begin{bmatrix} z_1^3 + z_1^5 \\ z_2^3 + z_2^5 \\ z_3^3 + z_3^5 \\ z_4^3 + z_4^5 \\ z_5^3 + z_5^5 \end{bmatrix},$$

$$\Theta(Y) = \begin{bmatrix} y_1 + y_1^3 \\ y_2 + y_2^3 \\ y_3 + y_3^3 \\ y_4 + y_4^3 \\ y_5 + y_5^3 \end{bmatrix} \text{ and } F(X) = \begin{bmatrix} -x_1 - x_1^3 \\ -x_2 - x_2^3 \\ -x_3 - x_3^3 \\ -x_4 - x_4^3 \\ -x_5 - x_5^3 \end{bmatrix}$$

then, respectively, we have

$$\lambda_1(\Psi(Z)) = 1 + z_1^2, \quad \lambda_2(\Psi(Z)) = 1 + z_2^2, \quad \lambda_3(\Psi(Z)) = 1 + z_3^2,$$

$$\lambda_4(\Psi(Z)) = 1 + z_4^2, \quad \lambda_5(\Psi(Z)) = 1 + z_5^2,$$

$$J_\Theta(Y) = \begin{bmatrix} 1+3y_1^2 & 0 & 0 & 0 & 0 \\ 0 & 1+3y_2^2 & 0 & 0 & 0 \\ 0 & 0 & 1+3y_3^2 & 0 & 0 \\ 0 & 0 & 0 & 1+3y_4^2 & 0 \\ 0 & 0 & 0 & 0 & 1+3y_5^2 \end{bmatrix},$$

$$\lambda_1(J_\Theta(Y)) = 1+3y_1^2 \quad \lambda_2(J_\Theta(Y)) = 1+3y_2^2 \quad \lambda_3(J_\Theta(Y)) = 1+3y_3^2,$$

$$\lambda_4(J_\Theta(Y)) = 1+3y_4^2 \quad \lambda_5(J_\Theta(Y)) = 1+3y_5^2,$$

$$J_F(X) = \begin{bmatrix} -1-3x_1^2 & 0 & 0 & 0 & 0 \\ 0 & -1-3x_2^2 & 0 & 0 & 0 \\ 0 & 0 & -1-3x_3^2 & 0 & 0 \\ 0 & 0 & 0 & -1-3x_4^2 & 0 \\ 0 & 0 & 0 & 0 & -1-3x_5^2 \end{bmatrix}$$

and

$$\lambda_1(J_F) = -1-3x_1^2 \quad \lambda_2(J_F) = -1-3x_2^2 \quad \lambda_3(J_F) = -1-3x_3^2,$$

$$\lambda_4(J_F) = -1-3x_4^2 \quad \lambda_5(J_F) = -1-3x_5^2\,.$$

Hence,

$\lambda_i(\Psi(Z)) \geq 1$ for all z_1, z_2, z_3, z_4 and z_5,

$\lambda_i(J_\Theta(Y)) \geq 1$ for all y_1, y_2, y_3, y_4 and y_5,

$\lambda_i(J_F(X)) \leq -1$ for all x_1, x_2, x_3, x_4, x_5, $(i = 1, 2, 3, 4, 5)$,

and

$$\sum_{i=1}^{3} z_i \Phi_i(Z) = z_1^4 + z_1^6 + z_2^4 + z_2^6 + z_3^4 + z_3^6 + z_4^4 + z_4^6 + z_5^4 + z_5^6 \geq 0$$

for all z_1, z_2, z_3, z_4 and z_5. So the assumptions of Theorem 3.1 are satisfied.

References

[1] J. O. C. Ezeilo, An instability theorem for a certain fourth order differential equation, *Bull. London Math. Soc.* **10** (1978), 184–185.

[2] J. O. C. Ezeilo, Instability theorems for certain fifth-order differential equations, *Math. Proc. Cambridge Philos. Soc.* **84** (1978), 343–350.

[3] J. O. C. Ezeilo, A further instability theorem for a certain fifth-order differential equation, *Math. Proc. Cambridge Philos. Soc.* **86** (1979), 491–493.

[4] J. O. C. Ezeilo, Extension of certain instability theorems for some fourth and fifth order differential equations, *Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.* **66** (1979), 239–242.

[5] J. O. C. Ezeilo, An instability theorem for a certain sixth order differential equation, *J. Austral. Math. Soc. Ser. A* **32** (1982), 129–133.

[6] A. Iggidr and G. Sallet, On the stability of non-autonomous systems, *Automatica J. IFAC* **39** (2003), 167–171.

[7] N. Krasovskii, On conditions of inversion of A. M. Lyapunov's theorems on instability for stationary systems of differential equations (Russian), *Dokl. Akad. Nauk. SSSR (N.S.)* **101** (1955), 17–20.

[8] Z. H. Liao, and D. Lu, Instability of solution for the third order linear differential equation with varied coefficient, *Appl. Math. Mech. (English Ed.)*, no. 10, 9 (1988), 969–984; translated from Appl. Math. Mech., no. 10, 9 (1988), 909–923 (Chinese).

[9] W. J. Li and Y. H. Yu, Instability theorems for some fourth-order and fifth-order differential equations, *(Chinese) J. Xinjiang Univ. Natur. Sci.* **7** (1990), 7–10.

[10] W. J. Li and K. C. Duan, Instability theorems for some nonlinear differential systems of fifth order, *J. Xinjiang Univ. Natur. Sci.* **17** (2000), 1–5.

[11] D. Lu, Instability of solution for a class of the third order nonlinear differential equation, *Appl. Math. Mech. (English Ed.)*, no. 12, 16 (1995), 1185–1200; translated from Appl. Math. Mech., no. 12, 16 (1995), 1101–1114 (Chinese).

[12] D. Lu and Z. H. Liao, Instability of solution for the fourth order linear differential equation with varied coefficient, *Appl. Math. Mech. (English Ed.)*, no. 5, 14 (1993), 481–497; translated from Appl. Math. Mech., no. 5, 14 (1993), 455–469 (Chinese).

[13] A. M. Lyapunov, *Stability of Motion*, Academic Press, London, 1966.

[14] L. Mirsky, *An introduction to Linear Algebra*, Dover Publications, Inc., New York, 1990.

[15] C. Qian, Asymptotic behavior of a third-order nonlinear differential equation, *J. Math. Anal. Appl.* **284** (2003), 191–205.

[16] A. I. Sadek, An instability theorem for a certain seventh-order differential equation, *Ann. Differential Equations* **19** (2003), 1–5.

[17] A. I. Sadek, Instability results for certain systems of fourth and fifth order differential equations, *Appl. Math. Comput.* **145** (2003), 541–549.

[18] W. A. Skrapek, Instability results for fourth-order differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **85** (1980), 247–250.

[19] W. A. Skrapek, Some instability theorems for third order ordinary differential equations, *Math. Nachr.* **96** (1980), 113–117.

[20] W. J. Sun and X. Hou, New results about instability of some fourth and fifth order nonlinear systems, *(Chinese) J. Xinjiang Univ. Natur. Sci.* **16** (1999), 14–17.

[21] H. O. Tejumola, Instability and periodic solutions of certain nonlinear differential equations of orders six and seven, Ordinary Differential Equations (Abuja, 2000), 56–65, Proc. Natl. Math. Cent. Abuja Niger., 1.1, Natl. Math. Cent., Abuja, 2000.

[22] C. Tunç, An instability theorem for a certain vector differential equation of the fourth order, *Journal of Inequalities in Pure and Applied Mathematics* **5** (2004), 1–5.

[23] C. Tunç, On the instability of solutions of certain nonlinear vector differential equations of fifth order, *Panamer. Math. J.* **14** (2004), 25–30

[24] C. Tunç, An instability result for certain system of sixth order differential equations, *Appl. Math. Comput.* **157** (2004), 477–481.

[25] C. Tunç, On the instability of certain sixth-order nonlinear differential equations, *Electron. J. Diff. Eqns.* **2004** (2004), 1–6.

[26] C. Tunç, An instability result for a certain non-autonomous vector differential equation of fifth-order, *Panamer. Math. J.* **15** (2005), 51–58.

[27] C. Tunç, Instability of solutions of a certain non-autonomous vector differential equation of eighth-order, *Annals of Differential Equations* **22** (2006), 7–12.

[28] C. Tunç, A further instability result for a certain vector differential equation of fourth order, *International Journal of Mathematics, Game Theory, and Algebra* **15** (2006), 489–495.

[29] C. Tunç, New results about instability of nonlinear ordinary vector differential equations of sixth and seventh orders, Dynamics of Continuous, Discrete and Impulsive Systems; DCDIS Series A: Mathematical Analysis, Volume 14, Number1, (2007) 123–136.

[30] C. Tunç and F. Erdogan, On the instability of solutions of certain non-autonomous vector differential equations of fifth order, *SUT Journal of Mathematics* **43** (2007), 1–14.

[31] C. Tunç and H. Sevli, On the instability of solutions of certain fifth order nonlinear differential equations, *Memoirs on Differential Equations and Mathematical Physics* **35** (2005), 147–156.

[32] C. Tunç and E. Tunç, A result on the instability of solutions of certain non-autonomous vector differential equations of fourth order, *East-West Journal of Mathematics* **6** (2004), 127–134.

[33] C. Tunç and E. Tunç, Instability of solutions of certain nonlinear vector differential equations of order seven, Iranian Journal of Science and Technology, *Transaction A. Science* **29** (2005), 515–521.

[34] C. Tunç and E. Tunç, An instability theorem for a certain eighth order differential equation, *Differential Equations (Differ. Uravn.)* **42** (2006), 150–154.

[35] C. Tunç and E. Tunç, Instability results for certain third order nonlinear vector differential equations, *Electron. J. Diff. Eqns.* **2006** (2006), 1–10.

[36] E. Tunç, Instability of solutions of certain nonlinear vector differential equations of third order, *Electron. J. Differential Equations* **51** (2005), 1–6 (electronic).