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Measure of Noncompactness in Banach Algebra and Application to the Solvability of Integral Equations in BC($\mathbb{R}_+$)

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Abstract: In this paper an attempt is made to prove a fixed point theorem for the product of two operators each of which satisfies a special conditions in Banach algebra, using the technique of measure of noncompactness. Also we show that how it can be used to investigate the solvability of integral equations.

Keywords: Measure of noncompactness, Fixed point.

1 Introduction

It is has been witnessed that the differential and integral equations that appear in many physical problems are generally nonlinear and fixed point theory presents a strong tool for obtaining the solutions of such equations which otherwise are hard to solve by other ordinary procedures (for example, see [1,2,3]). In this paper, we analyze solvability of a certain functional-integral equation which consist of many special cases of integral and functional-integral equations, which are applicable in various real world problems of engineering, economics, physics and similar fields (see [4,5]). Indeed, we are going to investigate the solvability of the integral equation

$$x(t) = \big((Tx)(t)\big)\big(f(t,x(t)) + \int_0^t g(t,s,x(s))ds\big), \qquad (1)$$

where $t \in \mathbb{R}_+$ and T is an operator acting from the Banach algebra $BC(\mathbb{R}_+)$ consisting of all functions $x : \mathbb{R}_+ \to \mathbb{R}$ which are continuous and bounded on $\mathbb{R}_+$ into itself and the functions f, g are continuous and satisfy certain conditions. Eq.1 includes many known integral equations as model cases. In the case $Tx \equiv 1$ the equation 1 turns into

$$x(t) = f(t,x(t)) + \int_0^t g(t,s,x(s))ds,$$

which has been investigated in [6]. What we are going to achieve in this paper, will extend the findings already obtained in [8,9,10,12,14]. The main tool used in our investigation is the technique associated with measure of noncompactness. For a discussion of existence of solution to the above-mentioned integral equation, the used measure of noncompactness must also satisfy an additional condition. Indeed, we will use a class of measures of noncompactness which satisfies a condition called (m). Such a condition will guarantee the solvability of operator equations in Banach algebra. It is worthwhile mentioning that the important measures of noncompactness in notable spaces satisfy condition (m) (see [8,9,10,12,18]). This condition had first been used in for Hausdorff measure of noncompactness in Banach algebra $C(I)$ consisting of real continuous functions defined on a closed and bounded interval I (see [15]).

2 Preliminaries

For this reason, suppose that E is a given Banach space which has the norm $\|.\|$ and zero element θ. If the closed ball in E is centered at x and has radius r, we show it by $B(x,r)$. In order to show $B(\theta,r)$, we write B_r. If X is a subset of E, in that case, we can show the closure and the closed convex hull of X with the symbols $\overline{X}$ and $ConvX$ respectively. Also $X+Y$ and λX $(\lambda \in \mathbb{R})$ are used to show the algebraic operation on sets. Moreover, by the

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symbol $\|X\|$ we will denote the norm of a bounded set X, i.e, $\|X\| = \sup\{\|x\| : \quad x \in X\}$.

Furthermore $\mathfrak{M}_E$ is used to denote the family of all nonempty bounded subsets of E and $\mathfrak{N}_E$ denote its subfamily includes all relatively compact sets.

Definition 1([11]). *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be measure of noncompactness in E if it is satisfies the following conditions*

(1)*The family* $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker\mu \subset \mathfrak{N}_E$.
(2)$X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3)$\mu(\overline{X}) = \mu(X)$.
(4)$\mu(ConvX) = \mu(X)$.
(5)$\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.
(6)*If* (X_n) *is a nested sequence of closed sets from* $\mathfrak{M}_E$ *such that* $\lim_{n\to\infty}\mu(X_n) = 0$, *then the intersection set* $X_\infty = \cap_{n=1}^{\infty} X_n$ *is nonempty.*

Observe that the intersection set X_∞ from axiom (6) is a member of the $\ker\mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we have that $\mu(X_\infty) = 0$. This yields that $X_\infty \in ker\mu$ (see [13]). Now suppose Banach space E has the structure of Banach algebra. For given subsets X and Y of a Banach algebra E, let

$$XY = \{xy : x \in X, y \in Y\}.$$

Definition 2. *We state that measure of noncompactness μ which has been defined in Banach algebra E satisfies the condition (m), if for arbitrary sets $X, Y \in \mathfrak{M}_E$, the following inequality is satisfied*

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X).$$

Now we present an example of a measure of noncompactness in Banach algebra which satisfies condition (m). Let us consider the Banach space $BC(\mathbb{R}_+)$ consisting of all functions $x : \mathbb{R}_+ \to \mathbb{R}$ which are continuous and bounded on $\mathbb{R}_+$. This space is endowed with the standard norm $\|x\| = \sup\{\|x(t)\| : \quad t \in \mathbb{R}_+\}$. Obviously $BC(\mathbb{R}_+)$ has also the structure of Banach algebra with the standard multiplication of functions. In addition, fix a set $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ and numbers $\varepsilon > 0$ and $L > 0$. For an arbitrary function $x \in X$ let us denote by $\omega^L(x,\varepsilon)$ the modulus of continuity of x on the interval $[0,L]$, i.e.

$$\omega^L(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0,L], [t-s] \leq \varepsilon\}.$$

In addition

$$\omega^L(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\},$$
$$\omega_0^L(X) = \lim_{\varepsilon\to 0}\omega(X,\varepsilon),$$
$$\omega_0^\infty(X) = \lim_{L\to\infty}\omega_0^L(X,\varepsilon).$$

Moreover, if $t \in \mathbb{R}_+$ is a fixed number, let us denote

$$X(t) = \{x(t) : x \in X\},$$
$$diamX(t) = \sup\{|x(t) - y(t)| : x, y \in X\},$$
$$c(X) = \limsup_{t\to\infty} diamX(t).$$

With help of the above mappings we denote the following measures of noncompactness in $BC(\mathbb{R}_+)$ (cf. [16,17])

$$\mu_c(X) = \omega_0^\infty(X) + c(X). \tag{2}$$

Theorem 1. *The measure of noncompactness μ_c defined by 2 satisfies condition (m) on the family of all nonempty and bounded subsets X of Banach algebra $BC(\mathbb{R}_+)$ such that functions belonging to X are nonnegative on $\mathbb{R}_+$.*

Proof. If $x, y \in C[a,b]$, $\varepsilon > 0$ then for $t, s \in [a,b]$ such that $|t-s| \leq \varepsilon$, we get

$$\begin{aligned} |x(t)y(t) - x(s)y(s)| &\leq |x(t)y(t) - x(t)y(s)| \\ &+ |x(t)y(s) - x(s)y(s)| \\ &\leq |x(t)||y(t) - y(s)| + |y(s)||x(t) - x(s)| \\ &\leq \|x\|\omega(y,\varepsilon) + \|y\|\omega(x,\varepsilon). \end{aligned}$$

As a result

$$\omega(xy,\varepsilon) \leq \|x\|\omega(y,\varepsilon) + \|y\|\omega(x,\varepsilon).$$

So $\omega_0^\infty(X)$ satisfies condition (m). Now, fix arbitrarily sets $X; Y \in \mathfrak{M}_{BC(\mathbb{R}_+)}$. Choose arbitrary functions $z_1, z_2 \in XY$. This means that there exist functions $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $z_1 = x_1y_1, z_2 = x_2y_2$. Next, for $t \in \mathbb{R}_+$ we get

$$\begin{aligned} |z_1(t) - z_2(t)| &= |x_1(t)y_1(t) - x_2(t)y_2(t)| \\ &\leq |x_1(t)y_1(t) - x_1(t)y_2(t)| \\ &+ |x_1(t)y_2(t) - x_2(t)y_2(t)| \\ &= |x_1(t)||y_1(t) - y_2(t)| + |y_2(t)||x_1(t) - x_2(t)| \\ &\leq \|X\|diamY(t) + \|Y\|diamX(t). \end{aligned}$$

Hence we obtain $diam(X(t)Y(t)) \leq \|X\|diamY(t) + \|Y\|diamX(t)$ and consequently $c(XY) \leq \|X\|c(Y) + \|Y\|c(X)$. So, that the measure of noncompactness μ_c satisfies condition (m).

In order to achieve the main purpose of this paper, the following theorem plays a crucial role

Theorem 2([7]). *Let Ω be a bounded, nonempty, convex and closed subset of a Banach space E. Then each continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in the set Ω.*

Obviously the above formulated theorem constitutes the well know Schauder fixed point principle.

3 Main result

Now it is time to put forward the main theorem of this paper.

Theorem 3. *Assume that Ω is a nonempty, bounded, closed and convex subset of the Banach algebra E and the operators P and T continuously transform the set Ω into E such that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, we assume that the operator $S = P.T$ transform Ω into itself.*

If the operator P and T on the set Ω satisfy the following conditions

$$\begin{cases} \mu(P(X)) \leq \psi_1(\mu(X)), \\ \mu(T(X)) \leq \psi_2(\mu(X)), \end{cases}$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness satisfying condition (m) and $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing functions such that

$$\begin{cases} \lim_{n\to\infty} \psi_1^n(t) = 0, \\ \lim_{n\to\infty} \psi_2^n(t) = 0, \\ \lim_{n\to\infty} \big(\|P(\Omega)\|\psi_2 + \|T(\Omega)\|\psi_1\big)^n(t) = 0, \end{cases}$$

for any $t \geq 0$, then S has at least fixed point in the set Ω.

Proof. Let us take an arbitrary nonempty subset X of the set Ω. Then in view of the assumption that μ satisfies condition (m) we obtain

$$\begin{aligned} \mu(S(X)) &\leq \mu(P(X).T(X)) \\ &\leq \|P(X)\|\mu(T(X)) + \|T(X)\|\mu(P(X)) \\ &\leq \|P(\Omega)\|\mu(T(X)) + \|T(\Omega)\|\mu(P(X)) \\ &\leq \|P(\Omega)\|\psi_2(\mu(X)) + \|T(\Omega)\|\psi_1(\mu(X)) \\ &= \big(\|P(\Omega)\|\psi_2 + \|T(\Omega)\|\psi_1\big)(\mu(X)). \end{aligned} \tag{3}$$

Now, letting $\varphi(t) = \big(\|P(\Omega)\|\psi_2 + \|T(\Omega)\|\psi_1\big)(t)$, then from **??** , we have $\mu(S(X)) \leq \varphi(\mu(X))$. Now, with regard to the fact that ψ_1 and ψ_2 are nondecreasing, we conclude φ is nondecreasing and in view of $\lim_{n\to\infty} \varphi^n(t) = 0$, we can apply main result in [6] , to get the desired result. But for the convenience of the reader, we add the scheme of proof of aforementioned theorem. We define sequence Ω_n as $\Omega_0 = \Omega$, $\Omega_n = ConvS\Omega_{n-1}$ for $n \geq 1$. Furthermore we assume $\mu(\Omega_n) > 0$ for all $n = 1, 2, \ldots$. Keeping this condition in mind, we get

$$\begin{aligned} \mu(\Omega_{n-1}) &= \mu(ConvS\Omega_n) \\ &= \mu(S\Omega_n) \\ &\leq \varphi(\mu(\Omega_n)) \\ &\leq \varphi^2(\mu(\Omega_{n-1})) \\ &\leq \ldots \\ &\leq \varphi^n(\mu(\Omega)). \end{aligned}$$

This showed that $\mu(\Omega_n) \to 0$ as $n \to \infty$. Now, we can use axiom 6 of definition of measure of noncompactness and conclude that $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, convex and closed subset of the set Ω. Moreover it is noteworthy that Ω_∞ is compact. With regard to the above discussion Schauder fixed point principle guarantees the existence of a fixed point for the operator S .

Remark. By letting

$$\begin{cases} \psi_1(t) = k_1, & 0 \leq k_1 < 1 \\ \psi_2(t) = k_2, & 0 \leq k_2 < 1 \end{cases}$$

in Theorem 3, we obtain a special case of above theorem which has already been studied in([9,10,12,18]), where the application of that special case in the existence of solutions of many integral equation has been investigated .

4 Application

In this section we use the main theorem of this paper to prove the solvability of integral equation

$$x(t) = (Tx)(t)(f(t,x(t)) + \int_0^t g(t,s,x(s))ds), \qquad t \in \mathbb{R}_+,$$

we define

$$(Fx)(t) = f(t,x(t)) + \int_0^t g(t,s,x(s))ds, \qquad t \in \mathbb{R}_+$$

where the operator T, F are defined on the Banach algebra $BC(\mathbb{R}_+)$. Notice that F represented the so-called Volterra integral operator. Now, we formulate the assumptions under which the equation 1 will be investigated. We will assume the following hypotheses:

(I) T is an operator acting continuously from Banach algebra $BC(\mathbb{R}_+)$ into itself which satisfies the following condition

$$\mu_c(T(X)) \leq \psi_1(\mu_c(X))$$

for any nonempty subset X of Ω in which Ω is a nonempty, bounded, closed and convex subset of the Banach algebra $BC(\mathbb{R}_+)$ and $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \psi_1^n(t) = 0$ for any $t \geq 0$.

(II) There exists a constant b such that

$$\|Tx\| \leq \psi_1(\|x\|) + b$$

(III) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Moreover, $t \to f(t,0)$ is a member of the space $BC(\mathbb{R}_+)$.

(IV) There exists an upper semicontinuous function $\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing function such that $\lim_{n\to\infty} \psi_2^n(t) = 0$ for any $t \geq 0$, we have that

$$|f(t,x) - f(t,y)| \leq \psi_2(|x-y|), \quad t \in \mathbb{R}_+, \quad x,y \in \mathbb{R}.$$

Moreover, we assume that ψ_2 is superadditive i.e., for each $t,s, \in \mathbb{R}_+$, $\psi_2(t) + \psi_2(s) \leqslant \psi_2(t+s)$.

(V) $g:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is a continuous function and there exist continuous functions $c,d:\mathbb{R}_+\to\mathbb{R}_+$ such that $\lim_{t\to\infty}c(t)\int_0^t d(s)ds=0$ and $|g(t,s,x)|\le c(t)d(s)$ for $t,s\in\mathbb{R}_+$ such that $s\le t$, and for each $x\in\mathbb{R}$.

(VI) The inequality $(\psi_1(r)+b)(\psi_2(r)+q)\le r$ has a positive solution r_0 in which q is constant and defined as

$$q=\sup\left\{|f(t,0)|+c(t)\int_0^t d(s)ds\}:t\ge 0\right\}.$$

Moreover, the number r_0 is such that $\big((\psi_2(r_0)+q)\psi_1+(\psi_1(r_0)+b)\psi_2\big)(t)<t$ for $t\in\mathbb{R}_+$.

The following lemma is necessary to prove the theorem 4.

Lemma 1([6]). *Let $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then the following two conditions are equivalent*

(1) $\lim_{n\to\infty}\varphi^n(t)=0$ *for each* $t\ge 0$.
(2) $\varphi(t)<t$ *for any* $t>0$.

Theorem 4. *Under the assumptions (I) to (VI), the integral equation 1 has at least one solution in the space $BC(\mathbb{R}_+)$.*

Proof. We define the operator A as follows

$$(Ax)(t)=(Tx)(t)(Fx)(t).$$

With regard to the above assumptions, the functions Tx and Fx are continuous functions on $\mathbb{R}_+$ for any $x\in BC(\mathbb{R}_+)$. For an arbitrary fixed function $x\in BC(\mathbb{R}_+)$, we have

$$\begin{aligned}|(Ax)(t)|&=|(Tx)(t)||(Fx)(t)|\\&\le(\psi_1(\|x\|)+b)(|f(t,x(t))-f(t,0)|\\&\quad+|f(t,0)|+|g(t,s,x(s))|ds)\\&\le(\psi_1(\|x\|)+b)(\psi_2(|x(t)|)\\&\quad+|f(t,0)|+|g(t,s,x(s))|ds)\\&\le(\psi_1(\|x\|)+b)(\psi_2(|x(t)|)\\&\quad+|f(t,0)|+c(t)\int_0^t d(s)ds)\\&\le(\psi_1(\|x\|)+b)(\psi_2(|x(t)|)+q).\end{aligned}$$

So, we get

$$\|Ax\|\le(\psi_1(\|x\|)+b)(\psi_2(\|x(t)\|)+q),$$

in which b and q are constant, defined in assumptions(II), (IV). So A maps the space $BC(\mathbb{R}_+)$ into itself. Moreover based of assumption (IV), we conclude that A maps the ball B_{r_0} into itself in which r_0 is a constant appearing in assumption (VI). Now we show that operator A is continuous on the ball B_{r_0}. To do this, let us first observe that the continuity of the operator T on the ball B_{r_0} is an easy consequence of the assumptions (I), (II), (VI). Thus, it suffices to show that the operator F is continuous on B_{r_0}. Fix an arbitrary $\varepsilon>0$ and $x,y\in B_{r_0}$ such that $\|x-y\|\le\varepsilon$. So we can conclude

$$\begin{aligned}|(Fx)(t)-(Fy)(t)|&\le\psi_2(|x(t)-y(t)|)\\&\quad+\int_0^t|g(t,s,x(s))-g(t,s,y(s))|ds\\&\le\psi_2(|x(t)-y(t)|)\\&\quad+\int_0^t|g(t,s,x(s))|ds\\&\quad+\int_0^t|g(t,s,y(s))|ds\\&\le\psi_2(\varepsilon)+2k(t),\end{aligned}\tag{4}$$

where we denoted

$$k(t)=c(t)\int_0^t d(s)ds.$$

Further, in view of assumption (V), we deduce that there exists a number $L>0$ such that

$$2k(t)=2c(t)\int_0^t d(s)ds\le\varepsilon,\tag{5}$$

for each $t\ge L$. Thus, taking into account Lemma 1 and linking 5 and 4, for an arbitrary $t\ge L$ we get

$$|(Fx)(t)-(Fy)(t)|\le 2\varepsilon.\tag{6}$$

Now, we define the quantity $\omega^L(g,\varepsilon)$ as follows

$$\begin{aligned}\omega^L(g,\varepsilon)=\sup\{&|g(t,s,x)-g(t,s,y)|:\\&t,s\in[0,L],x,y\in[-r_0,r_0],\|x-y\|\le\varepsilon\}.\end{aligned}$$

Now with regard to the fact that the function $g(t,s,x)$ is uniformly continuous on the set $[0,L]\times[0,L]\times[-r_0,r_0]$, so

$$\lim_{\varepsilon\to 0}\omega^L(g,\varepsilon)=0.$$

By considering 4 for an arbitrary fixed $t\in[0,L]$, we conclude that

$$\begin{aligned}|(Fx)(t)-(Fy)(t)|&\le\psi_2(\varepsilon)+\int_0^L\omega^L(g,\varepsilon)ds\\&=\psi_2(\varepsilon)+L\omega^L(g,\varepsilon).\end{aligned}\tag{7}$$

Combining 6 and 7, it is possible to conclude that the operator F is continuous on the ball B_{r_0}. Now, let X be an arbitrary nonempty subset of the ball B_{r_0}. Fix numbers $\varepsilon>0$ and $L>0$. Next, choose $t,s\in[0,L]$ such that $\|t-s\|\le\varepsilon$. Without loss of generality, we assume that

$s < t$. Then, for $x \in X$ we conclude

$$\begin{aligned}
&|Fx)(t) - (Fx)(s)| \le |f(t,x(t)) - f(s,x(s))| \\
&+ |\int_0^t g(t,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,x(\tau))d\tau| \\
&\le |f(t,x(t)) - f(s,x(t))| + |f(s,x(t)) - f(s,x(s))| \\
&+ |\int_0^t g(t,\tau,x(\tau))d\tau - \int_0^t g(s,\tau,x(\tau))d\tau| \\
&+ \int_0^t g(s,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,x(\tau))d\tau| \\
&\le \omega_1^L(f,\varepsilon) + \psi_2(|x(t) - x(s)|) \\
&+ \int_0^t |g(t,\tau,x(\tau)) - g(s,\tau,x(\tau))|d\tau \\
&+ \int_s^t |g(s,\tau,x(\tau))|d\tau \\
&\le \omega_1^L(f,\varepsilon) + \psi_2\big(\omega^L(x,\varepsilon)\big) \\
&+ \int_0^t \omega_1^L(g,\varepsilon)d\tau + c(s)\int_s^t d(\tau)d\tau \\
&\le \omega_1^L(f,\varepsilon) + \psi_2\big(\omega^L(x,\varepsilon)\big) \\
&+ L\omega_1^L(g,\varepsilon) + \varepsilon \sup\{c(s)d(t) : t,s \in [0,L]\} \qquad (8)
\end{aligned}$$

where we denote

$$\begin{aligned}
\omega_1^L(f,\varepsilon) = \sup\{&|f(t,x) - f(s,x)| : \\
&t,s \in [0,L], x \in [-r_0,r_0], |t-s| < \varepsilon\}, \\
\omega_1^L(g,\varepsilon) = \sup\{&|g(t,t,x) - g(s,t,x)| : \\
&t,s,t \in [0,L], x \in [-r_0,r_0], |t-s| < \varepsilon\}.
\end{aligned}$$

Now with regard to the fact that f is uniformly continuous on the set $[0,L] \times [-r_0,r_0]$ and g is uniformly continuous on the set $[0,L] \times [0,L] \times [-r_0,r_0]$, we can conclude $\omega_1^L(f,\varepsilon) \to 0$ and $\omega_1^L(g,\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, since $c = c(t)$ and $d = d(t)$ are continuous on $\mathbb{R}_+$, the quantity $\sup\{c(s)d(t) : t,s \in [0,L]\}$ is finite. From 8, we conclude

$$\omega_0^L(FX) \le \lim_{\varepsilon \to 0} \psi_2\big(\omega^L(X,\varepsilon)\big).$$

Now with regard to the fact that ψ_2 is upper semicontinuous, so

$$\omega_0^L(FX) \le \psi_2\big(\omega_0^L(X)\big),$$

and so

$$\omega_0^\infty(FX) \le \psi_2\big(\omega_0^\infty(X)\big). \qquad (9)$$

Now we choose two arbitrary functions $x,y \in X$. Then for $t \in \mathbb{R}$ we have

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\le |f(t,x(t)) - f(t,y(t))| \\
&+ \int_0^t |g(t,s,x(s))|ds + \int_0^t |g(t,s,y(s))|ds \\
&\le \psi_2\big(|x(t) - y(t)|\big) \\
&+ 2c(t)\int_0^t d(s)ds \\
&\le \psi_2\big(|x(t) - y(t)|\big) + 2k(t).
\end{aligned}$$

This estimate allows us to get the following one

$$diam(FX)(t) \le \psi_2\big(diamX(t)\big) + 2k(t).$$

Now with regard to the upper semicontinuity of the functions ψ_2 we obtain

$$c(FX) = \limsup_{t\to\infty} diam(FX)(t) \le \psi_2\big(\limsup_{t\to\infty} diamX(t)\big) = \psi_2(c(X)). \qquad (10)$$

So, combining 9 and 10, we can conclude

$$\begin{aligned}
\mu_c(FX) &= \omega_0^\infty(FX) + c(FX) \\
&= \omega_0^\infty(FX) + \limsup_{t\to\infty} diam(FX)(t) \\
&\le \psi_2\big(\omega_0^\infty(X(t))\big) + \psi_2\big(\limsup_{t\to\infty} diam(X)(t)\big) \\
&\le \psi_2\big(\omega_0^\infty(X(t)) + \limsup_{t\to\infty} diam(X)(t)\big) \\
&\le \psi_2\big(\omega_0^\infty(X(t)) + c(X)\big)
\end{aligned}$$

or, equivalently

$$\mu_c(FX) \le \psi_2\big(\mu_c(X)\big),$$

moreover, by considering assumption (I) we have

$$\mu_c(TX) \le \psi_1\big(\mu_c(X)\big),$$

in which μ_c is the defined measure of noncompactness on the space $BC(\mathbb{R}_+)$. Also, we get

$$\|TB_{r_0}\| \le \psi_1(r_0) + b \quad , \quad \|FB_{r_0}\| \le \psi_2(r_0) + q.$$

So, according to assumption (VI), we have

$$\begin{aligned}
\big(\|FB_{r_0}\|\psi_1 + \|TB_{r_0}\|\psi_2\big)(t) &< \big((\psi_2(r_0)+q)\psi_1 + (\psi_1(r_0)+b)\psi_2\big)(t) \\
&< t \quad \text{for all} \quad t \in \mathbb{R}_+ \qquad (11)
\end{aligned}$$

Now, linking 11 and lemma 1 we get

$$\lim_{n\to\infty} \big(\|FB_{r_0}\|\psi_1 + \|TB_{r_0}\|\psi_2\big)^n(t) = 0.$$

Thus, all the conditions of Theorem 4 hold. Therefore Eq.1 has at least one solution in the space $BC(\mathbb{R}_+)$.

5 Example

*Example 1.*Consider the following functional integral equation

$$\begin{aligned}
x(t) = &\left(\frac{t^2}{1+t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x(s)}{1+|\cos x(s)|}ds\right) \\
&\times \left(\frac{t^2}{5+5t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x(s)}{3+|\cos x(s)|}ds\right), \qquad (12)
\end{aligned}$$

we define

$$\begin{aligned}
(Tx)(t) &= \frac{t^2}{1+t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x(s)}{1+|\cos x(s)|}ds, \\
(Fx)(t) &= \frac{t^2}{5+5t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x(s)}{3+|\cos x(s)|}ds.
\end{aligned}$$

Now, we show that all the conditions of Theorem 4 are satisfied for the functional integral equation 12. To do so, first we checked out whether condition (IV) and (V) are satisfied. Similar fashion, by putting $\psi_1(t) = \frac{1}{4}\ln(1+t)$ condition (I) and (II) satisfied for the operator T, too. Moreover, we put

$f(t,x) = \frac{t^2}{1+t^6}\ln(1+|x|), g(t,s,x) = \frac{se^{-t}sinx}{1+|cosx|}$ and $\psi_2(t) = \frac{1}{5}\ln(1+t)$. obviously, ψ_2 is nondecreasing and concave on $\mathbb{R}_+$ and $\psi_2(t) < t$ for all $t > 0$. In addition, for arbitrarily fixed $x, y \in \mathbb{R}_+$ such that $|x| \geq |y|$ and for $t > 0$ we get

$$\begin{aligned}
|f(t,x) - f(t,y)| &= \frac{1}{5}\frac{t^2}{2+2t^4}\ln\big(\frac{1+|x|}{1+|y|}\big) \\
&\leq \frac{1}{5}\ln\big(1+\frac{|x|-|y|}{1+|y|}\big) \\
&< \frac{1}{5}\ln\big(1+|x-y|\big) \\
&< \frac{1}{4}\ln\big(1+2|x-y|\big) \\
&= \psi_2\big(|x-y|\big).
\end{aligned}$$

The case $|y| \geq |x|$ can be dealt with in the same way. Conditions (III) of Theorem 4 are clearly evident. In addition, pay close attention that the function g is continuous and maps the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$. Also, we have

$$|g(t,s,x)| \leq e^{-t}s$$

for $t, s \in \mathbb{R}$ and $x \in \mathbb{R}$. So, if we put $c(t) = e^{-t}$, and $d(s) = s$, then we can see that assumption (V) is satisfied. Indeed, we have

$$\lim_{t\to\infty} c(t)\int_0^t d(s)ds = 0.$$

Now, let us calculate the constant q which appears in assumption (VI). We obtain

$$\begin{aligned}
q &= \sup\{|f(t,0)| + c(t)\int_0^t d(s)ds : t \geq 0\} \\
&= \sup\{2t^2e^{-t/2} : t \geq 0\} = 2e^{-2}.
\end{aligned}$$

Just like the above way checked out condition (II), we get $b = 2e^2$ (also see [6]). Furthermore, we can check that the inequality from assumption (VI) takes the form

$$\big(\frac{1}{4}\ln(1+r)+b\big)\big(\frac{1}{5}\ln(1+r)+q\big) < r.$$

It is obvious that this inequality has a positive solution r_0, say $r_0 = 1$. Moreover, we have $\big((\psi_2(r_0)+q)\psi_1 + (\psi_1(r_0)+b)\psi_2\big)(t) < t$ for $t \in \mathbb{R}_+$. Consequently, all the conditions of Theorem 4 are satisfied. Therefore the functional integral equation 12 has at least one solution in the space $BC(\mathbb{R}_+)$.

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