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On the Boundedness of the First and Second Derivatives of a Type of the Spectral Problem

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Abstract: The behavior of eigenvalues, the boundedness of eigenfunctions and the first and second derivatives of eigenfunctions of the spectral problem

$$-y''(x)+y'(x) = \lambda^2\rho(x)y(x), \quad x\in[0,a], \quad a>0,$$

with the boundary conditions:

$$y'(a)=y'(0)-y(0) = 0,$$

$$\int_0^a \overline{y}(x)y'(x)dx = \alpha, \alpha>0$$

have been studied, where λ is a spectral parameter.

Keywords: Boundedness, spectral problem, spectral parameters, eigenfunctions.

Symbol

The following auxiliary materials are used through the whole work, and each one is defined as follows:
λ is an eigenvalue and $\lambda=\theta+i\gamma$, where $i=\sqrt{-1}$ and $\theta,\gamma\in\mathbb{R}$ and $\mathbb{R}$ is denoted to the set of all real numbers. Two positive real numbers m and M are chosen so that $0<m\le M$. The symbol $\rho(x)$ is refered to the positive weight function such that $0<m\le\rho\le M$. $I^+[0,a]$ refers to the set of all positive integrable functions and a and α are positive real numbers.

1 Introduction

Boundary value problems for differential equations of the second order with different boundary conditions were studied in [1-4,6-10,12 and 13], and various applications of such problems can be found in[5,11,and 14].
The behavior of eigenvalues and eigenfunctions and the boundedness of eigenfunctions of the boundary problems of Sturm-Liouville type for the second order differential equation, with different types of boundary conditions and different classes of the coefficients, were obtained in [1-4]and [6-10].
The behavior of eigenvalues and eigenfunctions boundedness of eigenfunctions of the boundary problems of Sturm-Liouville with the spectral parameter in the boundary condition were obtained in[1-3]and [6-10].

In this paper, we study the behavior of eigenvalues, the boundedness of eigenfunctions and the boundedness of the first and second derivatives of eigenfunctions of the spectral problem of the form:

$$-y''(x)+y'(x)=\lambda^2\rho y(x), x\in[0,a] \quad (1)$$

$$y'(a)=y'(0)-y(0) \quad =0, \quad (2)$$

$$\int_0^a \overline{y}(x)y'(x)dx=\alpha,\alpha>0 \quad (3)$$

where α is a positive constant, and λ is a spectral parameter.

The paper is organized into four sections. The study of

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the boundedness of eigenfunctions and determination of the behavior of eigenvalues are determined in Section (2). In Section (3), the boundedness for norm of the first and second derivatives of eigenfunctions for the given problem are presented.

2 Study the assessment of eigenfunctions and the behavior of eigenvalues to the problem (1)-(3)

This section concern to the study of the boundedness of eigenfunctions and determination of the behavior of eigenvalues to the problem (1)-(3).

Theorem 2.1. Let $\lambda = \theta + i\gamma$ be an eigenvalue where $\theta \neq 0$ and $\rho(x) \in I^{+}[0,a]$, then the eigenfunctions of the problem (1)-(3) satisfies the inequality $\max_{x\in[0,a]} |y(x)| \leq k|\lambda|^{1/2}$, where $k > 0$ and k does not depends on $\rho(x)$.

Proof. Let x be any point in $[0,a]$ and let us consider the identity:

$$|y(x)|^2 = y(x)\overline{y}(x)$$
$$= \int_0^x (\overline{y}(t)y'(t) + y(t)\overline{y}'(t))dt + |y(0)|^2$$
$$= \int_0^x \frac{\sqrt{\rho(t)}(\overline{y}(t)y'(t) + y(t)\overline{y}'(t))}{\sqrt{\rho(t)}}dt + |y(0)|^2.$$

From inequality $\rho(t) \geq m$, we obtain

$$|y(x)|^2 \leq \frac{1}{\sqrt{m}} \int_0^x \sqrt{\rho(t)}|\overline{y}(t)y'(t) + y(t)\overline{y}'(t)|dt + |y(0)|^2$$
$$\leq \frac{1}{\sqrt{m}}\left(\int_0^x \sqrt{\rho(t)}|\overline{y}(t)y'(t)|dt + \int_0^x \sqrt{\rho(t)}|y(t)\overline{y}'(t)|dt\right)$$
$$+|y(0)|^2$$
$$\leq \frac{1}{\sqrt{m}}\left(\int_0^x \sqrt{\rho(t)}|\overline{y}(t)||y'(t)|dt\right.$$
$$\left.+\int_0^x \sqrt{\rho(t)}|y(t)||\overline{y}'(t)|dt\right) + |y(0)|^2$$
$$|y(x)|^2 \leq \frac{2}{\sqrt{m}} \int_0^a \sqrt{\rho(t)}|y(t)||y'(t)|dt + |y(0)|^2.$$

Using Cauchy-Schwartz inequality on the last inequality, we deduce that

$$|y(x)|^2 \leq \frac{2}{\sqrt{m}}\sqrt{\int_0^a \rho(t)|y(t)|^2dt}\sqrt{\int_0^a |y'(t)|^2dt}$$
$$+|y(0)|^2.$$

Now, since $\rho(t), |y(t)|^2 > 0$ then $\int_0^a \rho(t)|y(t)|^2dt > 0$,

So let $\int_0^a \rho(t)|y(t)|^2dt = k_1$, where $k_1 > 0$, therefore the last inequality becomes

$$|y(x)|^2 \leq \frac{2\sqrt{k_1}}{\sqrt{m}}\left(\int_0^a |y'(t)|^2dt\right)^{1/2} + |y(0)|^2. \qquad (4)$$

Multiplying equation (1) by $\overline{y}(x)$ and integrating the obtained equation from 0 to a, yields

$$-\int_0^a \overline{y}(x)y''(x)dx + \int_0^a \overline{y}(x)y'(x)dx = \lambda^2 \int_0^a \rho(x)|y(x)|^2dx.$$

Integrating the first integral by parts and by using the boundary conditions (2)-(3) we get

$$|y(0)|^2 + \int_0^a |y'(x)|^2dx + \alpha = \lambda^2 k_1, \qquad (5)$$

$k_1 = \int_0^a \rho(x)|y(x)|^2dx > 0$.

Now, we rewrite equations (1)-(3) as follows:

$$-\overline{y}''(x) + \overline{y}'(x) = \overline{\lambda}^2\rho(x)\overline{y}(x), \qquad (6)$$

$$\overline{y}'(a) = \overline{y}'(0) - \overline{y}(0) = 0, \qquad (7)$$

$$\int_0^a y(x)\overline{y}'(x)dx = \alpha. \qquad (8)$$

By multiplying equation (6) by $y(x)$ and integrating from 0 up to a, we obtain

$$-\int_0^a y(x)\overline{y}''(x)dx + \int_0^a y(x)\overline{y}'(x)dx = \overline{\lambda}^2 k_1.$$

Again, integrating the first integral in the last equation by parts and using the boundary conditions (7)-(8), we gain

$$|y(0)|^2 + \int_0^a |y'(x)|^2dx + \alpha = \overline{\lambda}^2 k_1. \qquad (9)$$

By multiplying equation(5) by $\overline{\lambda}$ and equation (9) by λ and add them we get

$$(\lambda + \overline{\lambda})|y(0)|^2 + (\lambda + \overline{\lambda})\int_0^a |y'(x)|^2dx$$
$$+(\lambda + \overline{\lambda})\alpha = (\lambda + \overline{\lambda})|\lambda|^2 k_1$$

And since $\theta \neq 0$, then $(\lambda + \overline{\lambda}) \neq 0$, therefore

$$|y(0)|^2 + \int_0^a |y'(x)|^2dx + \alpha = |\lambda|^2 k_1.$$

Let $|y(0)|^2 = k_2$, $k_2 \in \mathbb{R}^+$, and α, $|y(0)|^2$ are positive real numbers, so is $\alpha + |y(0)|^2$,
therefore we assume $\alpha + |y(0)|^2 = k_3$, where $k_3 \in \mathbb{R}^+$, thus the last equation reduces to

$$\int_0^a |y'(x)|^2 dx = |\lambda|^2 k_1 - k_3.$$

By putting this equation in equation (4), we deduce

$$|y(x)|^2 \leq \frac{2\sqrt{k_1}}{\sqrt{m}}(|\lambda|^2 k_1 - k_3)^{1/2} + k_2$$

$$|y(x)|^2 \leq \frac{2\sqrt{k_1}}{\sqrt{m}}(|\lambda|^2 k_1(1 - \frac{k_3}{|\lambda|^2 k_1}))^{1/2} + k_2$$

$$|y(x)|^2 \leq \frac{2\lambda k_1}{\sqrt{m}} + k_2,$$

or

$$|y(x)| \leq |\lambda|^{1/2}\sqrt{\frac{2k_1}{\sqrt{m}} + \frac{k_2}{|\lambda|}}.$$

And since x is any value in the interval $[0,a]$, so

$$\max_{x\in[0,a]} |y(x)| \leq |\lambda|^{1/2}\sqrt{\frac{2k_1}{\sqrt{m}} + \frac{k_2}{|\lambda|}},$$

if we put $k = \sqrt{\frac{2k_1}{\sqrt{m}} + \frac{k_2}{|\lambda|}} > 0$ which does not depend on $\rho(x)$, we have

$$\max_{x\in[0,a]} |y(x)| \leq k|\lambda|^{1/2}.$$

Hence, the proof of theorem 2.1 is completed.

Lemma 2.1. For the presence of eigenvalues of the problem (1)-(3), must:

1. If $\theta = 0$, then the inequality $(2cm - 2k_1)^2 > 16k_1^2\gamma^2 m + 8k_1cm + 4c^2m^2$ holds, where $c = (y(0))^2$ and $k_1 = \int_0^a \rho(x)|y(x)|^2 dx$.
2. If $\theta \neq 0$, then the inequality $m(c_1 - c)^2 \leq 4k_1^2(\theta^2 - \gamma^2) - 4k_1(\alpha + c)$ holds, where $c_1 = |y(a)|^2$.

This lemma is understood as follows; on the imaginary axis eigenvalues are possible only if the inequality $(2cm - 2k_1)^2 > 16k_1^2\gamma^2 m + 8k_1cm + 4c^2m^2$ holds and the remainder of the complex plane only where the inequality $m(c_1 - c)^2 \leq 4k_1^2(\theta^2 - \gamma^2) - 4k_1(\alpha + c)$ holds.

Proof. (1) If $\theta = 0$, then $\lambda = i\gamma$, and $\lambda^2 = -\gamma^2$, so the given problem reduces to:

$$-y''(x) + y'(x) = -\gamma^2\rho(x)y(x), x \in [0,a] \tag{10}$$

$$y'(a) = y'(0) - y(0) = 0, \tag{11}$$

$$\int_0^a \bar{y}(x)y'(x)dx = \alpha.\alpha > 0 \tag{12}$$

We multiply equation(10) by $y(x)$ and integrate the resulting from 0 to a, we obtain

$$-\int_0^a y''(x)y(x)dx + \int_0^a y'(x)y(x)dx = -\gamma^2\int_0^a \rho(x)y^2(x)dx.$$

Integrating the last equation with using the boundary conditions (11) gives

$$\frac{1}{2}((y(0))^2 + (y(a))^2) + \int_0^a (y'(x))^2 dx = -k_1\gamma^2,$$

where $k_1 = \int_0^a \rho(x)y^2(x)dx$. Or

$$\int_0^a (y'(x))^2 dx = -k_1\gamma^2 - \frac{1}{2}((y(0))^2 + (y(a))^2). \tag{13}$$

Since from theorem (2.1) we have shown that $|y(a)|^2 \leq \frac{2\sqrt{k_1}}{\sqrt{m}}(\int_0^a |y'(x)|^2 dx)^{\frac{1}{2}} + |y(0)|^2$, then

$$(\int_0^a |y'(x)|^2 dx)^{\frac{1}{2}} \geq \frac{\sqrt{m}}{2\sqrt{k_1}}(|y(a)|^2 - |y(0)|^2)$$

It can be written as

$$\int_0^a |y'(x)|^2 dx \geq \frac{m}{4k_1}(|y(a)|^2 - |y(0)|^2)^2.$$

From the last inequality and equation (13) it follows that

$$-k_1\gamma^2 - \frac{1}{2}((y(0))^2 + (y(a))^2) \geq \frac{m}{4k_1}(|y(a)|^2 - |y(0)|^2)^2$$

$$-\frac{4k_1^2\gamma^2}{m} - \frac{2k_1}{m}(y(0))^2 - \frac{2k_1}{m}(y(a))^2 \geq (y(a))^4 - 2(y(a))^2(y(0))^2 + (y(0))^4.$$

If we assume $c = (y(0))^2$, and $u = (y(a))^2$, then the last inequality reduces to

$$-\frac{4k_1^2\gamma^2}{m} - \frac{2ck_1}{m} - \frac{2k_1}{m}u \geq u^2 - 2cu + c^2,$$

or

$$u^2 - (2c - \frac{2k_1}{m})u + (\frac{4k_1^2\gamma^2}{m} + \frac{2ck_1}{m} + c^2) \leq 0,$$

this is possible only if the discriminate

$$D = (2c - \frac{2k_1}{m})^2 - 4(\frac{4k_1^2\gamma^2}{m} + \frac{2ck_1}{m} + c^2) > 0,$$

or

$$(2c - \frac{2k_1}{m})^2 > \frac{16k_1^2\gamma^2}{m} + \frac{8ck_1}{m} + 4c^2,$$

or

$$(2c - 2k_1)^2 > 16k_1^2\gamma^2 m + 8ck_1 m + 4c^2m^2.$$

Hence, the proof of part (1) is completed.

(2) We consider the case $\theta = 0$.

We multiplying equation (1) by $\bar{y}(x)$ and the adjoint equation (6) by $y(x)$ and adding the resulting equations, we get

$-(y''(x)\overline{y}(x)+\overline{y}''(x)y(x))+(y'(x)\overline{y}(x)+\overline{y}'(x)y(x))$
$=(\lambda^2+\overline{\lambda}^2)\rho(x)|y(x)|^2.$

Integrating both sides of this equation from 0 up to a, we obtain

$$-\int_0^a (y''(x)\overline{y}(x)+\overline{y}''(x)y(x))dx+\int_0^a (y'(x)\overline{y}(x)+\overline{y}'(x)y(x))dx=(\lambda^2+\overline{\lambda}^2)\int_0^a \rho(x)|y(x)|^2dx.$$

Integrating by parts and using the equations (2)-(3) and (7)-(8), we conclude

$$2|y(0)|^2+2\int_0^a |y'(x)|^2dx+2\alpha=(\lambda^2+\overline{\lambda}^2)k_1, \qquad (14)$$

where $k_1=\int_0^a \rho(x)|y(x)|^2dx$.

Previously we have assumed that $c=|y(0)|^2>0$ and since $(\lambda^2+\overline{\lambda}^2)=2(\theta^2-\gamma^2)$,
thence equation (14) reduces to

$$2c+2\int_0^a |y'(x)|^2dx+2\alpha=2k_1(\theta^2-\gamma^2),$$

or

$$\int_0^a |y'(x)|^2dx=k_1(\theta^2-\gamma^2)-(\alpha+c). \qquad (15)$$

From theorem (2.1) we have proved that
$|y(a)|^2\le\frac{2\sqrt{k_1}}{\sqrt{m}}(\int_0^a |y'(x)|^2dx)^{1/2}+|y(0)|^2$,
if we put $|y(a)|^2=c_1>0$, then this inequality becomes

$$c_1\le\frac{2\sqrt{k_1}}{\sqrt{m}}(\int_0^a |y'(x)|^2dx)^{1/2}+c,$$

or

$$\frac{\sqrt{m}}{2\sqrt{k_1}}(c_1-c)\le(\int_0^a |y'(x)|^2dx)^{1/2},$$

then

$$\frac{m}{4k_1}(c_1-c)^2\le\int_0^a |y'(x)|^2dx. \qquad (16)$$

From equations (15) and (16), we conclude that

$$\frac{m}{4k_1}(c_1-c)^2\le k_1(\theta^2-\gamma^2)-(\alpha+c),$$

hence

$$m(c_1-c)^2\le 4k_1^2(\theta^2-\gamma^2)-4k_1(\alpha+c).$$

Thus, the proof of the second part is finished; thence the proof of Lemma 2.1 is ended.

3 Estimations of the first and second derivatives of eigenfunctions to the problem (1)-(3)

The boundedness for norm of the first and second derivatives of eigenfunctions for the problem (1)-(3) are presented.

Theorem 3.1. Suppose that $\theta\ne 0$ and the weight function $\rho(x)$ is integrable on the interval $[0,a]$ such that $0<m\le\rho(x)\le M$, then for all eigenvalues λ_n and the corresponding eigenfunctions $y_n(x)$ of the problem (1)-(3), there are positive constants A and B that do not depend on $\rho(x)$ such that the following inequalities holds:

$||y_n'(x)||_{C[0,a]}\le A|\lambda_n|^{1/2}$, and
$||y_n''(x)||_{C[0,a]}\le B|\lambda_n|^{5/2}$,
where $B=kM+\frac{A}{|\lambda_n|^2}$.

Proof. Let x be any point in the interval $[0,a]$.At the beginning, we try to prove the first inequality.
Let us consider the following identity

$$|y_n'(x)|^2=y_n'(x)\overline{y_n}'(x)=\int_0^x (\overline{y_n}'(s)y_n''(s)+y_n'(s)\overline{y_n}''(s))ds+|y_n'(0)|^2.$$

In view of boundary condition (2): $y'(0)=y(0)$, so

$$\begin{aligned}|y_n'(x)|^2&=\int_0^x (\overline{y_n}'(s)y_n''(s)+y_n'(s)\overline{y_n}''(s))ds+|y_n(0)|^2\\&\le|\int_0^x |\overline{y_n}'(s)y_n''(s)+y_n'(s)\overline{y_n}''(s)|ds|+|y_n(0)|^2\\&\le|\int_0^x (|\overline{y_n}'(s)y_n''(s)|+|y_n'(s)\overline{y_n}''(s)|)ds|+c_1,\end{aligned}$$

where $c_1=|y_n(0)|^2>0$.

$$|y_n'(x)|^2\le|\int_0^a (|y_n'(s)y_n''(s)|+|y_n'(s)y_n''(s)|)ds|+c_1$$

$$\begin{aligned}|y_n'(x)|^2&\le 2|\int_0^a |y_n'(s)y_n''(s)|ds|+c_1\\&\le 2|\int_0^a |y_n'(s)||y_n''(s)|ds|+c_1.\end{aligned}$$

Estimating the last integral by the Cauchy-Schwartz inequality, we obtain

$$|y_n'(x)|^2\le 2|(\int_0^a |y_n'(s)|^2ds)^{1/2}(\int_0^a |y_n''(s)|^2ds)^{1/2}|+c_1.$$

$$|y_n'(x)|^2 \le 2|(\int_0^a |y_n'(s)|^2 ds)^{1/2}||(\int_0^a |y_n''(s)|^2 ds)^{1/2}| + c_1. \quad (17)$$

From theorem (2.1) we have shown that (by what we have done in the same way we can going on to prove that)

$$\int_0^a |y'(x)|^2 dx = |\lambda|^2 k_1 - k_3,$$

where $k_1 = \int_0^a \rho(x)|y(x)|^2 dx > 0$ and $k_3 = |y(0)|^2 + \alpha > 0$ and since $|y_n''(s)|^2$ is positive real number, so is $\int_0^a |y_n''(s)|^2 ds$, therefore we assume $(\int_0^a |y_n''(s)|^2 ds)^{1/2} = b_1$, where b_1 is a positive real number, thus equation (17) becomes

$$|y_n'(x)|^2 \le 2b_1|(|\lambda_n|^2 k_1 - k_3)^{1/2}| + c_1$$
$$|y_n'(x)|^2 \le 2b_1|\lambda_n||(k_1 + \frac{k_3}{|\lambda_n|^2})^{1/2}| + c_1.$$

And since $(k_1 + \frac{k_3}{|\lambda_n|^2}) > 0$, so $|(k_1 + \frac{k_3}{|\lambda_n|^2})^{1/2}|$ $= (k_1 + \frac{k_3}{|\lambda_n|^2})^{1/2}$, thence

$$\begin{aligned}|y_n'(x)|^2 &\le 2b_1|\lambda_n|(k_1 + \frac{k_3}{|\lambda_n|^2})^{1/2} + c_1\\ &= |\lambda_n|(2b_1(k_1 + \frac{k_3}{|\lambda_n|^2})^{1/2} + \frac{c_1}{|\lambda_n|}),\end{aligned}$$

or

$$|y_n'(x)| \le |\lambda_n|^{1/2}\sqrt{2b_1(k_1 + \frac{k_3}{|\lambda_n|^2})^{1/2} + \frac{c_1}{|\lambda_n|}},$$

if we put $A = \sqrt{2b_1(k_1 + \frac{k_3}{|\lambda_n|^2})^{1/2} + \frac{c_1}{|\lambda_n|}}$ which does not depend on $\rho(x)$, the last inequality becomes

$|y_n'(x)| \le A|\lambda_n|^{1/2}$.

And since x is any point in the interval $[0,a]$, so

$$\max_{x\in[0,a]} |y_n'(x)| \le A|\lambda_n|^{1/2},$$

thereby

$$||y_n'(x)||_{C[0,a]} \le A|\lambda_n|^{1/2}.$$

Then, the proof of the first part is completed.

It remains to prove the second part. From equation (1) we have

$$\begin{aligned}|y_n''(x)| &= |\lambda_n^2 \rho(x) y_n(x) - y_n'(x)|\\ &= |\lambda_n^2 \rho(x) y_n(x) + (-y_n'(x))|\\ &\le |\lambda_n^2 \rho(x) y_n(x)| + |-y_n'(x)|\\ &= |\lambda_n|^2 \rho(x)|y_n(x)| + |y_n'(x)|\\ &\le |\lambda_n|^2 \rho(x) \max_{x\in[0,a]} |y_n(x)| + \max_{x\in[0,a]} |y_n'(x)|\\ &\le |\lambda_n|^2 M \max_{x\in[0,a]} |y_n(x)| + \max_{x\in[0,a]} |y_n'(x)|.\end{aligned}$$

In the first part we have proved that $\max_{x\in[0,a]} |y_n'(x)| \le A|\lambda_n|^{1/2}$, and from theorem (2.1) we have shown that $\max_{x\in[0,a]} |y_n(x)| \le k|\lambda_n|^{1/2}$,where k and A are not dependents on $\rho(x)$, therefore the last inequality reduces to

$$\begin{aligned}|y_n''(x)| &\le |\lambda_n|^2 Mk|\lambda_n|^{1/2} + A|\lambda_n|^{1/2}\\ &= kM|\lambda_n|^{5/2} + A|\lambda_n|^{1/2}\\ &= |\lambda_n|^{5/2}(kM + \frac{A}{|\lambda_n|^2})\end{aligned}$$

$|y_n''(x)| \le B|\lambda_n|^{5/2}$,

where $B = (kM + \frac{A}{|\lambda_n|^2})$ does not depend on $\rho(x)$. And since x is any point in the interval $[0,a]$, thus

$$\max_{x\in[0,a]} |y_n''(x)| \le B|\lambda_n|^{5/2},$$

from here we get that

$$||y_n''(x)||_{C[0,a]} \le B|\lambda_n|^{5/2}.$$

Hence, the proof of the second part has finished and thereby, the proof of theorem (3.1) has finished.

Theorem 3.2. Suppose $\{y_n(x)\}$ be the sequence of eigenfunctions corresponding to the sequence of eigenvalues $\{\lambda_n = \theta_n + i\gamma_n | n \in \mathbb{N}\}$, where the sequence of eigenvalues satisfying $\gamma_n \le b_1|\theta_n|$, where $b_1 > 0$ be a fixed number, then there are constants d_1 and d_2 where $0 < d_1 < d_2$ such that the following double inequality holds

$$d_1|\lambda_n|^{1/2} \le ||y_n(x)||_{C[0,a]} \le d_2|\lambda_n|^{1/2}$$

,for all natural numbers n.

Proof. Here we discuss two cases: **Case (1): If** $\theta \neq 0$. **Case (2): If** $\theta = 0$.

Case (1): If $\theta \neq 0$, since $\gamma_n \le b_1|\theta_n|$ it follows that $\gamma_n^2 \le b_1^2|\theta_n|^2$

Now, $|\lambda_n| = \sqrt{\theta_n^2 + \gamma_n^2} \le \sqrt{|\theta_n|^2 + b_1^2|\theta_n|^2}$
$= |\theta_n|\sqrt{1 + b_1^2} = |\theta_n|b_2,$

where $b_2 = \sqrt{1 + b_1^2} > 0$. Now let $|y_n(a)|^2 = b_3 > 0$. So

$$\begin{aligned}|y_n(a)|^2 = b_3 &= \frac{b_3}{|\theta_n|b_2}|\theta_n|b_2\\ &\ge \frac{b_3}{|\theta_n|b_2}|\lambda_n|,\end{aligned}$$

then

$$\begin{aligned}|y_n(a)| &\ge |\lambda_n|^{1/2}\sqrt{\frac{b_3}{|\theta_n|b_2}}\\ &\ge d_1|\lambda_n|^{1/2},\end{aligned}$$

where $d_1 = \sqrt{\frac{b_3}{|\theta_n|b_2}} > 0$.

Since $||y_n(x)||_{C[0,a]} \geq |y_n(a)| \geq d_1|\lambda_n|^{1/2}$, so

$$||y_n(x)||_{C[0,a]} \geq d_1|\lambda_n|^{1/2}. \tag{18}$$

And from theorem (2.1) we have $||y_n(x)||_{C[0,a]} \leq k|\lambda_n|^{1/2}$, where $k > 0$ and k does not depend on $\rho(x)$. And since k is any positive number, which means that the last inequality holds for any positive number which does not depends on $\rho(x)$, hence this means that $\exists \quad d_2 > 0$ such that

$$||y_n(x)||_{C[0,a]} \leq d_2|\lambda_n|^{1/2}. \tag{19}$$

So from equations (18) and (19) we conclude that

$$d_1|\lambda_n|^{1/2} \leq ||y_n(x)||_{C[0,a]} \leq d_2|\lambda_n|^{1/2}.$$

Thus the case $\theta \neq 0$ is completely discussed.

Case (2): If $\theta = 0$. then the given problem becomes

$$-y_n''(x) + y_n'(x) = -\gamma_n^2\rho(x)y_n(x), \tag{20}$$

$$y_n'(a) = y_n'(0) - y_n(0) = 0,$$
$$\int_0^a \overline{y_n}(x)y_n'(x)dx = \alpha.$$

Integrating equation (20) from 0 to a with respect to x gives

$$-y_n'(x)]_0^a + y_n(x)]_0^a = -\gamma_n^2 \int_0^a \rho(x)y_n(x)dx.$$

In view of boundary conditions, the last equations reduces to:

$$y_n(a) = -\gamma_n^2 \int_0^a \rho(x)y_n(x)dx$$

$$|y_n(a)| = |\gamma_n^2 \int_0^a \rho(x)y_n(x)dx|$$

$$|y_n(a)| = |\gamma_n|^2| \int_0^a \rho(x)y_n(x)dx|,$$

or

$$|y_n(a)| = |\lambda_n|^2| \int_0^a \rho(x)y_n(x)dx|$$

$$|y_n(a)| \geq |\lambda_n|^{1/2}| \int_0^a \rho(x)y_n(x)dx|$$
$$= d_1|\lambda_n|^{1/2},$$

where $d_1 = |\int_0^a \rho(x)y_n(x)dx| > 0$. Therefore
$|y_n(a)| \geq d_1|\lambda_n|^{1/2}, \forall n \in \mathbb{N}$. And since
$||y_n(x)||_{C[0,a]} \geq |y_n(a)| \forall n \in \mathbb{N}$, then

$$||y_n(x)||_{C[0,a]} \geq d_1|\lambda_n|^{1/2}, \forall n \in \mathbb{N}. \tag{21}$$

Again, from equation (20) we have

$$|y_n(x)| = |\frac{1}{\gamma_n^2\rho(x)}y_n''(x) - \frac{1}{\gamma_n^2\rho(x)}y_n'(x)|$$
$$= |\frac{1}{\gamma_n^2\rho(x)}y_n''(x) + (-\frac{1}{\gamma_n^2\rho(x)})y_n'(x)|$$
$$\leq |\frac{1}{\gamma_n^2\rho(x)}y_n''(x)| + |(-\frac{1}{\gamma_n^2\rho(x)})y_n'(x)|$$
$$= |\frac{1}{\gamma_n^2\rho(x)}||y_n''(x)| + |-\frac{1}{\gamma_n^2\rho(x)}||y_n'(x)|$$
$$= \frac{1}{\gamma_n^2\rho(x)}|y_n''(x)| + \frac{1}{\gamma_n^2\rho(x)}|y_n'(x)|$$

(since $\gamma_n^2, \rho(x) > 0$). From inequality $m \leq \rho(x)$ it follows that
$\frac{1}{m} \geq \frac{1}{\rho(x)}$, therefore

$$|y_n(x)| \leq \frac{1}{\gamma_n^2\rho(x)}|y_n''(x)| + \frac{1}{\gamma_n^2\rho(x)}|y_n'(x)|$$
$$\leq \frac{1}{m\gamma_n^2}|y_n''(x)| + \frac{1}{m\gamma_n^2}|y_n'(x)|$$
$$\leq \frac{1}{m\gamma_n^2}(|y_n''(x)| + |y_n'(x)|).$$

Since $\frac{1}{m\gamma_n^2}, |y_n'(x)|, |y_n''(x)| > 0$, thus we assume that
$d_2 = \frac{1}{m\gamma_n^2}(|y_n''(x)| + |y_n'(x)|) > 0$, hence the last inequality becomes
$|y_n(x)| \leq d_2, \forall n \in \mathbb{N}$ and since x is any point in the interval $[0,a]$, then
$\max_{x\in[0,a]} |y_n(x)| \leq d_2, \forall n \in \mathbb{N}$, it follows that

$$||y_n(x)||_{C[0,a]} \leq d_2 \leq d_2|\lambda|^{1/2}, \forall n \in \mathbb{N}. \tag{22}$$

From equations (21) and (22) we get

$$d_1|\lambda|^{1/2} \leq ||y_n(x)||_{C[0,a]} \leq d_2|\lambda|^{1/2}, \forall n \in \mathbb{N}.$$

Thence, the proof of theorem (3.2) is completed.

4 Conclusions

In this study, we obtain the assessment of eigenfunctions, the behavior of eigenvalues, and the boundedness of the first and second derivatives of eigenfunctions for the boundary value theorem. Which contains the first derivative with the boundary condition but doesnt contain the spectral parameter.

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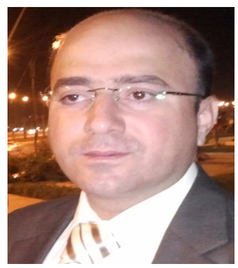

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