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Some Common Fixed Point Theorems for Four Self Maps in Complex Valued Metric Spaces

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Abstract: In this paper, first we prove a common fixed point theorem for two pairs of weakly compatible self maps in complex valued metric spaces. Secondly, we prove common fixed point theorems for weakly compatible mappings along with E.A. and (CLR) properties.

Keywords: Complex valued metric space, Partial order, Weakly compatible maps, E.A. property, (CLR) property.

1 Introduction

In 2011, Azam et al. [3] introduced the notion of complex valued metric space which is a generalization of the classical metric space. They established some fixed point results for mappings satisfying a rational inequality. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces; additionally, it offers numerous research activities in mathematical analysis.

A complex number $z \in C$ is an ordered pair of real numbers, whose first co-ordinate is called $\mathrm{Re}(z)$ and second coordinate is called $\mathrm{Im}(z)$. Thus a complex-valued metric d is a function from a set $X \times X$ into $\mathbb{C}$, where X is a nonempty set and $\mathbb{C}$ is the set of complex numbers.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$, that is $z_1 \precsim z_2$, if one of the following holds

(C1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;
(C2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;
(C3) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;
(C4) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3), and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Remark. We note that the following statements hold:

(i) $a, b \in \mathbb{R}$ and $a \le b \Rightarrow az \precsim bz \ \forall z \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$,
(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 1. *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*

(i) $0 \precsim d(x,y)$, *for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;*
(ii) $d(x,y) = d(y,x)$ *for all $x, y \in X$;*
(iii) $d(x,y) \precsim d(x,z) + d(z,y)$, *for all $x, y, z \in X$.*

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 1. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = 2i|z_1 - z_2|, \quad \text{for all } z_1, z_2 \in X.$$

Then (X,d) is a complex valued metric space.

Definition 2. *Let (X,d) be a complex valued metric space, $\{x_n\}$ be a sequence in X and $x \in X$.*

(i) *If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $k \in \mathbb{N}$ such that for all $n > k$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.*
(ii) *If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $k \in \mathbb{N}$ such that for all $n > k$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be a Cauchy sequence.*

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(iii) *If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued metric space.*

Lemma 1. *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 2. *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

In 1996, Jungck [4] introduced the concept of weakly compatible maps as follows:

Definition 3. *Two self maps f and g are said to be weakly compatible if they commute at coincidence points.*

In 2002, Aamri et al. [1] introduced the notion of E.A. property as follows:

Definition 4. *Two self-mappings f and g of a metric space (X,d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.*

In 2011, Sintunavarat et al. [5] introduced the notion of (CLR) property as follows:

Definition 5. *Two self-mappings f and g of a metric space (X,d) are said to satisfy (CLR_f) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some x in X.*

In the same way, we can introduce these notions in complex valued metric space.

Example 2. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1,z_2) = 2i|z_1 - z_2|, \quad \text{for all } z_1,z_2 \in X.$$

Then (X,d) is a complex valued metric space.
Define $S,T : X \to X$ by

$$Sz = z + i \text{ and } Tz = 2z, \quad \text{for all } z \in X.$$

Consider a sequence $\{z_n\} = \left\{i - \frac{1}{n}\right\}$, $n \in \mathbb{N}$, in X, then

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} (z_n + i) = \lim_{n\to\infty} i - \frac{1}{n} + i = 2i.$$

$$\lim_{n\to\infty} Tz_n = \lim_{n\to\infty} 2z_n = \lim_{n\to\infty} 2\left(i - \frac{1}{n}\right) = 2i,$$

where $2i \in X$.
Thus, S and T satisfies E.A. property.
Also, we have

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = 2i = S(i),$$

where $i \in X$.
Thus, S and T satisfies (CLRS) property.

Now, we shall prove our results relaxing the condition of complex valued metric space being complete.

2 Weakly compatible maps

Theorem 1. *Let A, B, S and T be self maps of a complex valued metric space (X,d) satisfying the followings:*

(2.1)$SX \subseteq BX$, $TX \subseteq AX$,
(2.2)$d(Sx,Ty) \precsim k\max\{d(Ax,By), d(Sx,Ax), d(Ty,By), \frac{1}{2}(d(Sx,By) + d(Ty,Ax))\}$, $0 < k < 1$.

If one of AX, BX, SX or TX is complete subspace of X, then the pair (A,S) or (B,T) have a coincidence point. Moreover, if pairs (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point of X. From (2.1), we can construct a sequence $\{y_n\}$ in X as follows:

(2.3)$y_{2n+1} = Sx_{2n} = Bx_{2n+1}$, $y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}$, for all $n = 0,1,2,\ldots$.

Define $d_n = d(y_n,y_{n+1})$. Suppose that $d_{2n} = 0$ for some n. Then $y_{2n} = y_{2n+1}$, that is, $Tx_{2n-1} = Ax_{2n} = Sx_{2n} = Bx_{2n+1}$, and so the pair (A,S) have a coincidence point.
Similarly, if $d_{2n+1} = 0$, then the pair (B,T) have a coincidence point.
Assume that $d_n \neq 0$ for each n.
From (2.2), we have

$$\begin{aligned}
&d(y_{2n+1},y_{2n+2}) \\
&= d(Sx_{2n},Tx_{2n+1}) \\
&\precsim k\max\{d(Ax_{2n},Bx_{2n+1}), d(Sx_{2n},Ax_{2n}), \\
&\quad d(Tx_{2n+1},Bx_{2n+1}), \tfrac{1}{2}(d(Sx_{2n},Bx_{2n+1}) \\
&\quad + d(Tx_{2n+1},Ax_{2n}))\} \qquad (2.1)\\
&= k\max\{d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n}), d(y_{2n+2},y_{2n+1}), \\
&\quad \tfrac{1}{2}(d(y_{2n+1},y_{2n+1}) + d(y_{2n+2},y_{2n}))\} \\
&\precsim k\max\{d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), \\
&\quad \tfrac{1}{2}(d(y_{2n+2},y_{2n+1}) + d(y_{2n+1},y_{2n}))\} \\
&= k\max\{d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2})\} \\
&= k\max\{d_{2n},d_{2n+1}\}. \qquad (2.2)
\end{aligned}$$

Now, if $d_{2n+1} \geq d_{2n}$, for some n, then from (2.5), we have

$$d(y_{2n+1},y_{2n+2}) \precsim kd(y_{2n+1},y_{2n+2}),$$

that is,

$$\begin{aligned}
|d(y_{2n+1},y_{2n+2})| &\leq k|d(y_{2n+1},y_{2n+2})| \\
&< |d(y_{2n+1},y_{2n+2})|,
\end{aligned}$$

since $0 < k < 1$, a contradiction.
Thus, $d_{2n} > d_{2n+1}$ for all n, and so, from (2.5), we have

$$d(y_{2n+1},y_{2n+2}) \precsim kd(y_{2n},y_{2n+1}).$$

Similarly,

$$d(y_{2n},y_{2n+1}) \precsim kd(y_{2n-1},y_{2n}).$$

In general, we have for all $n = 0, 1, 2, \ldots$,

$$d(y_n, y_{n+1}) \precsim kd(y_{n-1}, y_n) \precsim k^2 d(y_{n-2}, y_{n-1}) \ldots \precsim k^n d(y_0, y_1).$$

Now, for all $m > n$,

$$\begin{aligned} &d(y_m, y_n) \\ &\precsim d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_m, y_{m-1}) \\ &\precsim k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) + \ldots + k^{m-1} d(y_0, y_1) \\ &\precsim \frac{k^n}{1-k} d(y_0, y_1). \end{aligned}$$

Therefore, we have

$$|d(y_m, y_n)| \leq \frac{k^n}{1-k} |d(y_0, y_1)|.$$

Hence,

$$\lim_{n\to\infty} |d(y_m, y_n)| = 0.$$

Hence, $\{y_n\}$ is a Cauchy sequence.
Now, suppose that $A(X)$ is complete. Note that $\{y_{2n}\}$ is contained in $A(X)$ and has a limit in $A(X)$, say u, that is, $\lim_{n\to\infty} y_{2n} = u$. Let $v \in A^{-1}u$. Then $Av = u$.
Now, we shall prove that $Sv = u$.
Let, if possible, $Sv \neq u$.
Putting $x = v$ and $y = x_{2n-1}$ in (2.2), we have

$$\begin{aligned} d(Sv, Tx_{2n-1}) \precsim k \max\{&d(Av, Bx_{2n-1}), d(Sv, Av), \\ &d(Tx_{2n-1}, Bx_{2n-1}), \tfrac{1}{2}(d(Sv, Bx_{2n-1}) \\ &+ d(Tx_{2n-1}, Av))\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &|d(Sv, Tx_{2n-1})| \\ &\leq k|\max\{(d(Av, Bx_{2n-1}), d(Sv, Av), \\ &\quad d(Tx_{2n-1}, Bx_{2n-1}), \\ &\quad \tfrac{1}{2}(d(Sv, Bx_{2n-1}) + d(Tx_{2n-1}, Av)))\}|. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} |d(Sv, u)| &\leq k|\max\{d(u, u), d(Sv, u), d(u, u), \\ &\quad \tfrac{1}{2}(d(Sv, u) + d(u, u))\}| \\ &= k|d(Sv, u)| < |d(Sv, u)|, \text{ a contradiction.} \end{aligned}$$

Thus, $Sv = u = Av$, that is, v is the coincidence point of the pair (A, S).
Since $SX \subseteq BX$, $Sv = u$, implies that, $u \in BX$.
Let $w \in B^{-1}u$. Then $Bw = u$. By using the same arguments as above, one can easily verify that, $Tw = u = Bw$, that is, w is the coincidence point of the pair (B, T).
The same result holds, if we assume that BX is complete instead of AX.
Now, if TX is complete, then by (2.1), $u \in TX \subseteq AX$.
Similarly, if SX is complete, then $u \in SX \subseteq BX$.
Now, since the pairs (A, S) and (B, T) are weakly compatible, so

$$u = Sv = Av = Tw = Bw,$$

then

$$Au = ASv = SAv = Su, \quad Bu = BTw = TBw = Tu. \quad (2.3)$$

Now, we claim that $Tu = u$.
Let, if possible, $Tu \neq u$.
From (2.2), we have

$$\begin{aligned} d(u, Tu) &= d(Sv, Tu) \\ &\precsim k \max\{d(Av, Bu), d(Sv, Av), d(Tu, Bu), \\ &\quad \tfrac{1}{2}(d(Sv, Bu) + d(Tu, Av))\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &|d(u, Tu)| \\ &\leq k|\max\{d(Av, Bu), d(Sv, Av), d(Tu, Bu), \\ &\quad \tfrac{1}{2}(d(Sv, Bu) + d(Tu, Av))\}| \\ &= k|\max\{d(u, Tu), d(u, u), 0, \tfrac{1}{2}(d(u, Tu) + d(Tu, u))\}| \\ &= k|d(u, Tu)| < |d(u, Tu)|, \quad \text{a contradiction.} \end{aligned}$$

Thus, we have $Tu = u$.
Similarly, $Su = u$.
Thus, we get $Au = Su = Bu = Tu = u$.
Hence u is the common fixed point of A, B, S and T.
For the uniqueness, let z be another common fixed point of A, B, S and T.
Now, we claim that $u = z$.
Let, if possible, $u \neq z$.
From (2.2), we have

$$\begin{aligned} d(u, z) &= d(Su, Tz) \\ &\precsim k \max\{d(Au, Bz), d(Su, Au), d(Tz, Bz), \\ &\quad \tfrac{1}{2}(d(Su, Bz) + d(Tz, Au))\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &|d(u, z)| \\ &\leq k|\max\{d(Au, Bz), d(Su, Au), d(Tz, Bz), \\ &\quad \tfrac{1}{2}(d(Su, Bz) + d(Tz, Au))\}| \\ &= k|\max\{d(u, z), d(u, u), d(z, z), \tfrac{1}{2}(d(u, z) + d(z, u))\}| \\ &= k|d(u, z)| < |d(u, z)|, \quad \text{a contradiction.} \end{aligned}$$

Thus, we get, $u = z$.
Hence u is the common fixed point of A, B, S and T.

Corollary 1. *Let B and S be two self maps of a complex valued metric space (X, d) satisfying the following:*

(i) $SX \subseteq BX$,

(ii) $d(Sx,Sy) \precsim k\max\{d(Bx,By),d(Sx,Bx),d(Sy,By),$ $\frac{1}{2}(d(Sx,By)+d(Sy,Bx))\}$, *for all x, y in X and* $0<k<1$.

If one of SX or BX is complete subspace of X, then the pair (B,S) *have a coincidence point. Moreover, if B and S are weakly compatible, then B and S have a unique common fixed point.*

Proof. By putting $A=B$ and $S=T$, we get the Corollary 2.2.

3 E.A. property

Theorem 2. *Let A, B, S and T be self mappings of a complex valued metric space* (X,d) *satisfying (2.1), (2.2) and the followings:*

(3.1)pairs (A,S) *and* (B,T) *are weakly compatible,*
(3.2)pair (A,S) *or* (B,T) *satisfy the E.A. property.*

If any one of AX, BX, SX and TX is a complete subspace of X, then A, B, S and T have a unique common fixed point.

Proof. Suppose that (A,S) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \quad \text{for some } z \text{ in } X.$$

Since $SX \subseteq BX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$.
Hence $\lim_{n\to\infty} By_n = z$.
We shall show that $\lim_{n\to\infty} Ty_n = z$.
Let, if possible, $\lim_{n\to\infty} Ty_n = t \neq z$.
From (2.2), we have

$$\begin{aligned} &d(Sx_n,Ty_n)\\ &\precsim k\max\{d(Ax_n,By_n),d(Sx_n,Ax_n),d(Ty_n,By_n),\\ &\quad \tfrac{1}{2}(d(Sx_n,By_n)+d(Ty_n,Ax_n))\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &|d(Sx_n,Ty_n)|\\ &\leq k|\max\{(d(Ax_n,By_n),d(Sx_n,Ax_n),d(Ty_n,By_n),\\ &\quad \tfrac{1}{2}(d(Sx_n,By_n)+d(Ty_n,Ax_n)))\}|. \end{aligned}$$

Letting $n\to\infty$, we have

$$\begin{aligned} |d(z,t)| \leq k|\max\{&d(z,z),d(z,z),d(t,z),\\ &\tfrac{1}{2}(d(z,z)+d(t,z))\}|. \end{aligned}$$

Thus, we have

$$|d(z,t)| \leq k|d(z,t)| < |d(z,t)|, \quad \text{a contradiction.}$$

Therefore, $t=z$, that is, $\lim_{n\to\infty} Ty_n = z$.

Suppose that BX is a complete subspace of X.
Then $z=Bu$ for some u in X.
Subsequently, we have

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = z = Bu.$$

Now, we shall show that $Tu=Bu$.
Let, if possible, $Tu\neq Bu$.
From (2.2), we have

$$\begin{aligned} d(Sx_n,Tu) \precsim k\max\{&d(Ax_n,Bu),d(Sx_n,Ax_n),d(Tu,Bu),\\ &\tfrac{1}{2}(d(Sx_n,Bu)+d(Tu,Ax_n))\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &|d(Sx_n,Tu)|\\ &\leq k|\max\{(d(Ax_n,Bu),d(Sx_n,Ax_n),d(Tu,Bu),\\ &\quad \tfrac{1}{2}(d(Sx_n,Bu)+d(Tu,Ax_n)))\}|. \end{aligned}$$

Letting $n\to\infty$, we have

$$\begin{aligned} &|d(z,Tu)|\\ &\leq k|\max\{d(z,z),d(z,z),d(Tu,z),\tfrac{1}{2}(d(z,z)+d(Tu,z))\}|.\\ &= k|d(z,Tu)| < |d(z,Tu)|, \quad \text{a contradiction.} \end{aligned}$$

Therefore, $Tu=z=Bu$.
Since B and T are weakly compatible, therefore, $BTu = TBu$, implies that, $TTu=TBu=BTu=BBu$.
Since $TX\subseteq AX$, there exists $v\in X$, such that, $Tu=Av$.
Now, we claim that $Av=Sv$.
Let, if possible, $Av\neq Sv$.
From (2.2), we have

$$\begin{aligned} d(Sv,Tu) \precsim k\max\{&d(Av,Bu),d(Sv,Av),d(Tu,Bu),\\ &\tfrac{1}{2}(d(Sv,Bu)+d(Tu,Av))\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} |d(Sv,Tu)| \leq k|\max\{&d(Av,Bu),d(Sv,Av),d(Tu,Bu),\\ &\tfrac{1}{2}(d(Sv,Bu)+d(Tu,Av))\}|\\ &= k|\max 0,d(Sv,Tu),0,\tfrac{1}{2}(d(Sv,Tu)+0)|\\ &= k|d(Sv,Tu)| < |d(Sv,Tu)|, \text{ a contradiction.} \end{aligned}$$

Therefore, $Sv=Tu=Av$.
Thus, we have, $Tu=Bu=Sv=Av$.
The weak compatibility of A and S implies that $ASv=SAv=SSv=AAv$.
Now, we claim that Tu is the common fixed point of A, B, S and T.
Suppose that, $TTu\neq Tu$.
From (2.2), we have

$$\begin{aligned} d(Tu,TTu) &= d(Sv,TTu)\\ &\precsim k\max\{d(Av,BTu),d(Sv,Av),d(TTu,BTu),\\ &\quad \tfrac{1}{2}(d(Sv,BTu)+d(TTu,Av))\}. \end{aligned}$$

Thus, we have

$$|d(Tu,TTu)| \leq k|\max\{d(Av,BTu),d(Sv,Av),d(TTu,BTu),\frac{1}{2}(d(Sv,BTu)+d(Tu,Av))\}| = k|\max\{d(Tu,TTu),0,0,d(Tu,TTu)\}| = k|d(Tu,TTu)| < |d(Tu,TTu)|,$$
a contradiction.

Therefore, $Tu = TTu = BTu$.
Hence Tu is the common fixed point of B and T.
Similarly, we prove that Sv is the common fixed point of A and S. Since $Tu = Sv$, Tu is the common fixed point of A, B, S and T. The proof is similar when AX is assumed to be a complete subspace of X. The cases in which TX or SX is a complete subspace of X are similar to the cases in which AX or BX, respectively is complete subspace of X, since $TX \subseteq AX$ and $SX \subseteq BX$.
Now, we shall prove that the common fixed point is unique.
If possible, let p and q be two common fixed points of A, B, S and T, such that, $p \neq q$.
From (2.2), we have

$$d(p,q) = d(Sp,Tq) \precsim k\max\{d(Ap,Bq),d(Sp,Ap),d(Tq,Bq),\frac{1}{2}(d(Sp,Bq)+d(Tq,Ap))\}.$$

Thus, we have

$$|d(p,q)| \leq k|\max\{d(Ap,Bq),d(Sp,Ap),d(Tq,Bq),\frac{1}{2}(d(Sp,Bq)+d(Tq,Ap))\}| = k|\max\{d(p,q),d(p,p),d(q,q),\frac{1}{2}(d(p,q)+d(q,p))\}| = k|d(p,q)| < |d(p,q)|,$$
a contradiction.

Thus, we get, $p = q$.
Hence the mappings A, B, S and T have a unique common fixed point.

Corollary 2. *Let B and S be two weakly compatible self maps of a complex valued metric space (X,d) satisfying the following:*

(i) $SX \subseteq BX$,
(ii) $d(Sx,Sy) \precsim k\max\{d(Bx,By),d(Sx,Bx),d(Sy,By),\frac{1}{2}(d(Sx,By)+d(Sy,Bx))\}$, *for all x, y in X and* $0 < k < 1$.
(iii) *B and S satisfies the E.A. property*

If SX or BX is complete subspace of X, then B and S have a unique common fixed point.

Proof. By putting $A = B$ and $S = T$, we get the Corollary 3.2.

4 (CLR) property

Theorem 3. *Let A, B, S and T be self maps of a metric space (X,d) satisfying (2.2), (3.1) and the following:*

(4.1) *$SX \subseteq BX$ and the pair (A,S) satisfies (CLR$_A$) property, or $TX \subseteq AX$ and the pair (B,T) satisfies (CLR$_B$) property.*

Then A, B, S and T have a unique common fixed point.

Proof. Without loss of generality, assume that $SX \subseteq BX$ and the pair (A,S) satisfies (CLR$_A$) property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Ax$, for some x in X.
Since $SX \subseteq BX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$.
Hence $\lim_{n\to\infty} By_n = Ax$.
We shall show that $\lim_{n\to\infty} Ty_n = Ax$.
Let, if possible, $\lim_{n\to\infty} Ty_n = z \neq Ax$.
From (2.2), we have

$$d(Sx_n,Ty_n) \precsim k\max\{d(Ax_n,By_n),d(Sx_n,Ax_n),d(Ty_n,By_n),\frac{1}{2}(d(Sx_n,By_n)+d(Ty_n,Ax_n))\}$$

Thus, we have

$$|d(Sx_n,Ty_n)| \leq k|\max\{(d(Ax_n,By_n),d(Sx_n,Ax_n),d(Ty_n,By_n),\frac{1}{2}(d(Sx_n,By_n)+d(Ty_n,Ax_n)))\}|.$$

Letting $n \to \infty$, we have

$$|d(Ax,z)| \leq k|\max\{d(Ax,Ax),d(Ax,Ax),d(Ax,z),\frac{1}{2}(d(Ax,Ax)+d(z,Ax))\}|.$$

Thus, we have

$$|d(Ax,z)| \leq k|d(Ax,z)| < |d(Ax,z)|,$$
a contradiction.

Therefore, $Ax = z$, that is, $\lim_{n\to\infty} Ty_n = Ax$.
Subsequently, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Ax = z.$$

Now, we shall show that $Sx = z$.
Let, if possible, $Sx \neq z$.
From (2.2), we have

$$d(Sx,Ty_n) \precsim k\max\{d(Ax,By_n),d(Sx,Ax),d(Ty_n,By_n),\frac{1}{2}(d(Sx,By_n)+d(Ty_n,Ax))\}.$$

Thus, we have

$$|d(Sx,Ty_n)| \leq k|\max\{(d(Ax,By_n),d(Sx,Ax),d(Ty_n,By_n),\frac{1}{2}(d(Sx,By_n)+d(Ty_n,Ax)))\}|.$$

Letting $n \to \infty$, we have

$$|d(Sx,z)| \leq k|\max\{d(z,z), d(Sx,z), d(z,z), \frac{1}{2}(d(Sx,z)+d(z,z))\}|.$$

Thus, we have

$$|d(Sx,z)| \leq k|d(Sx,z)| < |d(Sx,z)|,$$
which is not possible.

Therefore, $Sx = z = Ax$.
Since, the pair (A,S) is weakly compatible, it follows that $Az = Sz$.
Also, since $SX \subseteq BX$, there exists some y in X such that $Sx = By$, that is, $By = z$.
Now, we show that $Ty = z$.
Let, if possible, $Ty \neq z$.
From (2.2), we have

$$d(Sx_n,Ty) \precsim k\max\{d(Ax_n,By), d(Sx_n,Ax_n), d(Ty,By), \frac{1}{2}(d(Sx_n,By)+d(Ty,Ax_n))\}.$$

Thus, we have

$$|d(Sx_n,Ty)| \leq k|\max\{(d(Ax_n,By), d(Sx_n,Ax_n), d(Ty,By), \frac{1}{2}(d(Sx_n,By)+d(Ty,Ax_n)))\}|.$$

Letting $n \to \infty$, we have

$$|d(z,Ty)| \leq k|\max\{d(z,z), d(z,z), d(z,Ty), \frac{1}{2}(d(z,z)+d(Ty,z))\}|.$$

Thus, we have

$$|d(z,Ty)| \leq k|d(z,Ty)| < |d(z,Ty)|,$$
which is not possible.

Thus, $z = Ty = By$.
Since the pair (B,T) is weakly compatible, it follows that $Tz = Bz$.
Now, we claim that $Sz = Tz$.
Let, if possible, $Sz \neq Tz$.
From (2.2), we have

$$d(Sz,Tz) \precsim k\max\{d(Az,Bz), d(Sz,Az), d(Bz,Tz), \frac{1}{2}(d(Sz,Bz)+d(Tz,Az))\}.$$

Thus, we have

$$\begin{aligned}
&|d(Sz,Tz)| \\
&\leq k|\max\{d(Az,Bz), d(Sz,Az), d(Bz,Tz), \\
&\quad \frac{1}{2}(d(Sz,Bz)+d(Tz,Az))\}| \\
&= k|\max\{d(Sz,Tz), 0, 0, \frac{1}{2}(d(Sz,Tz)+d(Tz,Sz))\}| \\
&= k|d(Sz,Tz)| < |d(Sz,Tz)|, \quad \text{a contradiction.}
\end{aligned}$$

Therefore, $Sz = Tz$, that is, $Az = Sz = Tz = Bz$.
Now, we shall show that $z = Tz$.
Let, if possible, $z \neq Tz$.
From (2.2), we have

$$d(Sx,Tz) \precsim k\max\{d(Ax,Bz), d(Sx,Ax), d(Bz,Tz), \frac{1}{2}(d(Sx,Bz)+d(Tz,Ax))\}.$$

Thus, we have

$$\begin{aligned}
|d(z,Tz)| &\leq k|\max\{d(Ax,Bz), d(Sx,Ax), d(Bz,Tz), \\
&\quad \frac{1}{2}(d(Sx,Bz)+d(Tz,Ax))\}| \\
&= k|\max\{d(z,Tz), 0, 0, \frac{1}{2}(d(z,Tz)+d(Tz,z))\}| \\
&= k|d(z,Tz)| < |d(z,Tz)|, \quad \text{a contradiction.}
\end{aligned}$$

Therefore, $z = Tz = Bz = Az = Sz$.
Hence, z is the common fixed point of A, B, S and T.
Now, we shall prove that the common fixed point is unique.
Let u be another common fixed point of A, B, S and T.
Let, if possible, $z \neq u$.
From (2.2), we have

$$\begin{aligned}
d(u,z) &= d(Su,Tz) \\
&\precsim k\max\{d(Au,Bz), d(Su,Au), d(Tz,Bz), \\
&\quad \frac{1}{2}(d(Su,Bz)+d(Tz,Au))\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|d(u,z)| &\leq k|\max\{d(Au,Bz), d(Su,Au), d(Tz,Bz), \\
&\quad \frac{1}{2}(d(Su,Bz)+d(Tz,Au))\}| \\
&= k|\max\{d(u,z), d(u,u), d(z,z), \\
&\quad \frac{1}{2}(d(u,z)+d(z,u))\}| \\
&= k|d(u,z)| < |d(u,z)|, \quad \text{a contradiction.}
\end{aligned}$$

Thus, we get, $u = z$.
Hence z is the unique common fixed point of A, B, S and T.

Corollary 3. *Let B and S be two weakly compatible self maps of a complex valued metric space (X,d) satisfying the following:*

(i) $SX \subseteq BX$,
(ii) $d(Sx,Sy) \precsim k\max\{d(Bx,By), d(Sx,Bx), d(Sy,By), \frac{1}{2}(d(Sx,By)+d(Sy,Bx))\}$, *for all x, y in X and* $0 < k < 1$,
(iii) *B and S satisfies the (CLR_B) property.*

Then B and S have a unique common fixed point.

Proof. By putting $A = B$ and $S = T$, we get the Corollary 4.2.

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