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Preserving Properties of Subordination and Superordination of Analytic Functions Associated with the Srivastava-Khairnar-More Integral Operator

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Abstract: In the present paper, we investigate some subordination and superordination preserving properties of analytic functions associated with the Srivastava-Khairnar-More integral operator defined on the space of normalized analytic functions in the open unit disk $\mathbb{U}$. Several sandwich-type results associated with this transformation are derived. Some useful consequences of the main results are mentioned and relevance with some of the earlier results are also pointed out.

Keywords: Analytic functions, univalent functions, differential subordination and superordination, Srivastava-Khairnar-More integral operator

1 Introduction and Definitions

Let $\mathscr{H} := \mathscr{H}(\mathbb{U})$ be the linear space of all analytic functions in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let

$$\mathscr{H}[d,n] = \left\{ f \in \mathscr{H} : f(z) = d + d_n z^n + d_{n+1} z^{n+1} + \cdots \right\} \quad (d \in \mathbb{C},\ n \in \mathbb{N} = \{1,2,3,\cdots\}). \tag{1}$$

Let $\mathscr{A}$ the subclass of the functions $\mathscr{H}[d,1]$ with usual *normalization* given by $f(0) = f'(0) - 1 = 0$. A function f in $\mathscr{A}$ is said to be *univalent* in $\mathbb{U}$ if f is one to one in $\mathbb{U}$.

For two functions f and g are in $\mathscr{H}$, we say that f is *subordinate* to g in $\mathbb{U}$, write as

$$f \prec g \text{ in } \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ satisfying the condition of the Schwarz lemma such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

It is well known (see, for details [3,11,19]) that if the function g is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

We need the following definitions for our present investigation:

Definition 1 (see [11]). Let $\psi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination:

$$\psi(p(z), zp'(z)) \prec h(z), \tag{2}$$

then p is called a solution of the differential subordination (2). A univalent function q is called a dominant of the solutions of the differential subordination (2) or more simply, a dominant if $p(z) \prec q(z)$ for all p satisfying (2). A dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants q of (2) is said to be the best dominant of (2).

Definition 2 (see [12]). Let $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\phi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination:

$$h(z) \prec \phi(p(z), zp'(z)), \tag{3}$$

then p is called a solution of the differential superordination (3). An analytic function q is called a

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subordinant of the solutions of the differential superordination (3) or more simply, a subordinant if $q(z) \prec p(z)$ for all p satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants q of (3) is said to be the best subordinant of (3).

Definition 3 (see [11], Definition 2.2b, p. 21). We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \{\xi : \xi \in \partial\mathbb{U} \text{ and } \lim_{z \longrightarrow \xi} f(z) = \infty\}$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(f)$.

Definition 4 (see [12]). A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is called a subordination (or a Löwner) chain if $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 < t_2$ and $z \in \mathbb{U}$.

Let $\mathcal{A}$ denote the family of *normalized* functions of the form:

$$f(z) = z + \sum_{k=1}^{\infty} d_{k+1} z^{k+1} \tag{4}$$

which are analytic in $\mathbb{U}$. Let $f, g \in \mathcal{A}$, where $f(z)$ is defined by (4) and $g(z)$ is given by

$$g(z) = z + \sum_{k=1}^{\infty} e_{k+1} z^{k+1},$$

then the Hadamard product (or convolution) of f and g denoted by $f * g$ is defined as

$$(f*g)(z) := z + \sum_{k=1}^{\infty} d_{k+1} e_{k+1} z^{k+1} =: (g*f)(z).$$

Recently, Srivastava et al. [18] introduced and investigated the following integral operator:

$$\mathcal{I}_{\mu}^{\lambda}(a,b,c)f(z) = z + \sum_{k=1}^{\infty} \frac{(\lambda+1)_k (c)_k}{(\mu k+1)(a)_k (b)_k} d_{k+1} z^{k+1} \tag{5}$$

$(a,b,c \in \mathbb{C} \setminus \mathbb{Z}_0^- := \{0,-1,-2,\cdots\}; \lambda > -1, \mu \geq 0; z \in \mathbb{U})$, where $(\alpha)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the familiar Gamma function) by

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} 1 & (k=0) \\ \alpha(\alpha+1)\ldots(\alpha+k-1) & (k \in \mathbb{N}). \end{cases} \tag{6}$$

In particular, we have

$$\mathcal{I}_0^{\lambda}(a,\lambda+1,a)f(z) = f(z) \quad \text{and} \quad \mathcal{I}_0^{1}(a,1,a)f(z) = zf'(z).$$

Clearly, the integral operator $\mathcal{I}_0^{\lambda}(a,b,c)$ is the Noor integral operator [7]. It can be easily shown from (5) that

$$z\left(\mathcal{I}_{\mu}^{\lambda}(a,b,c)f(z)\right)' = (\lambda+1)\mathcal{I}_{\mu}^{\lambda+1}(a,b,c)f(z) - \lambda \mathcal{I}_{\mu}^{\lambda}(a,b,c)f(z), \tag{7}$$

and

$$z\left(\mathcal{I}_{\mu}^{\lambda}(a+1,b,c)f(z)\right)' = a\mathcal{I}_{\mu}^{\lambda}(a,b,c)f(z) - (a-1)\mathcal{I}_{\mu}^{\lambda}(a+1,b,c)f(z). \tag{8}$$

Recently, Wang et al. [21] investigated various properties and characteristics of the Srivastava-Khairnar-More integral operator $\mathcal{I}_{\mu}^{\lambda}(a,b,c)$ defined by the equation (5) (for recent expository work on integral operator in analytic function theory see [2,6,17,20]).

2 Preliminaries lemmas

To prove our main results, we need each of the following lemmas.

Lemma 1 *([8], also see [11])*. Suppose that the function $H : \mathbb{C}^2 \longrightarrow \mathbb{C}$ satisfies the condition:

$$\Re\{H(is,t)\} \leq 0,$$

for all real s and $t \leq \frac{-n(1+s^2)}{2}$, where n is a positive integer. If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$ and

$$\Re\{H(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then $\Re\{p(z)\} > 0$ in $\mathbb{U}$.

Lemma 2 *(see [10])*. Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If $\Re\{\beta h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U};\ q(0) = c),$$

is analytic in $\mathbb{U}$ and satisfies $\Re\{\beta q(z) + \gamma\} > 0 \quad (z \in \mathbb{U})$.

Lemma 3 *(see [11])*. Let $p \in \mathcal{Q}$ with $p(0) = d$ and let $q(z) = d + d_n z^n + d_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{U}$ with $q(z) \neq d$ and $n \geq 1$. If q is not subordinate to p, then there exists the points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(f)$ for which $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,

$$q(z_0) = p(\xi_0), \text{ and } z_0 q'(z_0) = m\xi_0 p'(\xi_0) \quad (m \geq n \geq 1),$$

where $\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

Lemma 4 *(see [15])*. The function $L(z,t) : \mathbb{U} \times [0,\infty) \longrightarrow \mathbb{C}$ of the form

$$L(z,t) = d_1(t)z + d_2(t)z^2 + \cdots$$

with $d_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t \longrightarrow \infty} |d_1(t)| = \infty$ is a subordination chain if and only if

$$\Re\left[\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right] > 0 \quad (z \in \mathbb{U}; 0 \leq t < \infty).$$

Lemma 5 *(see [12])*. Let $\mathcal{H}[d,1] = \{f \in \mathcal{H} : f(0) = d, f'(0) \neq 0\}$ and $q \in \mathcal{H}[d,1]$, $\psi : \mathbb{C}^2 \longrightarrow \mathbb{C}$. Also set

$\psi(q(z), zq'(z)) \equiv h(z)$. If $L(z,t) = \psi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathscr{H}[d,1] \cap \mathscr{Q}$, then

$$h(z) \prec \psi(p(z), zp'(z)) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathbb{U}).$$

Furthermore, if the differential equation $\psi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathscr{Q}$, then q is the best subordinant.

Using the principle of subordination between analytic functions, Miller et al. [9] investigated some subordination theorems involving certain integral operators for analytic (multivalent) functions in $\mathbb{U}$ (see, also [1,4,13]). Very recently, Prajapat [14] and Kwon and Cho [5] investigated subordination and superordination preserving properties for multivalent functions associated with the generalized multiplier transformation operator and differintegral operator respectively.

Motivated by aforementioned work, in this paper the authors obtain the subordination and superordination-preserving properties of the Srivastava-Khairnar-More integral operator $\mathscr{I}_{\mu}^{\lambda}(a,b,c)$ defined by (5). Several sandwich-type results involving this operator are also derived.

3 Main results

Unless otherwise mentioned, we assume throughout the sequel that

$$a, b, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda > -1 \quad \text{and} \quad \mu \geq 0.$$

Theorem 1 contains subordination results for the integral operator $\mathrm{I}_{\mu}^{\lambda}(a,b,c)$ defined by (5).

Theorem 1. Let $f, g \in \mathscr{A}$ and Suppose that

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\beta \quad (z \in \mathbb{U}) \tag{9}$$

where

$$\phi(z) := \left[\frac{\mathscr{I}_{\mu}^{\lambda}(a,b,c)g(z)}{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)g(z)}\right]\left[\frac{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)g(z)}{z}\right]^{\alpha} \tag{10}$$

and β is given by

$$\beta = \frac{1 + \alpha^2|a|^2 - |1 - \alpha^2 a^2|}{4\alpha\Re(a)} \quad (\alpha > 0,\ \Re(a) > 0). \tag{11}$$

Then the subordination condition:

$$\left[\frac{\mathscr{I}_{\mu}^{\lambda}(a,b,c)f(z)}{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)f(z)}\right]\left[\frac{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathscr{I}_{\mu}^{\lambda}(a,b,c)g(z)}{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)g(z)}\right]\left[\frac{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)g(z)}{z}\right]^{\alpha} \tag{12}$$

implies that

$$\left[\frac{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)f(z)}{z}\right]^{\alpha} \prec \left[\frac{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)g(z)}{z}\right]^{\alpha}, \tag{13}$$

where $\mathscr{I}_{\mu}^{\lambda}(a,b,c)$ is the integral operator defined by (5). Moreover, the function $\left[\frac{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)g(z)}{z}\right]^{\alpha}$ is the best dominant.

Proof. Let us define the functions $F(z)$ and $G(z)$ in $\mathbb{U}$ by

$$F(z) := \left[\frac{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)f(z)}{z}\right]^{\alpha}$$

and

$$G(z) := \left[\frac{\mathscr{I}_{\mu}^{\lambda}(a+1,b,c)g(z)}{z}\right]^{\alpha} \quad (z \in \mathbb{U}) \tag{14}$$

respectively. Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and that $G'(\xi) \neq 0 \quad (|\xi| = 1)$. Otherwise, we replace the functions $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$ respectively for $0 < \rho < 1$. These new functions have the desired properties on $\overline{\mathbb{U}}$, we can use them in the proof of our result. Therefore, our result would follow by letting $\rho \longrightarrow 1$.

Now, we show that, if the function $q(z)$ is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{15}$$

then

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}). \tag{16}$$

Taking logarithmic differentiation on both sides of the second equation in (14) and using the identity (8) for $g \in \mathscr{A}$ in the resulting equation, we get

$$\phi(z) = G(z) + \frac{zG'(z)}{\alpha a} \tag{17}$$

where the function $\phi(z)$ is defined in (10).

Differentiating both sides of (17) with respect to z gives

$$\phi'(z) = \left(1 + \frac{1}{\alpha a}\right)G'(z) + \frac{zG''(z)}{\alpha a}. \tag{18}$$

From (15) and (18) after simplification yields

$$\begin{aligned}1 + \frac{z\phi''(z)}{\phi'(z)} &= 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + \alpha a} \\ &= q(z) + \frac{zq'(z)}{q(z) + \alpha a} \equiv h(z) \quad (z \in \mathbb{U}).\end{aligned} \tag{19}$$

Therefore, it follows from (9) and (19) that

$$\Re\{h(z) + \alpha a\} > 0 \quad (z \in \mathbb{U}). \tag{20}$$

Hence by Lemma 2 we deduce that the differential equation (19) has a solution $q\in\mathscr{H}(\mathbb{U})$ with $h(0)=q(0)=1$.
Let us define the function

$$\mathscr{H}(u,v)=u+\frac{v}{u+\alpha a}+\beta, \tag{21}$$

where β is given by (11). From (9), (19) and (21), we have

$$\Re\{\mathscr{H}\left(q(z),zq'(z)\right)\}>0 \quad (z\in\mathbb{U}).$$

Now for all real s and $t\leq -\frac{(1+s^2)}{2}$ we want to verify that

$$\Re\{\mathscr{H}(is,t)\}\leq 0. \tag{22}$$

From (21), we have

$$\begin{aligned}\Re\{\mathscr{H}(is,t)\} &=\Re\left\{is+\frac{t}{is+\alpha a}+\beta\right\}\\ &=\frac{t\alpha\Re(a)}{|\alpha a+is|^2}+\beta\\ &\leq -\frac{H_\beta(s)}{2\,|\alpha a+is|^2}\end{aligned} \tag{23}$$

where

$$H_\beta(s)=(\alpha\Re(a)-2\beta)s^2-4\alpha\beta\Im(a)s \\ -2\alpha^2\beta|a|^2+\alpha\Re(a). \tag{24}$$

For β given by (11), we observe that the coefficient of s^2 in the quadratic expression $H_\beta(s)$ given by (24) is positive or equal to zero. To check this, put $\alpha a=r$, so that

$$\alpha\Re(a)=r_1 \quad\text{and}\quad \alpha\Im(a)=r_2.$$

Thus we have to verify that

$$r_1-2\beta\geq 0,$$

or

$$r_1\geq 2\beta=\frac{1+|r|^2-|1-r^2|}{2r_1}.$$

This inequality will hold true if

$$2r_1^2+|1-r^2|\geq 1+|r|^2=1+r_1^2+r_2^2,$$

that is, if

$$|1-r^2|\geq 1-\Re(r^2),$$

which is obviously true. Moreover, the quadratic expression for s in $H_\beta(s)$ given by (24) is a perfect square for the assumed value of β given by (11). Therefore, it follows from (23) that

$$\Re\{\mathscr{H}(is,t)\}\leq 0,$$

for all real s and $t\leq -\frac{1+s^2}{2}$. Thus by application of Lemma 1, we conclude that

$$\Re\{q(z)\}>0 \quad (z\in\mathbb{U}),$$

that is, the function $G(z)$ defined by (14) is convex (univalent) in $\mathbb{U}$.
Next, we prove that the subordination condition (12) implies that

$$F(z)\prec G(z) \quad (z\in\mathbb{U}) \tag{25}$$

for the functions F and G defined by (14).
To prove (25), let us define the function $L(z,t)$ by

$$L(z,t):=G(z)+\frac{1+t}{\alpha a}zG'(z) \quad (z\in\mathbb{U};\ 0\leq t<\infty). \tag{26}$$

We note that

$$\frac{\partial L(z,t)}{\partial z}\Big|_{z=0}=G'(0)\left[1+\frac{1+t}{\alpha a}\right]\neq 0 \quad (z\in\mathbb{U};\ 0\leq t<\infty).$$

This shows that the function

$$L(z,t)=d_1(t)z+\cdots$$

satisfies the conditions $d_1(t)\neq 0$ for all $t\in[0,\infty)$. Furthermore,

$$\Re\left\{\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\}=\Re\left(\alpha a+(1+t)q(z)\right)>0 \quad (z\in\mathbb{U}),$$

since G is convex and $\Re(a)>0$. Thus, by virtue of Lemma 4, $L(z,t)$ is a subordination chain. Hence, it follows from Definition 4 that

$$\phi(z)=G(z)+\frac{zG'(z)}{\alpha a}=L(z,0)$$

and

$$L(z,0)\prec L(z,t) \quad (z\in\mathbb{U};\ 0\leq t<\infty).$$

This implies that

$$L(\xi,t)\notin L(\mathbb{U},0)=\phi(\mathbb{U}) \quad (\xi\in\partial\mathbb{U}; 0\leq t<\infty). \tag{27}$$

Now suppose that the function F is not subordinate to G, then by Lemma 3 there exists two points $z_0\in\mathbb{U}$ and $\xi_0\in\partial\mathbb{U}$ such that

$$F(z_0)=G(\xi_0)\ \ \textit{and}\ \ z_0F'(z_0)=(1+t)\xi_0G'(\xi_0) \tag{28}$$

$$(0\leq t<\infty).$$

Hence, we have

$$\begin{aligned}L(\xi_0,t)= &\quad G(\xi_0)+(1+t)\frac{\xi_0G'(\xi_0)}{\alpha a}\\ = &\quad F(z_0)+\frac{z_0F'(z_0)}{\alpha a}\\ = &\quad \left[\frac{\mathscr{I}_\mu^\lambda(a+1,b,c)f(z_0)}{z_0}\right]^\alpha\\ &\left[\frac{\mathscr{I}_\mu^\lambda(a,b,c)f(z_0)}{\mathscr{I}^\lambda(a+1,b,c)f(z_0)}\right]\in\phi(\mathbb{U}),\end{aligned}$$

by virtue of the subordination condition (12). This contradicts (27). Thus, the subordination condition (12) must imply the subordination given by (25). Considering

$F = G$, we see that the function $G(z)$ is the best dominant. Thus, the prove of Theorem 1 is completed. □

Next theorem gives subordination result with respect to the variation of parameter λ.
By employing the same technique as in the proof of Theorem 1 and using the identity (7) instead of (8), we obtain the following result in form of the theorem.

Theorem 2. Let $f, g \in \mathscr{A}$ and suppose that

$$\Re\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\gamma \quad (z \in \mathbb{U}) \tag{29}$$

where

$$\psi(z) = \left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)g(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha \tag{30}$$

and γ is given by

$$\gamma = \frac{1+\alpha^2(1+\lambda)^2 - |1-\alpha^2(1+\lambda)^2|}{4\alpha(1+\lambda)} \tag{31}$$

$$(\alpha > 0,\ \lambda > -1).$$

Then the subordination condition:

$$\left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)f(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{z}\right]^\alpha$$
$$\prec \left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)g(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha, \tag{32}$$

implies that

$$\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha, \tag{33}$$

where $\mathscr{I}_\mu^{\lambda}(a,b,c)$ is the integral operator defined by (5). Moreover, the function $\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha$ is the best dominant.

Taking $\lambda = \mu = 0$, $a = 1$ and $b = c$ in Theorem 2 , we get a result of Shenan (see [16], Corollary 2) for $p = 1$ as follows:

Corollary 1. Let $f,\ g \in \mathscr{A}$ and suppose that

$$\Re\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\gamma \quad (z \in \mathbb{U}),$$

where

$$\psi(z) = \left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha \quad (\alpha > 0; z \in \mathbb{U}) \tag{34}$$

and γ is given by

$$\gamma = \frac{1+\alpha^2 - |1-\alpha^2|}{4\alpha}. \tag{35}$$

Then the subordination condition:

$$\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\alpha \prec \left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha,$$

implies that

$$\left[\frac{f(z)}{z}\right]^\alpha \prec \left[\frac{g(z)}{z}\right]^\alpha,$$

and the function $\left[\frac{g(z)}{z}\right]^\alpha$ is the best dominant.

Next theorem provides a solution to a dual problem of Theorem 1 in the sense that the subordinations are replaced by superordinations.

Theorem 3. Let $f, g \in \mathscr{A}$ and $\Re(a) > 0$. Suppose that

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\beta$$

where $\phi(z)$ and β are given by (10) and (11) respectively. If the function $\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g(z)}{z}\right]^\alpha$ is univalent in $\mathbb{U}$ and $\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g(z)}{z}\right]^\alpha \in \mathscr{Q}$, then the superordination condition:

$$\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g(z)}{z}\right]^\alpha$$
$$\prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}{z}\right]^\alpha, \tag{36}$$

implies that

$$\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}{z}\right]^\alpha. \tag{37}$$

Moreover, the function $\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g(z)}{z}\right]^\alpha$ is the best subordinant.

Proof. The proof of the theorem follows the same lines as that of Theorem 1. We will give only main steps.
Let the functions F, G and q are defined by (14) and (15) respectively. As in the proof of Theorem 1, we have

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}),$$

that is G is defined by (14) is convex (univalent) in $\mathbb{U}$.

Next, to arrive at our desired result, we show that

$$G \prec F \quad (z \in \mathbb{U}). \tag{38}$$

For this purpose, we defined the function $L(z,t)$ as (26).

Since G is convex and $\Re(a) > 0$, by applying a similar method as in Theorem 1 we conclude that $L(z,t)$ is a subordination chain. Therefore, by making using of Lemma 5, we deduce that the superordination condition

(36) must imply (38). Furthermore, since the differential equation

$$\phi(z) = G(z) + \frac{zG'(z)}{\alpha a} = \varphi(G(z), zG'(z))$$

has a univalent solution G, it is the best subordinant of the given differential superordination. This completes the proof of Theorem 3. □

The following theorem provides a solution to a dual problem of Theorem 2.

Theorem 4. Let $f, g \in \mathscr{A}$ and suppose that

$$\Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\gamma \quad (z \in \mathbb{U}),$$

where $\psi(z)$ and γ are given by (30) and (31) respectively. If the function $\left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)g(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha$ is univalent in $\mathbb{U}$ and $\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha \in \mathscr{Q}$, then the superordination condition:

$$\left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)g(z)}{\mathrm{I}_\mu^{\lambda}(a,b,c)g(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)f(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{z}\right]^\alpha,$$

implies that

$$\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{z}\right]^\alpha.$$

Moreover, the function $\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g(z)}{z}\right]^\alpha$ is the best subordinant.

Putting $\lambda = \mu = 0$, $a = 1$ and $b = c$ in Theorem 4, we get a result of Shenan (see [16], Corollary 4) for $p = 1$ as follows:

Corollary 2. Let $f,\ g \in \mathscr{A}$ and suppose that

$$\Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\gamma$$

where $\psi(z)$ and γ are given by (34) and (35) respectively. If the function $\left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha$ is univalent in $\mathbb{U}$ and $\left[\frac{g(z)}{z}\right]^\alpha \in \mathscr{Q}$, then the superordination condition

$$\left[\frac{zg'(z)}{g(z)}\right]\left[\frac{g(z)}{z}\right]^\alpha \prec \left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\alpha,$$

implies that

$$\left[\frac{g(z)}{z}\right]^\alpha \prec \left[\frac{f(z)}{z}\right]^\alpha,$$

and the function $\left[\frac{g(z)}{z}\right]^\alpha$ is the best subordinant.

Combining Theorems 1, 3 and Theorems 2, 4, we obtain Theorem 5 and Theorem 6 respectively, so called "sandwich-type results".

Theorem 5. Let $f, g_i \in \mathscr{A}$ $(i = 1,2)$ and suppose that

$$\Re\left\{1 + \frac{z\phi_i''(z)}{\phi_i'(z)}\right\} > -\beta$$

where

$$\phi_i(z) = \left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_i(z)}{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_i(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_i(z)}{z}\right]^\alpha$$

and β is given by (11). If the function $\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}{z}\right]^\alpha$ is univalent in $\mathbb{U}$ and $\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}{z}\right]^\alpha \in \mathscr{Q}$, then the condition:

$$\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_1(z)}{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_1(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_1(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_2(z)}{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_2(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_2(z)}{z}\right]^\alpha$$

implies that

$$\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_1(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)f(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_2(z)}{z}\right]^\alpha.$$

Moreover, the function $\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_1(z)}{z}\right]^\alpha$ and $\left[\frac{\mathscr{I}_\mu^{\lambda}(a+1,b,c)g_2(z)}{z}\right]^\alpha$ are respectively the best subordinant and the best dominant.

Theorem 6. Let $f, g_i \in \mathscr{A}$ $(i = 1,2)$ and suppose that

$$\Re\left\{1 + \frac{z\psi_i''(z)}{\psi_i'(z)}\right\} > -\gamma \quad (z \in \mathbb{U})$$

where

$$\psi_i(z) = \left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)g_i(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)g_i(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_i(z)}{z}\right]^\alpha,$$

$$(\lambda > -1,\ \alpha > 0, \mu \geq 0;\ z \in \mathbb{U})$$

and γ is given by (31). If the function $\left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)f(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{z}\right]^\alpha$ is univalent in $\mathbb{U}$ and $\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{z}\right]^\alpha \in \mathscr{Q}$, then the condition:

$$\left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)g_1(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)g_1(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_1(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)f(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda+1}(a,b,c)g_2(z)}{\mathscr{I}_\mu^{\lambda}(a,b,c)g_2(z)}\right]\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_2(z)}{z}\right]^\alpha$$

implies that

$$\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_1(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)f(z)}{z}\right]^\alpha \prec \left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_2(z)}{z}\right]^\alpha.$$

Moreover, the function $\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_1(z)}{z}\right]^\alpha$ and $\left[\frac{\mathscr{I}_\mu^{\lambda}(a,b,c)g_2(z)}{z}\right]^\alpha$ are respectively the best subordinant and the best dominant.

Letting $\lambda = \mu = 0$, $a = 1$ and $b = c$ in Theorem 6, we get the following result due to Shenan (see [16], Corollary 6) for $p = 1$.

Corollary 3. Let $f,\ g_i \in \mathscr{A}$ $(i = 1,2)$ and suppose that

$$\Re\left\{1 + \frac{z\psi_i''(z)}{\psi_i'(z)}\right\} > -\gamma$$

where

$$\psi_i(z) = \left[\frac{zg_i'(z)}{g_i(z)}\right]\left[\frac{g_i(z)}{z}\right]^\alpha \quad (\alpha > 0;\ z \in \mathbb{U}),$$

and γ is given by (35). If the function $\left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\alpha$ is univalent in $\mathbb{U}$ and $\left[\frac{f(z)}{z}\right]^\alpha \in \mathscr{Q}$, then the condition:

$$\left[\frac{zg_1'(z)}{g_1(z)}\right]\left[\frac{g_1(z)}{z}\right]^\alpha \prec \left[\frac{zf'(z)}{f(z)}\right]\left[\frac{f(z)}{z}\right]^\alpha \prec \left[\frac{zg_2'(z)}{g_2(z)}\right]\left[\frac{g_2(z)}{z}\right]^\alpha,$$

implies that

$$\left[\frac{g_1(z)}{z}\right]^\alpha \prec \left[\frac{f(z)}{z}\right]^\alpha \prec \left[\frac{g_2(z)}{z}\right]^\alpha.$$

Moreover, the function $\left[\frac{g_1(z)}{z}\right]^\alpha$ and $\left[\frac{g_2(z)}{z}\right]^\alpha$ are respectively the best subordinant and the best dominant.

References

[1] T. Bulboacă, Integral operators that preserve the subordination, Bull. Korean Math. Soc., **34**, 627-636 (1997).

[2] N. E. Cho, I. H. Kim and H. M. Srivastava, Sandwich-type theorems for multivalent functions associaed with the Srivastava-Attiya operator, Appl. Math. Comput., **217**, 918-928 (2010).

[3] P. L. Duren, Univalent Functions, Graduate Texts in Mathematics; vol. 259, Springer-Verlag, New York, 1983.

[4] S. P. Goyal, P. Goswami and H. Silverman, Subordination and superordination results for a class of analytic multivalent functions, Int. J. Math. Math. Sci. (2008), doi: 10.1155/2008/561638.

[5] O. S. Kwon and N. E. Cho, Subordination and superordination for multivalent functions associated with the differintegral operator, Kyungpook Math. J., **53**, 1-12 (2013).

[6] J.-L. Liu and H. M. Srivastava, Certain properties of the Dziok-Srivastava operator, Appl. Math. Comput., **159**, 485-493 (2004).

[7] K. I. Noor, Integral operators defined by convolution with hypergeometric functions, Appl. Math. Comput., **182**, 1872-1881 (2006).

[8] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., **28**, 157-172 (1981).

[9] S. S. Miller, P. T. Mocanu and M. O. Reade, Subordination-preserving integral operators, Trans. Amer. Math. Soc., **283**, 605-615 (1984).

[10] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, J. Differential Equations, **56**, 297-309 (1985).

[11] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, in: Monographs and Text Books in Pure and Applied Mathematics, **225**, Marcel Dekker, New York, 2000.

[12] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Var. Theory Appl. **48**(10), 815-826 (2003).

[13] S. Owa and H. M. Srivastava, Some subordination theorems involving a certain family of integral operators, Integral Transforms Spec. Funct, **45**, 445-454 (2004).

[14] J. K. Prajapat, Subordination and superordination preserving properties for generalized multiplier transformation operator, Math. Comput. Modelling, **55**, 1456-1465 (2012).

[15] Ch. Pommerenke, Univalent Functions, Vanderhoeck and Ruprecht, Göttingen, Germany, 1975.

[16] J. M. Shenan, Preserving properties of subordination and superordination of analytic functions associated with fractional differintegral operator, Tamkang J. Math., **45**(1), 63-75 (2014).

[17] H. M. Srivastava, E. Deniz and H. Orhan, Some general univalence criteria for a family of integral operators, Appl. Math. Comput., **215**, 3696-3701 (2010).

[18] H. M. Srivastava, S. M. Khairnar and Meena More, Inclusion properties of a subclass of analytic functions defined by an integral operator involving the Gauss hypergeometric function, Appl. Math. Comput., **218**, 3810-3821 (2011).

[19] H. M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific, Singapore, 1992.
[20] Z.-G. Wang, N.-E. Xu and M. Acu, Certain subclasses of multivalent analytic functions defined by multiplier transforms, Appl. Math. Comput., **216**, 192-204 (2010).
[21] Z.-G. Wang, G.-W. Zhang and F.-H. Wen, Properties and charateristics of the Srivastava-Khairnar-More integral operator, Appl. Math. Comput., **218**, 7747-7758 (2012).

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