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Variational Iteration Method and He's Polynomials for Time-Fractional Partial Differential Equations

Abdolali Neamaty[1,*], *Bahram Agheli*[1] *and Rahmat Darzi*[2]

[1] Department of Mathematics, University of Mazandaran, Babolsar, Iran
[2] Department of Mathematics, Neka Branch, Islamic Azad University, Neka, Iran

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Abstract: In this work, we have applied the variational iteration method and He's polynomials to solve partial differential equation (PDEs) with time-fractional derivative. The variational homotopy perturbation iteration method (VHPIM) is presented in two steps. Some illustrative examples are given in order to show the ability and simplicity of the approach. All numerical calculations in this manuscript were performed on a PC applying some programs written in $Maple18$.

Keywords: Variational-iteration method (VIM), Homotopy-perturbation method (HPM), fractional partial differential equation (FPDEs).

1 Introduction

In recent years, fractional area has been one of the most interesting issues that has attracted many scientists, specially in the fields of mathematics and engineering sciences. Many natural phenomena can be presented by boundary value problems of fractional differential equations.

Many authors in various fields such as chemical physics, fluid flows, electrical networks, viscoelasticity, have tried to present a model of these phenomena by fractional differential equations [1,2]. In order to achieve extra information in fractional calculus, interested readers can refer to more valuable books that are written by other authors [3,4,5]. Most fractional differential equations do not have accurate analytical solutions. Therefore, direct and iterative estimate methods are used. In the recent years, several methods have been used to solve FDEs and FPDEs as ADM [6,7,8], VIM [9,10,11, 12], HPM[13,14,15,16], HAM[17,18] and so on[19,20,21].

In this research work, combining two different methods VIM and HPM is purposed for solving the time-fractional partial differential equation.

The purpose behind this research work is to use He's homotopy offered by He [22,23] and extend its application in order to solve FPDEs. First using an algorithm, after combining VIM and HPM methods, in two steps, we arrive to the following equation:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) + p\left\{\sum_{n=1}^{\infty} p^n u_n(x,t) + I^{\beta}\left(\lambda(t)(\sum_{n=0}^{\infty} p^n D^{\alpha} u_n(x,t) - \sum_{n=0}^{\infty} p^n \Re(\breve{u}_n, (\breve{u}_n)_x, (\breve{u}_n)_{xx}) - f(x,t))\right)\right\}.$$

Lagrange multiplier λ is calculated with VIM method. Eventually using HPM method for $p \to 1$, we get an estimate of the solution, that is:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

The present research work is arranged in five sections. In Section 2, the preliminaries and a reliable algorithm of VHPIM are given. In Section 3, we describe the estimate of solution. In Section 4, as the applications of this method,

* Corresponding author e-mail: namaty@umz.ac.ir

time-fractional advection, hyperbolic and Fisher equations have been sensibly solved. A conclusion is presented in Section 5.

2 Preliminaries and a reliable algorithm of VHPIM

In this part of the paper, we present and define Riemann-Liouville fractional integral and Caputo's fractional derivative [5]. Then VHIPM method is introduced and explained in detail.

Definition 1. A real function $f(x)$, $x>0$, is purposed to be in the space C_ν, $(\nu \in R)$, if there exists a real number $n(>\nu)$, so that $f(x)=x^n f_1(x)$, where $f_1(x)\in C[0,\infty)$, and $f\in C_\nu^k$ iff $f^{(k)}\in C_\nu$, $k\in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order of $\alpha>0$, $f\in C_\nu$, $\nu\geq -1$, is given:

$$I_a^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-r)^{\alpha-1} f(r)dr,$$

$$I^\alpha f(x)=I_0^\alpha f(x), \quad I^0 f(x)=f(x).$$

Definition 3. The Caputo's fractional derivative of f is defined:

$$D^\alpha f(x)=I^{k-\alpha}D^k f(x)=\frac{1}{\Gamma(k-\alpha)}\int_0^x (x-r)^{k-\alpha-1} f^{(k)}(r)dr, \quad x>0.$$

where, $f\in C_{-1}^k$, $k-1<\alpha\leq k$ and $k\in\mathbb{N}$.

Property 1. For $k-1<\alpha\leq k$, $k\in\mathbb{N}$, $f\in C_\nu^k$, $\nu\geq -1$ and $x>0$, the following properties satisfy

i) $D_a^\alpha I_a^\alpha f(x)=f(x)$.

ii) $I_a^\alpha D_a^\alpha f(x)=f(x)-\sum\limits_{j=0}^{k-1} f^{(j)}(a^+)\frac{(x-a)^j}{j!}$.

First of all, we purpose the following nonlinear problem, with two variables x and t:

$$D_t^\alpha u(x,t)=\mathfrak{R}\, u(x,t)+f(x,t), \tag{1}$$

where D_t^α is the fractional Caputo derivative with respect to t, $\alpha>0$, $\mathfrak{R}$ is an operator in x, and t which might include derivatives with respect to "x", $u(x,t)$ is an uncertain function, and $f(x,t)$ is the origin in homogeneous sentence.
Following this, VHPM is introduced and explained in two steps.

Step 1. Pursuant to VIM, we create the revision functional on Eq.(1):

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t)+I^\beta\left[\lambda(t)\left(D^\alpha u_n(x,t)-\mathfrak{R}\big(\breve{u}_n,(\breve{u}_n)_x,(\breve{u}_n)_{xx}\big)-f(x,t)\right)\right] \\ &= u_n(x,t)+\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\lambda(\tau)\left(D^\alpha u_n(x,\tau)-\mathfrak{R}\big(\breve{u}_n,(\breve{u}_n)_x,(\breve{u}_n)_{xx}\big)-f(x,\tau)\right)d\tau, \end{aligned} \tag{2}$$

where I^β is the Riemann-Liouville fractional integral operator of under $\beta=\alpha-floor(\alpha)$, that is $\beta=\alpha+1-m$, and λ is a common Lagrange multiplier, which may be discerned through variational principle.
The function $\breve{u}_k$ is regarded on a confined variation, that is $\delta\breve{u}_k=0$.

Step 2. By applying the VIM and HPM, we create the following repetition equation:

$$\sum_{n=0}^\infty p^n u_n(x,t)=u_0(x,t)+p\left\{\sum_{n=1}^\infty p^n u_n(x,t)+I^\beta\left(\lambda(t)(\sum_{n=0}^\infty p^n D^\alpha u_n(x,t)-\sum_{n=0}^\infty p^n\mathfrak{R}(\breve{u}_n,(\breve{u}_n)_x,(\breve{u}_n)_{xx})-f(x,t))\right)\right\}. \tag{3}$$

which is named as VHPIM.

In the Eq.(3), u_0 is primary estimation of Eq.(1), and $p\in[0,1]$ is embedded parameter.
Equating the sentences with identical powers of p in the Eq.(3), we can acquire u_i $(i=0,1,2,\ldots)$.
Finally, pursuant to HPM, for $p\to 1$, we estimate the solution:

$$u(x,t)=\sum_{n=0}^\infty u_n(x,t). \tag{4}$$

3 Estimate of solution

Pursuant to VIM, the revision functional Eq.(2) may be estimable:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\left(\frac{d^m}{dt^m}u(x,\tau) - \Re\left(\breve{u}_n, (\breve{u}_n)_x, (\breve{u}_n)_{xx}\right) - f(x,\tau)\right)d\tau,$$

where $\breve{u}_n$ is a revision functional, but $\breve{u}_n$ is regarded on a confined variation, i.e $\delta \breve{u}_n(x,t) = 0$.
Next, by making functional still:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) \ + \delta \int_0^t \lambda(\tau)\left(\frac{d^m}{dt^m}u(x,\tau) - f(x,\tau)\right)d\tau,$$

we acquire the Lagrange multiplier as

1) $m = 1: \quad \lambda = -1,$
2) $m = 2: \quad \lambda = \tau - t.$

Case 1. If $m = 1$, i.e in case $0 < \alpha \leq 1$, substituting $\lambda = -1$ into Eq.(2.2), we acquire the repetition formula:

$$u_{n+1}(x,t) = u_n(x,t) - I^\alpha\left(D^\alpha u_n(x,t) - \Re\, u_n(x,t) - f(x,t)\right).$$

Using Eq.(3), we may construct the repetition formula as follows:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) + p\left\{\sum_{n=1}^{\infty} p^n u_n(x,t) - I^\alpha\Big(\sum_{n=0}^{\infty} p^n D^\alpha u_n(x,t) - \sum_{n=0}^{\infty} p^n \Re\left(\breve{u}_n, (\breve{u}_n)_x, (\breve{u}_n)_{xx}\right) - f(x,t)\Big)\right\}. \tag{5}$$

contrasting the sentences with identical powers of p in the Eq.(5), we can acquire $u_i(x,t),\ (i = 0, 1, 2, 3, \ldots)$.
Pursuant to HPM, we have:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots,$$

which is an estimate solution for Eq.(1).
Case 2. For $m = 2$, i.e in the case $1 < \alpha \leq 2$, we substitute $\lambda = \tau - t$ into functional Eq.(2), to acquire:

$$u_{n+1}(x,t) = u_n(x,t) - \frac{\alpha - 1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(D^\alpha u_n(x,\tau) - \Re\, u_n(x,\tau) - f(x,\tau)\right)d\tau.$$

Therefore, we have:

$$u_{n+1}(x,t) = u_n(x,t) - \beta I^\alpha[D^\alpha u_n(x,t) - \Re\, u_n(x,t) - f(x,t)].$$

where $\beta = \alpha - 1$.
Using Eq.(3), we have the following repetition formula:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) + p\left\{\sum_{n=1}^{\infty} p^n u_n(x,t) - \beta I^\alpha\Big(\sum_{n=0}^{\infty} p^n D^\alpha u_n(x,t) - \sum_{n=0}^{\infty} p^n \Re\left(\breve{u}_n, (\breve{u}_n)_x, (\breve{u}_n)_{xx}\right) - f(x,t)\Big)\right\}. \tag{6}$$

4 Applications and results

This part, we apply VIM and He's polynomials to solve nonlinear time-fractional advection partial differential equation, time-fractional hyperbolic equation,time-fractional Fisher's equation . All of the plots and computations for this equations have been done on a PC applying some programs written in *Maple*18.
Example 1. We purpose the time-fractional advection partial differential equation:

$$\frac{d^\alpha}{dt^\alpha}u(x,t) + u(x,t)\ u_x(x,t) = x(1+t^2), \quad t > 0,\ x \in \mathbb{R},\ 0 < \alpha \leq 1, \tag{7}$$

with the primary condition:

$$u(x,0) = 0. \tag{8}$$

Now if we substitute the primary amount $u(x,0)$ into the repetition formulation Eq. (5), the result will be:

$$\begin{aligned}
p^0: &\quad u_0(x,t) = 0\\
p^1: &\quad u_1(x,t) = \frac{xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{2!xt^{\alpha+2}}{\Gamma(\alpha+3)},\\
p^2: &\quad u_2(x,t) = -\frac{x\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4x\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} -\\
&\qquad \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)\Gamma(\alpha+3)\Gamma(3\alpha+5)},\\
p^3: &\quad u_3(x,t) = -x\left\{\left(\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)^2 \frac{\Gamma(6\alpha+1)}{\Gamma(7\alpha+1)}t^{7\alpha}\right.\\
&\qquad + \frac{8\Gamma(2\alpha+1)\Gamma(2\alpha+3)\Gamma(6\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+1)\Gamma(3\alpha+3)\Gamma(7\alpha+3)}t^{7\alpha+2}\\
&\qquad + \frac{8\Gamma(2\alpha+1)\Gamma(2\alpha+5)\Gamma(6\alpha+5)}{[\Gamma(\alpha+3)]^2\Gamma(3\alpha+1)\Gamma(3\alpha+5)\Gamma(7\alpha+5)}t^{7\alpha+4}\\
&\qquad + \frac{16[\Gamma(2\alpha+3)]^2\Gamma(6\alpha+5)}{[\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)]^2\Gamma(7\alpha+5)}t^{7\alpha+4}\\
&\qquad + \frac{32\Gamma(2\alpha+3)\Gamma(2\alpha+5)\Gamma(6\alpha+7)}{[\Gamma(\alpha+3)]^3\Gamma(\alpha+1)\Gamma(3\alpha+3)\Gamma(3\alpha+5)\Gamma(7\alpha+7)}t^{7\alpha+6}\\
&\qquad \left. + \frac{16[\Gamma(2\alpha+5)]^2\Gamma(6\alpha+9)}{[\Gamma(\alpha+3)]^4[\Gamma(3\alpha+5)]^2\Gamma(7\alpha+9)}t^{7\alpha+8}\right\},\\
&\ldots .
\end{aligned}$$

Then, with purpose the first three sentence as estimate of solution for Eq.(7) is:

$$u(x,t) = \frac{xt^{\alpha}}{\Gamma(\alpha+1)} + \frac{2!xt^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{x\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4x\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} - \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)\Gamma(\alpha+3)\Gamma(3\alpha+5)}.$$

The solution that we have found is match to the accurate solution $u(x,t) = xt$, which is the same third order sentence estimate solution for Eq. (7)-(8) acquired from [25] applying VIM.

We can also solve the advection partial differential equation with time-fractional derivative Eq.(7) in [25] through applying ADM. Accordingly, the third order estimate of the decomposition series solution is presented in [25] on Eq.(7):

$$u(x,t) = x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)}\right).$$

In Table 1, we can see the estimate solutions toward $\alpha = 1$ which is derived for various values of x and t applying VHPIM, VIM, HPM and ADM.

Table 1: Numerical values when $\alpha = 1$ for Eq.(7)

t	x	u_{VIM}	u_{ADM}	u_{HPM}	u_{VHPIM}	u_{Exact}
0.2	0.25	0.050309	0.050000	0.0499876	0.0499876	0.050000
	0.50	0.100619	0.100000	0.099978	0.0999746	0.100000
	0.75	0.150928	0.150001	0.149968	0.149962	0.150000
	1.0	0.201237	0.200001	0.199957	0.199951	0.200000
0.4	0.25	0.101894	0.100023	0.099645	0.0995290	0.100000
	0.50	0.203787	0.200046	0.199290	0.199059	0.200000
	0.75	0.305681	0.300069	0.298935	0.298588	0.300000
	1.0	0.407575	0.400092	0.398580	0.398118	0.400000
0.6	0.25	0.153094	0.150411	0.147158	0.145690	0.150000
	0.50	0.306188	0.300823	0.294317	0.291380	0.300000
	0.75	0.459282	0.451234	0.441475	0.437070	0.450000
	1.0	0.612376	0.601646	0.588634	0.582759	0.600000

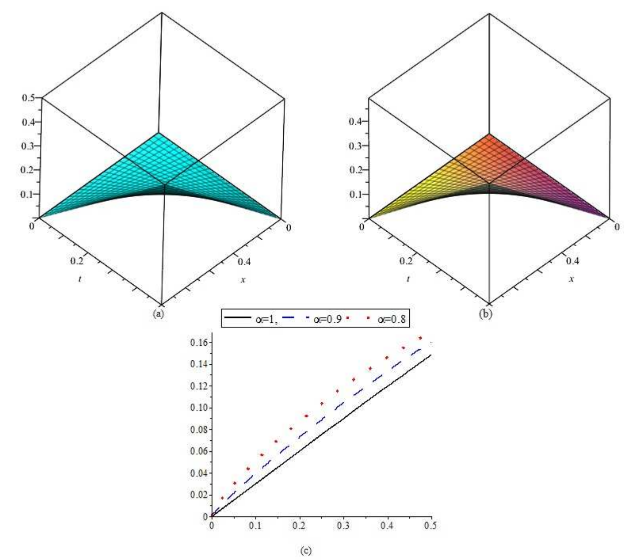

Fig. 1: (a) The accurate solution (b) The estimate solution in the case $\alpha = 1.0$ (c) The third-order Eq.(8) for various value of α when $x = 0.3$.

Example 2. Now we purpose the time-fractional hyperbolic equation:

$$\frac{d^{\alpha}}{dt^{\alpha}}u(x,t) = \frac{\partial}{\partial x}(u(x,t)u_x(x,t)), \quad t > 0,\ x \in \mathbb{R},\ 1 < \alpha \leq 2, \tag{9}$$

with the primary condition:

$$u(x,0) = x^2, \quad u_t(x,o) = -2x^2. \tag{10}$$

Now when we substitute the primary amount $u(x,0)$ in to the repetition formulation (3), we can get:

$$p^0: \quad u_0(x,t) = x^2(1-2t),$$

$$p^1: \quad u_1(x,t) = 6\beta x^2\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right\}$$

$$\begin{aligned} p^2: \quad u_2(x,t) = & 6\beta(1-\beta)^2x^2\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right\} \\ & +216\beta^3x^2\Big\{\frac{\Gamma(2\alpha+1)t^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} - \frac{8\Gamma(2\alpha+2)t^{3\alpha+1}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+2)} \\ & + \frac{64\Gamma(2\alpha+5)t^{3\alpha+4}}{[\Gamma(\alpha+3)]^2\Gamma(3\alpha+5)} + \frac{16\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} \\ & + \frac{16\Gamma(2\alpha+3)t^{3\alpha+2}}{[\Gamma(\alpha+2)]^2\Gamma(3\alpha+3)} - \frac{64\Gamma(2\alpha+4)t^{3\alpha+3}}{\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(3\alpha+4)}\Big\} \end{aligned}$$

$$\cdots .$$

The first three sentence as estimate of solution for (9) is presented:

$$\begin{aligned} u(x,t) = \quad & x^2(1-2t) + 6\beta x^2\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right\} \\ & + 6\beta(1-\beta)^2x^2\left\{\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right\} \\ & +216\beta^3x^2\Big\{\frac{\Gamma(2\alpha+1)t^{3\alpha}}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)} - \frac{8\Gamma(2\alpha+2)t^{3\alpha+1}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+2)} \\ & + \frac{64\Gamma(2\alpha+5)t^{3\alpha+4}}{[\Gamma(\alpha+3)]^2\Gamma(3\alpha+5)} + \frac{16\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} \\ & + \frac{16\Gamma(2\alpha+3)t^{3\alpha+2}}{[\Gamma(\alpha+2)]^2\Gamma(3\alpha+3)} - \frac{64\Gamma(2\alpha+4)t^{3\alpha+3}}{\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(3\alpha+4)}\Big\}. \end{aligned}$$

The resulting solution is match to the accurate solution $u(x,t) = (x/t+1)^2$, which is the same third order sentence estimate solution for (9)-(10) derived from [25] applying VIM.
Through applying ADM, time-fractional hyperbolic differential equation (9) can also be solved in [25]. The first three sentences of the decomposition series solution on (9) in [25]:

$$u(x,t) = x^2(1-2t) + 6x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right) + 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right).$$

Table 2: Numerical values when $\alpha = 2$ for Eq.(9)

t	x	u_{VIM}	u_{ADM}	u_{HPM}	u_{VHPIM}	u_{Exact}
0.2	0.25	0.043400	0.0433951	0.043400	0.0432049	0.043403
	0.50	0.173600	0.173580	0.173600	0.172820	0.173611
	0.75	0.390600	0.390556	0.390600	0.388844	0.390625
	1.0	0.694400	0.694321	0.694400	0.691278	0.694444
0.4	0.25	0.031779	0.031567	0.031779	0.0299125	0.031888
	0.50	0.127118	0.126268	0.127118	0.119650	0.127551
	0.75	0.286015	0.284103	0.286015	0.269212	0.286990
	1.0	0.508471	0.505072	0.508471	0.478600	0.508471
0.6	0.25	0.023665	0.022005	0.023665	0.0188604	0.024414
	0.50	0.094660	0.088018	0.094660	0.0754415	0.097656
	0.75	0.212984	0.198040	0.212984	0.169743	0.219727
	1.0	0.378638	0.352071	0.378638	0.301766	0.390625

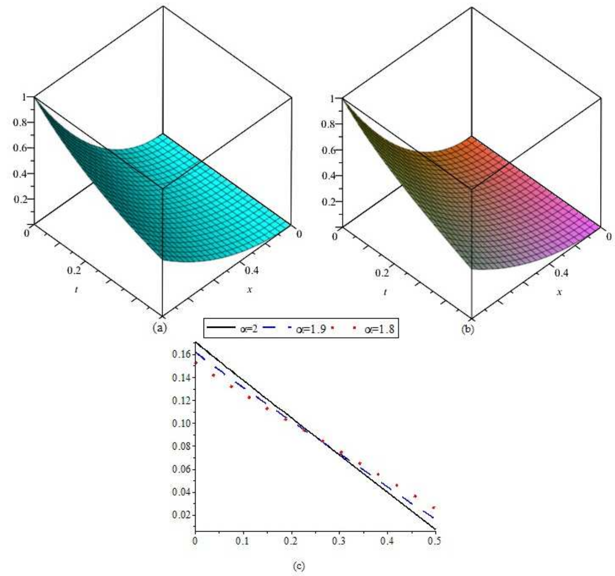

Fig. 2: (a) The accurate solution (b) The estimate solution in the case $\alpha = 2.0$ (c) The third-order Eq.(9) for various value of α when $x = 0.3$.

The estimate solutions for $\alpha = 2$ acquired for various values of x and t applying VHPIM, VIM, HPM and ADM, is shown in Table 2.

Example 3. In this example, we choose the time-fractional Fisher's equation:

$$\frac{d^{\alpha}}{dt^{\alpha}}u(x,t) = u_{xx}(x,t) + 6u(x,t)(1-u(x,t)), \quad t > 0,\ x \in \mathbb{R},\ 0 < \alpha \le 1, \tag{11}$$

with the primary condition:

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{12}$$

Now by substituting the primary amount $u(x,0)$ in to the repetition formulation Eq.(3), we can get:

$$p^0: \quad u_0(x,t) = \frac{1}{(1+e^x)^2},$$

$$p^1: \quad u_1(x,t) = \frac{10e^x}{(1+e^x)^3}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$p^2: \quad u_2(x,t) = \frac{(70e^x+50e^{2x}+100e^{3x})}{(1+e^x)^6}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{600e^{2x}}{(1+e^x)^6}\frac{\Gamma(2\alpha+1)(t^{3\alpha})}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)},$$

$\dots$.

The third order sentence estimate solution for (4.5) is given by:

$$u(x,t) = \frac{1}{(1+e^x)^2} + \frac{(70e^x+50e^{2x}+100e^{3x})}{(1+e^x)^6}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{10e^x}{(1+e^x)^3}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{600e^{2x}}{(1+e^x)^6}\frac{\Gamma(2\alpha+1)(t^{3\alpha})}{[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)}.$$

The solution found here is match to the accurate solution $u(x,t) = 1/(1+e^{x-5t})^2$, which results in the same third order sentence estimate solution on Eq.(11)-(12) acquired from [25] through applying VIM.

We can further solve the time-fractional advection partial differential equation (11) in [25] applying ADM. The third order sentences of the decomposition series solution on Eq.(11) in [25]:

$$u(x,t) = \frac{1}{(1+e^x)^2} + \frac{10e^x}{(1+e^x)^3}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{15e^x(2e^x-1)}{(1+e^x)^4}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

The estimate solutions for $\alpha = 1$ acquired for various values of x and t applying VHPIM, VIM, HPM and ADM, can be seen in Table 3.

Table 3: Numerical values when $\alpha = 1$ for Eq.(11)

t	x	u_{VIM}	u_{ADM}	u_{HPM}	u_{VHPIM}	u_{Exact}
0.1	0.25	0.315940	0.317948	0.315940	0.328019	0.316042
	0.50	0.249926	0.250500	0.249926	0.256513	0.250000
	0.75	0.191606	0.190964	0.191606	0.194303	0.191689
	1.0	0.142411	0.140979	0.142411	0.142715	0.142537
0.2	0.25	0.459320	0.481199	0.459320	0.512193	0.461284
	0.50	0.386450	0.396941	0.386450	0.414697	0.387456
	0.75	0.315478	0.315266	0.315478	0.324716	0.316042
	1.0	0.249092	0.241175	0.249092	0.245881	0.250000
0.3	0.25	0.591179	0.681440	0.591179	0.630275	0.604195
	0.50	0.527635	0.581861	0.527635	0.507643	0.534447
	0.75	0.459719	0.475833	0.459719	0.488298	0.461284
	1.0	0.387025	0.372917	0.387025	0.378472	0.387456

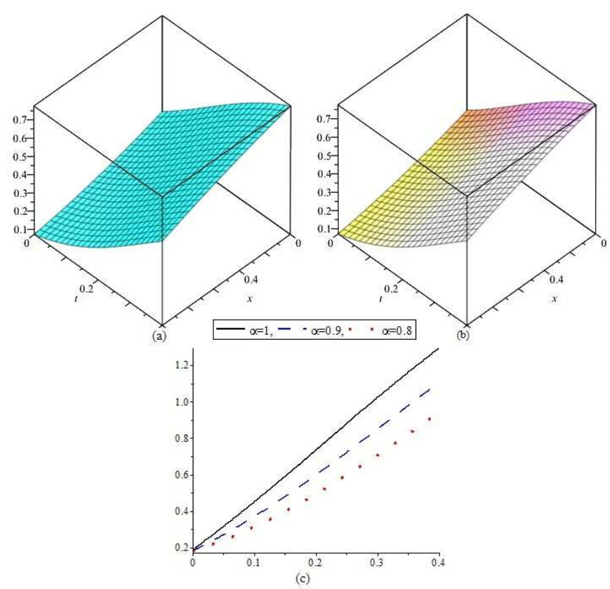

Fig. 3: (a) The accurate solution (b) The estimate solution in case $\alpha = 1.0$ (c) The third-order Eq.(11) for various value of α when $x = 0.3$.

5 Conclusions

In this paper,the combination of two different methods VIM and HPM is proposed for solving the time-fractional partial differential equation. As some applications of this method, the time-fractional advection, the hyperbolic and the Fisher equations were solved. The estimate of results show that the VHPIM is strong, new and impressive. All numerical computations in this study were done within $Maple18$.

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