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Second-Order Approximation for the Space Fractional Diffusion Equation with Variable Coefficient

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Abstract: In this paper, we consider a type of fractional diffusion equation (FDE) with variable coefficient on a finite domain. Firstly, we utilize a second-order scheme to approximate the Riemann-Liouville fractional derivative and present the finite difference scheme. Specifically, we discuss the Crank-Nicolson scheme and solve it in matrix form. Secondly, we prove the stability and convergence of the scheme and conclude that the scheme is unconditionally stable and convergent with the accuracy of $\mathscr{O}(\tau^2+h^2)$. Finally, comparing to the general first order scheme, two numerical examples are given to show the effectiveness and accuracy of our numerical method, and the results are in excellent agreement with theoretical analysis.

Keywords: finite difference method, Riemann-Liouville fractional derivative, fractional diffusion equation, Crank-Nicolson scheme, variable coefficient.

1 Introduction

In the past decades, fractional derivatives have been successfully in physics [1], biology [2], chemistry [3], hydrology [4,5,6], and finance [7]. Considerable numerical methods for solving the FDE have been proposed. Meerschaert and Tadjeran [8] developed practical numerical methods to solve the one-dimensional space FDE with variable coefficients on a finite domain. Liu et al. [6] transformed the space fractional Fokker-Planck equation into a system of ordinary differential equations (method of lines), which was then solved using backward differentiation formulas. Momani and Odibat [9] developed two reliable algorithms, the Adomian decomposition method and variational iteration method, to construct numerical solutions of the space-time FDE in the form of a rapidly convergent series with easily computable components. Zhuang et al. [10] discussed a variable-order fractional advection-diffusion equation with a nonlinear source term on a finite domain. Hristov [11] solved the fractional (half-time) sub-model of the heat diffusion equation by the heat-balance integral method and a parabolic profile with unspecified exponent. Yang et al. [12] analyzed the diffusion equation on Cantor space-time and obtained the approximation solutions by using the local fractional Adomian decomposition method. Yang et al. [13] also employed the local fractional variational iteration method to handle the sub-diffusion and wave equations on Cantor sets. In addition, high-order finite difference methods [14,15], finite element method [16], finite volume method [17], homotopy perturbation method [18] and spectral method [19] are also employed to approximate the FDE. In terms of computation, Wang et al. [20] developed a fast finite difference method for FDE, which require less storage and computation cost while retaining the same accuracy and approximation property.

However, less focuses are on the variable coefficient FDE in conservative form. The diffusion coefficient is generally space or time dependent in practical problems. Hence, we aim at deriving a more generalized fractional diffusion model with variable coefficients in conservative form. According to the principle of conservation of mass, the equation of continuity in 1-D form is given by

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial q(x,t)}{\partial x}=0, \tag{1}$$

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in which $u(x,t)$ is the distribution function of the diffusing quantity and $q(x,t)$ denotes the diffusion flux. Suppose that the flux term possesses the following form

$$q(x,t) = -\left(K_1(x)\frac{\partial}{\partial x}\int_a^x k_+(x,\xi)u(x,\xi)\ \mathrm{d}\xi + K_2(x)\frac{\partial}{\partial x}\int_x^b k_-(x,\xi)u(x,\xi)\ \mathrm{d}\xi\right), \tag{2}$$

on the closed interval $[a,b]$, where $k_+(x,\xi)$, $k_-(x,\xi)$ are the kernel functions, $K_1(x)$ and $K_2(x)$ are the nonnegative diffusion coefficients. We can regard (2) as a special case of Fick's first law by taking $k_+(x,\xi) = k_-(x,\xi) = \delta(x-\xi)$. We consider the most common case

$$\begin{cases} k_+(x,\xi) = \frac{1}{\Gamma(1-\alpha)(x-\xi)^{-\alpha}}, & \text{for } a \le \xi \le x, \\ k_-(x,\xi) = \frac{1}{\Gamma(1-\alpha)(x-\xi)^{-\alpha}}, & \text{for } x \le \xi \le b, \end{cases} \tag{3}$$

where $0 < \alpha < 1$. Combining Eqs.(1)-(3), we obtain a conservative form of the two-sided space fractional diffusion equation with variable coefficients:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_1(x)_aD_x^\alpha u(x,t) - K_2(x)_xD_b^\alpha u(x,t)\right), \tag{4}$$

where the operators ${}_aD_x^\alpha$, ${}_xD_b^\alpha$ are the left and right Riemann-Liouville fractional derivatives (see [21]) defined as

$${}_aD_x^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_a^x (x-\xi)^{n-\alpha-1}u(\xi,t)\ \mathrm{d}\xi,$$

$${}_xD_b^\alpha u(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_x^b (\xi-x)^{n-\alpha-1}u(\xi,t)\ \mathrm{d}\xi,$$

respectively, for $n-1 \le \alpha < n$, where n is an integer.

In this paper, we consider the following onesided space FDE with variable coefficient.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big[K(x)_aD_x^\alpha u(x,t)\Big] + f(x,t), \tag{5}$$

subject to the initial condition

$$u(x,0) = \varphi(x), \qquad a \le x \le b, \tag{6}$$

and the boundary conditions

$$u(a,t) = \psi_1(t), \quad u(b,t) = \psi_2(t), \quad 0 \le t \le T \tag{7}$$

where $0 < \alpha < 1$, $f(x,t)$ is a source term. The left Riemann-Liouville fractional derivative ${}_aD_x^\alpha u(x,t)$ on the finite domain $[a,b]$ is defined as

$${}_aD_x^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_a^x (x-\xi)^{-\alpha}u(\xi,t)d\xi. \tag{8}$$

The papers investigating the variable-coefficient FDE is sparse and most of which lack theoretical analysis. Chen et al. [22] proposed a wavelet method to the solution for a class of space-time fractional convection-diffusion equation with variable coefficients. Bhrawy et al. [23] applied an efficient Legendre-Gauss-Lobatto collocation (L-GL-C) method to solve the space-fractional advection diffusion equation with nonhomogeneous initial-boundary conditions. Liu et al. [24] considered a new weighted fractional finite volume method with a nonlocal operator (using nodal basis functions) for solving a two-sided space fractional diffusion equation. Zhao et al. [25] proposed a compact difference scheme for solving the time fractional sub-diffusion equation with the variable coefficient subject to both Dirichlet boundary conditions and Neumann boundary conditions. Chen et al. [26] discussed the practical alternating directions implicit method to solve a two-dimensional two-sided space fractional convection diffusion equation on a finite domain.

Generally, the shifted Grünwald-Letnikov derivative (see [8]) is utilized to approximate Riemann-Liouville fractional derivative, which is only first order accuracy. Totally different, in this paper, based on a second-order numerical scheme to approximate the Riemann-Liouville fractional derivative, we obtain the second order approximation of the FDE. Furthermore, we derive the Crank-Nicolson scheme and prove the scheme is unconditionally stable and convergent with the accuracy of $\mathscr{O}(\tau^2+h^2)$.

The outline of the paper is as follows. In Section 2, the second-order scheme and some lemmas are given. In Section 3, we present the finite difference method for the FDE and derive the Crank-Nicolson scheme. We proceed with the proof of the stability and convergence of the Crank-Nicolson scheme in Section 4. In order to verify the effectiveness of our theoretical analysis, two numerical examples are carried out and the results are compared with the general first order scheme in Section 5. Finally, the conclusions are drawn.

2 Preliminary knowledge

First, in the interval $[a,b]$, we take the mesh points $x_i = a+ih$, $i = 0,1,\cdots,m$, and $t_n = n\tau$, $n = 0,1,\cdots,N$, where $h = (b-a)/m$, $\tau = T/N$, i.e., h and τ are the uniform spatial step size and temporal step size. In the following, we suppose the symbol C ia a generic positive constant, which may take different values at different places. Now, we give some useful lemmas.

Lemma 2.1. If function $v(x) \in C^2[a,b]$, then we have

$$v(\eta) = \frac{(x_{i+1}-\eta)v(x_i)+(\eta-x_i)v(x_{i+1})}{h} - \frac{1}{2}(\eta-x_i)(x_{i+1}-\eta)v''(\varsigma), \tag{9}$$

where $x_i \le \varsigma \le x_{i+1}$, $i = 0,\,1,\,\cdots,\,m-1$.

Proof. By the Taylor expansion, we expand $v(x_i)$ and $v(x_{i+1})$ at $x = \eta$ and obtain

$$v(x_i) = v(\eta)+(x_i-\eta)v'(\eta)+\frac{1}{2}(x_i-\eta)^2 v''(\varsigma_{i1}) \tag{10}$$

$$v(x_{i+1}) = v(\eta)+(x_{i+1}-\eta)v'(\eta)+\frac{1}{2}(x_{i+1}-\eta)^2 v''(\varsigma_{i2}) \tag{11}$$

where $x_i \le \varsigma_{i1} \le \eta \le \varsigma_{i2} \le x_{i+1}$. From (10) and (11), we have

$$\frac{(x_{i+1}-\eta)v(x_i)+(\eta-x_i)v(x_{i+1})}{h} = v(\eta)+\frac{1}{2}(\eta-x_i)(x_{i+1}-\eta)\left[v''(\varsigma_{i2})\frac{x_{i+1}-\eta}{h}+v''(\varsigma_{i1})\frac{\eta-x_i}{h}\right].$$

Since $v(x) \in [a,b]$, then there exists a $\varsigma \in [\varsigma_{i1},\varsigma_{i2}]$ such that

$$v''(\varsigma) = v''(\varsigma_{i2})\frac{x_{i+1}-\eta}{h}+v''(\varsigma_{i1})\frac{\eta-x_i}{h}.$$

Thus, the Eq.(9) holds.

Lemma 2.2. Suppose that $0<\alpha<1$, $v(x) \in C^2[a,b]$, by the definition of Riemann-Liouville fractional derivative, we have

$$\begin{aligned}
{}_aD_x^{\alpha-1}v(x_i) &= \frac{1}{\Gamma(1-\alpha)}\int_a^{x_i}\frac{v(\eta)}{(x_i-\eta)^\alpha}\,\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}}\frac{\frac{(x_{k+1}-\eta)v(x_k)+(\eta-x_k)v(x_{k+1})}{h}}{(x_i-\eta)^\alpha}\,\mathrm{d}\eta + R_i \\
&= \frac{1}{\Gamma(1-\alpha)h}\sum_{k=0}^{i-1}\left[\mathscr{A}_{i,k}v(x_k)+\mathscr{B}_{i,k}v(x_{k+1})\right]+R_i,
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\mathscr{A}_{i,k} &= \int_{x_k}^{x_{k+1}}\frac{(x_{k+1}-\eta)}{(x_i-\eta)^\alpha}\,\mathrm{d}\eta = \int_{x_k}^{x_{k+1}}\frac{(x_{k+1}-x_i)+(x_i-\eta)}{(x_i-\eta)^\alpha}\,\mathrm{d}\eta \\
&= \frac{(i-k-1)}{1-\alpha}[(i-k-1)^{1-\alpha}-(i-k)^{1-\alpha}]\cdot h^{2-\alpha} - \frac{1}{2-\alpha}[(i-k-1)^{2-\alpha}-(i-k)^{2-\alpha}]\cdot h^{2-\alpha},
\end{aligned}$$

$$\begin{aligned}
\mathscr{B}_{i,k} &= \int_{x_k}^{x_{k+1}}\frac{(\eta-x_k)}{(x_i-\eta)^\alpha}\,\mathrm{d}\eta = \int_{x_k}^{x_{k+1}}\frac{(\eta-x_i)+(x_i-x_k)}{(x_i-\eta)^\alpha}\,\mathrm{d}\eta \\
&= \frac{(i-k)}{1-\alpha}[(i-k)^{1-\alpha}-(i-k-1)^{1-\alpha}]\cdot h^{2-\alpha} + \frac{1}{2-\alpha}[(i-k-1)^{2-\alpha}-(i-k)^{2-\alpha}]\cdot h^{2-\alpha},
\end{aligned}$$

and

$$R_i = -\frac{1}{2\Gamma(1-\alpha)}\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}}\frac{(\eta-x_k)(x_{k+1}-\eta)v''(\varsigma_k)}{(x_i-\eta)^\alpha}\,\mathrm{d}\eta.$$

It is easy to conclude that

$$|R_i| \le \frac{h^2}{2\Gamma(1-\alpha)}\max_{x\in[a,b]}|v''(x)|\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}}\frac{1}{(x_i-\eta)^\alpha}\,\mathrm{d}\eta \le Ch^2.$$

Lemma 2.3. If function $v(x) \in C^3[a,b]$, then

$$v'(x) = \frac{v(x+\frac{h}{2}) - v(x-\frac{h}{2})}{h} - \frac{h^2}{24} v'''(\xi), \tag{13}$$

where $x - \frac{h}{2} \leq \xi \leq x + \frac{h}{2}$.

3 The finite difference method for the FDE

In this section, we utilize the finite difference method to approximate the equation (5) and derive the Crank-Nicolson scheme. First, we consider the discretization of $\frac{\partial}{\partial x}\left[K(x)_a D_x^{\alpha} u(x,t)\right]$ at node (x_i, t_n). Applying Lemma 2.3., we have

$$\begin{aligned}\left\{\frac{\partial}{\partial x}\left[K(x)_a D_x^{\alpha} u(x,t)\right]\right\}\Big|_{(x_i,t_n)} = {} & \frac{K(x_{i+1/2})}{h}\left[{}_a D_x^{\alpha} u(x,t_n)\right]\Big|_{x_{i+1/2}} \\ & - \frac{K(x_{i-1/2})}{h}\left[{}_a D_x^{\alpha} u(x,t_n)\right]\Big|_{x_{i-1/2}} - \frac{h^2}{24}\left\{\frac{\partial^3}{\partial x^3}\left[K(x)_a D_x^{\alpha} u(x,t_n)\right]\right\}\Big|_{x=\zeta_i},\end{aligned} \tag{14}$$

where $x_{i-1/2} \leq \zeta_i \leq x_{i+1/2}$. By the definition of the Riemann-Liouville fractional derivative, we have

$${}_a D_x^{\alpha} u(x,t_n) = \frac{\partial}{\partial x} {}_a D_x^{\alpha-1} u(x,t_n). \tag{15}$$

Then, we can obtain

$$\left\{\frac{\partial}{\partial x}\left[K(x)_a D_x^{\alpha} u(x,t)\right]\right\}\Big|_{(x_i,t_n)} = \frac{K(x_{i+1/2})}{h}\left[\frac{\partial}{\partial x} {}_a D_x^{\alpha-1} u(x_{i+1/2},t_n)\right] - \frac{K(x_{i-1/2})}{h}\left[\frac{\partial}{\partial x} {}_a D_x^{\alpha-1} u(x_{i-1/2},t_n)\right] + \mathscr{O}(h^2). \tag{16}$$

For the items $\frac{\partial}{\partial x} {}_a D_x^{\alpha-1} u(x_{i+1/2},t_n)$ and $\frac{\partial}{\partial x} {}_a D_x^{\alpha-1} u(x_{i-1/2},t_n)$, we use Lemma 2.3. again, yielding

$$\frac{\partial}{\partial x} {}_a D_x^{\alpha-1} u(x_{i+1/2},t_n) = \frac{1}{h}\left[{}_a D_x^{\alpha-1} u(x_{i+1},t_n) - {}_a D_x^{\alpha-1} u(x_i,t_n)\right] + \mathscr{O}(h^2) \tag{17}$$

$$\frac{\partial}{\partial x} {}_a D_x^{\alpha-1} u(x_{i-1/2},t_n) = \frac{1}{h}\left[{}_a D_x^{\alpha-1} u(x_i,t_n) - {}_a D_x^{\alpha-1} u(x_{i-1},t_n)\right] + \mathscr{O}(h^2) \tag{18}$$

Then Eq.(16) can be written as

$$\begin{aligned}\left\{\frac{\partial}{\partial x}\left[K(x)_a D_x^{\alpha} u(x,t)\right]\right\}\Big|_{(x_i,t_n)} = {} & \frac{K(x_{i+1/2})}{h^2}\left[{}_a D_x^{\alpha-1} u(x_{i+1},t_n) - {}_a D_x^{\alpha-1} u(x_i,t_n)\right] \\ & - \frac{K(x_{i-1/2})}{h^2}\left[{}_a D_x^{\alpha-1} u(x_i,t_n) - {}_a D_x^{\alpha-1} u(x_{i-1},t_n)\right] + \mathscr{O}(h^2),\end{aligned} \tag{19}$$

Using Lemma 2.2., we have

$$\mathscr{I}_1 \triangleq {}_a D_x^{\alpha-1} u(x_{i+1},t_n) - {}_a D_x^{\alpha-1} u(x_i,t_n) = \frac{h^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{k=1}^{i-1} u(x_k,t_n) \mathscr{C}(i-k-1) + \mathscr{P}_{i,n} + \mathscr{K}_{i,n}, \tag{20}$$

where

$$\mathscr{C}(k) = 3(k+1)^{2-\alpha} + (k+3)^{2-\alpha} - 3(k+2)^{2-\alpha} - k^{2-\alpha}, \quad k \geq 0, \tag{21}$$

$$\mathscr{P}_{i,n} \triangleq \frac{h^{1-\alpha}(2^{2-\alpha}-3)u(x_i,t_n)}{\Gamma(3-\alpha)} + \frac{h^{1-\alpha}u(x_{i+1},t_n)}{\Gamma(3-\alpha)} + \frac{h^{1-\alpha}u(x_0,t_n)}{\Gamma(1-\alpha)} \mathscr{L}(i), \tag{22}$$

$$\mathscr{L}(i) = (i+1)^{1-\alpha}\left(\frac{i}{\alpha-1} - \frac{i+1}{\alpha-2}\right) - \frac{(i-1)^{2-\alpha}}{(\alpha-1)(\alpha-2)} + i^{1-\alpha}\left(\frac{2i}{\alpha-2} - \frac{1}{\alpha-1}\right) \tag{23}$$

and

$$|\mathscr{K}_{i,n}| \leq 2Ch^2 \max_{x\in[a,b]} |u''(x,t_n)|.$$

Similarly, we obtain

$$\mathscr{I}_2 \triangleq {}_aD_x^{\alpha-1}u(x_i,t_n) - {}_aD_x^{\alpha-1}u(x_{i-1},t_n) = \frac{h^{1-\alpha}}{\Gamma(3-\alpha)}\sum_{k=1}^{i-2} u(x_k,t_n)\mathscr{C}(i-k-2) + \mathscr{P}_{i-1,n} + \mathscr{K}_{i-1,n}, \tag{24}$$

and

$$|\mathscr{K}_{i-1,n}| \le 2Ch^2 \max_{x\in[a,b]} |u''(x,t_n)|.$$

Combining (19), (20) and (24), we find

$$\begin{aligned}\left\{\frac{\partial}{\partial x}\left[K(x)_aD_x^{\alpha}u(x,t)\right]\right\}\Big|_{(x_i,t_n)} &= A(i)u(x_0,t_n) + \sum_{k=1}^{i-2} u(x_k,t_n)V(i,k) \\ &+ G(i)u(x_{i-1},t_n) + B(i)u(x_i,t_n) + D(i)u(x_{i+1},t_n) + \mathscr{O}(h^2),\end{aligned} \tag{25}$$

where

$$\begin{aligned}A(i) &= \frac{1}{h^{1+\alpha}\Gamma(1-\alpha)}\Big[K(x_{i+1/2})\mathscr{L}(i) - K(x_{i-1/2})\mathscr{L}(i-1)\Big],\\ V(i,k) &= \frac{1}{h^{1+\alpha}\Gamma(3-\alpha)}\Big[K(x_{i+1/2})\mathscr{C}(i-k-1) - K(x_{i-1/2})\mathscr{C}(i-k-2)\Big],\\ G(i) &= \frac{1}{h^{1+\alpha}\Gamma(3-\alpha)}\Big[K(x_{i+1/2})\mathscr{C}(0) - K(x_{i-1/2})(2^{2-\alpha}-3)\Big],\\ B(i) &= \frac{1}{h^{1+\alpha}\Gamma(3-\alpha)}\Big[K(x_{i+1/2})(2^{2-\alpha}-3) - K(x_{i-1/2})\Big],\\ D(i) &= \frac{1}{h^{1+\alpha}\Gamma(3-\alpha)}K(x_{i+1/2}).\end{aligned} \tag{26}$$

Now we consider the discretization of the time. It is easy to conclude that,

$$\left(\frac{\partial u(x,t)}{\partial t}\right)\Big|_{(x_i,t_{n-1/2})} = \frac{u(x_i,t_n)-u(x_i,t_{n-1})}{\tau} + \mathscr{O}(\tau^2), \tag{27}$$

and

$$\begin{aligned}\left\{\frac{\partial}{\partial x}\Big[K(x)_aD_x^{\alpha}u(x,t)\Big] + f(x,t)\right\}\Big|_{(x_i,t_{n-1/2})} &= \frac{1}{2}\left\{\frac{\partial}{\partial x}\Big[K(x)_aD_x^{\alpha}u(x,t)\Big] + f(x,t)\right\}\Big|_{(x_i,t_n)} \\ &+ \frac{1}{2}\left\{\frac{\partial}{\partial x}\Big[K(x)_aD_x^{\alpha}u(x,t)\Big] + f(x,t)\right\}\Big|_{(x_i,t_{n-1})} + \mathscr{O}(\tau^2).\end{aligned} \tag{28}$$

Therefore, the Crank-Nicolson scheme of the fractional diffusion equation (5) at $(x_i,t_{n-1/2})$ can be

$$\frac{u(x_i,t_n)-u(x_i,t_{n-1})}{\tau} = \frac{\Omega_{i,n}+\Omega_{i,n-1}}{2} + f(x_i,t_{n-1/2}) + \mathscr{O}(\tau^2+h^2), \tag{29}$$

where

$$\Omega_{i,n} = A(i)u(x_0,t_n) + \sum_{k=1}^{i-2} u(x_k,t_n)V(i,k) + G(i)u(x_{i-1},t_n) + B(i)u(x_i,t_n) + D(i)u(x_{i+1},t_n) \tag{30}$$

and

$$f(x_i,t_{n-1/2}) = \frac{1}{2}\Big[f(x_i,t_n) + f(x_i,t_{n-1})\Big]. \tag{31}$$

Let u_i^n be the approximation solution of $u(x_i,t_n)$ and $f_i^{n-1/2} = f(x_i,t_{n-1/2})$, then we obtain the difference scheme of (29)

$$u_i^n - \frac{\tau}{2}\Omega_i^n = u_i^{n-1} + \frac{\tau}{2}\Omega_i^{n-1} + \tau f_i^{n-1/2} + \mathscr{O}(\tau^3+\tau h^2). \tag{32}$$

The boundary and initial conditions are discretized as

$$u_i^0 = \varphi(x_i), \quad u_0^n = \psi_1(t_n), \quad u_m^n = \psi_2(t_n). \tag{33}$$

Define $U^n = [\, u_1^n, u_2^n, \cdots, u_{m-1}^n]$,

$$F^n = \begin{pmatrix} A_1 + G_1 \\ A_2 \\ \vdots \\ A_{m-2} \\ A_{m-1} \end{pmatrix} (u_0^n + u_0^{n-1}) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ D_{m-1} \end{pmatrix} (u_m^n + u_m^{n-1}) + 2 \begin{pmatrix} f_1^{n-1/2} \\ f_2^{n-1/2} \\ \vdots \\ f_{m-2}^{n-1/2} \\ f_{m-1}^{n-1/2} \end{pmatrix} \tag{34}$$

and

$$Q = -\frac{\tau}{2} \begin{pmatrix} B_1 & D_1 & 0 & 0 & \cdots & 0 & 0 \\ G_2 & B_2 & D_2 & 0 & \cdots & 0 & 0 \\ V_{3,1} & G_3 & B_3 & D_3 & \cdots & 0 & 0 \\ V_{4,1} & V_{4,2} & G_4 & B_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{m-2,1} & V_{m-2,2} & V_{m-2,3} & V_{m-3,4} & \cdots & B_{m-2} & D_{m-2} \\ V_{m-1,1} & V_{m-1,2} & V_{m-1,3} & V_{m-1,4} & \cdots & G_{m-1} & B_{m-1} \end{pmatrix}, \tag{35}$$

eliminating the error term, then we can rewrite (32) in the matrix form

$$(I+Q)U^n = (I-Q)U^{n-1} + \frac{\tau}{2}F^n. \tag{36}$$

4 Theoretical analysis of the finite difference method

4.1 Stability

Here, we consider the stability of the Crank-Nicolson scheme (36). Before the proof, we start with some useful lemmas.
Lemma 4.1. Supposing that $1 < \beta_1 < 2$, we define function $p(x)$, $x \in (0, +\infty)$ as

$$p(x) = 2(x+1)^{\beta_1} - x^{\beta_1} - (x+2)^{\beta_1}$$

then

$$p'(x) = \beta_1 \left[2(x+1)^{\beta_1 - 1} - x^{\beta_1 - 1} - (x+2)^{\beta_1 - 1} \right] > 0, \tag{37}$$

and

$$p''(x) = \beta_1(\beta_1 - 1) \left[2(x+1)^{\beta_1 - 2} - x^{\beta_1 - 2} - (x+2)^{\beta_1 - 2} \right] < 0. \tag{38}$$

Proof. Let $w(x) = (x+1)^r - x^r$, $x \in (0, +\infty)$, $r \in (-1, 0) \cup (0, 1)$, and

$$g(x) = w(x) - w(x+1) = 2(x+1)^r - x^r - (x+2)^r.$$

It is easy to obtain that

$$w'(x) = r[(x+1)^{r-1} - x^{r-1}].$$

We can observe that, when $0 < r < 1$, $w'(x) < 0$, which means $w(x)$ is decreasing monotonically when x increases, then $w(x) > w(x+1)$. Hence, $w(x) - w(x+1) > 0$, namely, $g(x) > 0$ for all $x \in (0, +\infty)$ when $0 < r < 1$. Similarly, we can obtain $g(x) < 0$ for all $x \in (0, +\infty)$ when $-1 < r < 0$. Since $1 < \beta_1 < 2$, then $\beta_1 - 1 \in (0, 1)$ and $\beta_1 - 2 \in (-1, 0)$. According to above discussion, it is easy to obtain that (37) and (38) hold.
Lemma 4.2. Assuming that $0 < \alpha < 1$, we define $\mathscr{C}(k)$, $k = 0, 1, 2, \cdots$ as

$$\mathscr{C}(k) = 3(k+1)^{2-\alpha} + (k+3)^{2-\alpha} - 3(k+2)^{2-\alpha} - k^{2-\alpha}, \tag{39}$$

then

$$\mathscr{C}(k) < 0, \tag{40}$$

$\mathscr{C}(k)$ is increasing monotonically when k increases, i.e.,

$$\mathscr{C}(k) < \mathscr{C}(k+1), \tag{41}$$

and

$$\lim_{k\to+\infty} \mathscr{C}(k) = 0. \tag{42}$$

Proof. Since $0 < \alpha < 1$, it is easy to verify

$$\mathscr{C}(0) = 3 + 3^{2-\alpha} - 3\cdot 2^{2-\alpha} < 0. \tag{43}$$

We define function $\mathscr{C}(x)$, $x \in (0,+\infty)$ as

$$\mathscr{C}(x) = 3(x+1)^{2-\alpha} + (x+3)^{2-\alpha} - 3(x+2)^{2-\alpha} - x^{2-\alpha},$$

Let $\beta_1 = 2-\alpha$, it is easy to check that

$$\mathscr{C}(x) = p(x) - p(x+1),$$

where $p(x)$ is defined in Lemma 4.1. Thanks to (37), we obtain $p(x) - p(x+1) < 0$, i.e., for all $x \in (0,+\infty)$

$$\mathscr{C}(x) < 0.$$

Hence, (40) holds. By using the Taylor expansion and (38), we have

$$p(x+2) = p(x+1) + p'(x+1) + \frac{1}{2}p''(\xi) < p(x+1) + p'(x+1), \quad \xi \in (x+1, x+2),$$

$$p(x) = p(x+1) - p'(x+1) + \frac{1}{2}p''(\eta) < p(x+1) - p'(x+1), \quad \eta \in (x, x+1).$$

To sum each side of the inequalities respectively, we obtain

$$p(x) + p(x+2) < 2p(x+1),$$

which means

$$p(x) - p(x+1) < p(x+1) - p(x+2),$$

i.e,

$$\mathscr{C}(x) < \mathscr{C}(x+1).$$

Thus, (41) holds. Now we consider the following limit, $x > 0$, $1 < \beta_2 < 2$

$$\begin{aligned}
&\lim_{x\to+\infty} \left(3(x+1)^{\beta_2} + (x+3)^{\beta_2} - 3(x+2)^{\beta_2} - x^{\beta_2}\right)\\
=\ &\lim_{x\to+\infty} \frac{3(1+\frac{1}{x})^{\beta_2} + (1+\frac{3}{x})^{\beta_2} - 3(1+\frac{2}{x})^{\beta_2} - 1}{(\frac{1}{x})^{\beta_2}}\\
=\ &\lim_{t\to0^+} \frac{3(1+t)^{\beta_2} + (1+3t)^{\beta_2} - 3(1+2t)^{\beta_2} - 1}{t^{\beta_2}}.
\end{aligned}$$

By using the Taylor expansion, we obtain

$$\begin{aligned}
&3(1+t)^{\beta_2} + (1+3t)^{\beta_2} - 3(1+2t)^{\beta_2} - 1\\
=\ &3[1+\beta_2 t + \frac{\beta_2(\beta_2-1)}{2}t^2 + o(t^2)] + 1 + 3\beta_2 t + \frac{\beta_2(\beta_2-1)}{2}(3t)^2 + o(t^2) - 3[1+2\beta_2 t + \frac{\beta_2(\beta_2-1)}{2}(2t)^2 + o(t^2)] - 1\\
=\ &o(t^2).
\end{aligned}$$

Then,

$$\lim_{x\to+\infty} \left(3(x+1)^{\beta_2} + (x+3)^{\beta_2} - 3(x+2)^{\beta_2} - x^{\beta_2}\right) = \lim_{t\to0^+} \frac{o(t^2)}{t^{\beta_2}} = \lim_{t\to0^+} \frac{o(t^2)}{t^2}\cdot t^{2-\beta_2} = 0.$$

As $1 < 2-\alpha < 2$, therefore, (42) holds.

Corollary 4.1. It is easy to conclude that

$$\sum_{k=0}^{+\infty} \mathscr{C}(k) = 2 - 2^{2-\alpha}. \quad (44)$$

Now we consider the property of matrix Q.

Theorem 4.1. Suppose that $K(x)$ is decreasing monotonically when x increases and $K(x) > 0$ on $[a,b]$ and $0 < \alpha < 1$. When $6 + 3^{2-\alpha} - 2^{4-\alpha} \geq 0$ $(or\ \alpha \geq 0.5546)$, the coefficients Q_{ij} satisfy

$$|Q_{ii}| > \sum_{j=1, j\neq i}^{m-1} |Q_{ij}|, \quad i = 1, 2, \cdots, m-1. \quad (45)$$

i.e., Q is strictly diagonally dominant.

Proof. It is easy to obtain

$$Q_{ij} = k_0 \begin{cases} 0, & j > i+1 \\ K(x_{i+1/2}), & j = i+1 \\ K(x_{i+1/2})(2^{2-\alpha} - 3) - K(x_{i-1/2}), & j = i \\ K(x_{i+1/2})\mathscr{C}(0) - K(x_{i-1/2})(2^{2-\alpha} - 3), & j = i-1 \\ K(x_{i+1/2})\mathscr{C}(i-j-1) - K(x_{i-1/2})\mathscr{C}(i-j-2). & j < i-1 \end{cases} \quad (46)$$

where $k_0 = -\frac{\tau}{2h^{1+\alpha}\Gamma(3-\alpha)} < 0$. Since $K(x) > 0$, then $K(x_{i+1/2}) > 0$, so $k_0 K(x_{i+1/2}) < 0$, i.e., $Q_{i,i+1} < 0$. As $0 < \alpha < 1$, then $2^{2-\alpha} - 3 < 1$ and $0 < K(x_{i+1/2}) < K(x_{i-1/2})$, so

$$(2^{2-\alpha} - 3)K(x_{i+1/2}) < K(x_{i-1/2}),$$

hence,

$$k_0[K(x_{i+1/2})(2^{2-\alpha} - 3) - K(x_{i-1/2})] > 0,$$

i.e., $Q_{ii} > 0$. For the item $Q_{i,i-1}$,

$$\begin{aligned} Q_{i,i-1} &= k_0\Big[K(x_{i+1/2})\mathscr{C}(0) - K(x_{i-1/2})(2^{2-\alpha} - 3)\Big] \\ &= k_0\Big[(K(x_{i+1/2}) - K(x_{i-1/2}))\mathscr{C}(0) + K(x_{i-1/2})(6 + 3^{2-\alpha} - 2^{4-\alpha})\Big] \end{aligned}$$

Since $k_0 < 0$, $K(x_{i+1/2}) - K(x_{i-1/2}) < 0$, $\mathscr{C}(0) < 0$, $K(x_{i-1/2}) > 0$ and $6 + 3^{2-\alpha} - 2^{4-\alpha} \geq 0$, then $Q_{i,i-1} < 0$. When $j < i-1$,

$$\begin{aligned} Q_{ij} &= k_0\Big[K(x_{i+1/2})\mathscr{C}(i-j-1) - K(x_{i-1/2})\mathscr{C}(i-j-2)\Big] \\ &= k_0\Big[K(x_{i+1/2})(\mathscr{C}(i-j-1) - \mathscr{C}(i-j-2)) + (K(x_{i+1/2}) - K(x_{i-1/2}))\mathscr{C}(i-j-2)\Big] \end{aligned}$$

According to Lemma 4.2., $\mathscr{C}(i-j-1) - \mathscr{C}(i-j-2) > 0$ and $\mathscr{C}(i-j-2) < 0$, then it is easy to obtain $Q_{ij} < 0$. Now, for a given i, we consider the sum

$$\begin{aligned} &\sum_{j=1, j\neq i}^{m-1} |Q_{ij}| = \sum_{j=1}^{i-2} |Q_{ij}| + \sum_{j=i+2}^{m-1} |Q_{ij}| + |Q_{i,i-1}| + |Q_{i,i+1}| \\ &= -k_0\left\{\sum_{j=1}^{i-2}\Big[K(x_{i+1/2})\mathscr{C}(i-j-1) - K(x_{i-1/2})\mathscr{C}(i-j-2)\Big] + K(x_{i+1/2}) + K(x_{i+1/2})\mathscr{C}(0) - K(x_{i-1/2})(2^{2-\alpha} - 3)\right\} \\ &= -k_0\left\{K(x_{i+1/2})\sum_{j=1}^{i-2}\mathscr{C}(j) - K(x_{i-1/2})\sum_{j=0}^{i-3}\mathscr{C}(j) + K(x_{i+1/2}) + K(x_{i+1/2})\mathscr{C}(0) - K(x_{i-1/2})(2^{2-\alpha} - 3)\right\} \end{aligned}$$

$$= -k_0\left\{[K(x_{i+1/2})-K(x_{i-1/2})]\sum_{j=0}^{i-2}\mathscr{C}(j)+K(x_{i+1/2})+K(x_{i-1/2})[\mathscr{C}(i-2)-2^{2-\alpha}+3)]\right\}$$
$$< -k_0\left\{[K(x_{i+1/2})-K(x_{i-1/2})]\sum_{j=0}^{+\infty}\mathscr{C}(j)+K(x_{i+1/2})+K(x_{i-1/2})[\mathscr{C}(i-2)-2^{2-\alpha}+3)]\right\}$$
$$= -k_0\left\{[K(x_{i+1/2})-K(x_{i-1/2})](2-2^{2-\alpha})+K(x_{i+1/2})+K(x_{i-1/2})[\mathscr{C}(i-2)-2^{2-\alpha}+3)]\right\}$$
$$= -k_0\left\{K(x_{i+1/2})(3-2^{2-\alpha})+K(x_{i-1/2})[1+\mathscr{C}(i-2)]\right\}$$
$$< -k_0\left\{K(x_{i+1/2})(3-2^{2-\alpha})+K(x_{i-1/2})\right\}=Q_{i,i}=|Q_{i,i}|$$

i.e.,

$$\sum_{j=1,j\neq i}^{m-1}|Q_{ij}|<|Q_{ii}|.$$

Thus, the proof is completed.

Corollary 4.2. Let $\lambda=c+d\,\mathrm{i}$, $\mathrm{i}=\sqrt{-1}$ be any one eigenvalue of the matrix Q, then the real part of λ satisfies

$$c>0. \tag{47}$$

Proof. By the Gerschgorin's circle theorem (see [27]) , we have

$$|Q_{ii}-c-d\,\mathrm{i}|\leq r_i=\sum_{j=1,j\neq i}^{m-1}|Q_{ij}|.$$

Then

$$(Q_{ii}-c)^2+(d\,\mathrm{i})^2\leq r_i^2,$$

therefore,

$$|Q_{ii}-c|\leq r_i.$$

Using (45) and $Q_{ii}>0$, it is easy to derive (47) must hold.

Corollary 4.3. The matrix $I+Q$ is strictly diagonally dominant. Therefore, $I+D$ is invertible and the equation (36) is solvable.

Theorem 4.2. The difference scheme (36) is unconditionally stable.

Proof. Since the eigenvalues λ of matrix Q satisfy Corollary 4.2., then the eigenvalues of the matrix $(I+Q)^{-1}(I-Q)$ satisfy

$$\left|\frac{1-\lambda}{1+\lambda}\right|=\left|\frac{1-c-d\,\mathrm{i}}{1+c+d\,\mathrm{i}}\right|<1.$$

Hence, the spectral radius of the matrix $(I+Q)^{-1}(I-Q)$ is less than one. Therefore, the difference scheme (36) is unconditionally stable.

4.2 Convergence

By (32), we see that the local truncation error of the Crank-Nicolson scheme gives,

$$\mathscr{R}_i^n=\mathscr{O}(\tau^3+\tau h^2). \tag{48}$$

Theorem 4.3. Let u^n be the exact solution of the problem (5)–(7). Then the numerical solution U^n unconditionally converges to the exact solution u^n as h and τ tend to zero, and

$$||U^n-u^n||\leq C(\tau^2+h^2).$$

Proof. Let e_i^n denote the error at grid points (x_i,t_n) and $e_i^n = U_i^n - u(x_i,t_n)$. Substituting $u(x_i,t_n) = U_i^n - e_i^n$ into Eq. (29) and combining Eqs. (32) yields

$$e_i^n - \frac{\tau}{2}\Theta_i^n = e_i^{n-1} + \frac{\tau}{2}\Theta_i^{n-1} + \mathscr{O}(\tau^3+\tau h^2).$$

where

$$\Theta_i^n = A(i)e_0^n + \sum_{k=1}^{i-2} e_k^n V(i,k) + G(i)e_{i-1}^n + B(i)e_i^n + D(i)e_{i+1}^n.$$

Using the conditions (6), (7) and (33), we obtain the errors $e_i^0 = 0$ and $e_0^n = e_m^n = 0$ for $i = 1,2,\cdots,m-1$ and $j = 0,1,\cdots,N$. We can write the system in matrix-vector form as

$$(I+Q)E^n = (I-Q)E^{n-1} + \mathscr{O}(\tau^3+\tau h^2)\chi$$

or

$$E^n = ME^{n-1} + \boldsymbol{b}$$

where $\chi = [1,1,\cdots,1]^T$, $E^n = (e_1^n, e_2^n, \cdots, e_{m-1}^n)^T$, $M = (I+Q)^{-1}(I-Q)$ and $\boldsymbol{b} = \mathscr{O}(\tau^3+\tau h^2)(I+Q)^{-1}$. By iterating and noting that $E^0 = \mathbf{0}$, we obtain

$$E^n = (M^{n-1} + M^{n-2} + \cdots + I)\boldsymbol{b}.$$

Now, from Corollary 4.2., Corollary 4.3. and Theorem 4.2., we have $\rho((I+Q)^{-1}) < 1$ and $\rho(M) < 1$. Therefore, we can choose a vector norm and induced matrix norm $||\cdot||$ such that $||M|| < 1$ and $||(I+Q)^{-1}|| < 1$. Then upon taking norms,

$$\begin{aligned} ||E^n|| &\le (||M^{n-1}|| + ||M^{n-2}|| + \cdots + 1)||\boldsymbol{b}|| \\ &\le (1+1+\cdots+1)||\boldsymbol{b}|| \\ &\le n\mathscr{O}(\tau^3+\tau h^2) = T\mathscr{O}(\tau^2+h^2). \end{aligned}$$

Thus,

$$||E^n|| \le C(\tau^2+h^2),$$

which completes the proof.

5 Numerical examples

In order to demonstrate the effectiveness of the finite difference method, two examples are presented.
Example 5.1. First, we consider the following fractional diffusion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[K(x)_a D_x^{\alpha} u(x,t)\right] + f(x,t), \\ u(x,0) = x^3, \qquad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = e^t, \qquad 0 \le t \le T, \end{cases} \tag{49}$$

where $0 < \alpha < 1$,

$$f(x,t) = x^3 e^t - \frac{6K'(x)}{\Gamma(4-\alpha)}x^{3-\alpha}e^t - \frac{6K(x)}{\Gamma(3-\alpha)}x^{2-\alpha}e^t,$$

and the exact solution is $u(x,t) = x^3 e^t$.

Here we take $K(x) = 2-x$, the related numerical results are given in Table 1. Table 1 describes the L_2 error and convergence order of the Crank-Nicolson scheme at $t = 1$ with $\tau = h$, where α is corresponding to six distinct values. It can be seen that no matter $\alpha > 0.5546$ or $\alpha < 0.5546$ the numerical results are all in excellent agreement with the exact solution.

Example 5.2. Now, we consider the following fractional diffusion equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[K(x)_a D_x^{\alpha} u(x,t)\right] + f(x,t), \\ u(x,0) = x^2(1-x)^2, \qquad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \qquad 0 \le t \le T, \end{cases} \tag{50}$$

Table 1: The error and convergence order for different α.

$\tau = h$	$\alpha = 0.1$		$\alpha = 0.3$		$\alpha = 0.5$	
	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order
1/16	1.6257E-03		8.1478E-04		4.5315E-04	
1/32	3.5858E-04	2.18	1.9173E-04	2.09	1.1054E-04	2.04
1/64	8.3223E-05	2.11	4.6713E-05	2.04	2.7408E-05	2.01
1/128	2.0024E-05	2.06	1.1560E-05	2.01	6.8378E-06	2.00
1/256	4.9162E-06	2.03	2.8784E-06	2.01	1.7092E-06	2.00
$\tau = h$	$\alpha = 0.6$		$\alpha = 0.75$		$\alpha = 0.9$	
	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order
1/16	3.9998E-04		4.5870E-04		5.6288E-04	
1/32	1.0101E-04	1.99	1.1906E-04	1.95	1.4438E-04	1.96
1/64	2.5544E-05	1.98	3.0667E-05	1.96	3.6935E-05	1.97
1/128	6.4520E-06	1.99	7.8562E-06	1.96	9.4286E-06	1.97
1/256	1.6266E-06	1.99	2.0040E-06	1.97	2.4025E-06	1.97

where $0 < \alpha < 1$,

$$f(x,t) = -e^{-t}\left[x^2(1-x)^2 + K'(x)P(x,\alpha) + K(x)P(x,1+\alpha)\right],$$

$$P(x,\alpha) = \frac{\Gamma(5)}{\Gamma(5-\alpha)}x^{4-\alpha} - \frac{2\Gamma(4)}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha}$$

and the exact solution is $u(x,t) = x^2(1-x)^2e^{-t}$.

In [28], Chen et al. proposed a first order scheme of $\mathscr{O}(\tau + h)$ for a nonlinear two-sided space-fractional diffusion equation with variable diffusivity coefficients by using the backward Euler difference scheme for the temporal derivative and the shifted left and standard right Grünwald-Letnikov fractional derivatives to approximate the left and right Riemann-Liouville fractional derivatives respectively. Here we give some comparisons of the first order scheme and our second order scheme for different $K(x)$. Table 2 shows the comparison of the two schemes at $t = 1$ with $\tau = h$ and different α for the Parabolic case, where $K(x) = \frac{1-x^2}{2}$. Table 3 shows the comparison of the two schemes at $t = 1$ with $\tau = h$ and different α for the Exponential case, where $K(x) = \frac{1+e^{-x}}{2}$. Table 4 shows the comparison of the two schemes at $t = 1$ with $\tau = h$ and different α for the Asymptotic case, where $K(x) = 1 + \frac{1}{1+x}$. We can observe that the second order scheme is more accurate than the first order scheme for the same time and space steps, and for the different $K(x)$ the numerical results are still in excellent agreement with the exact solution, which further demonstrate the effectiveness of our numerical method.

6 Conclusions

In this paper, we have developed and demonstrated a second order finite difference method for solving a class of fractional diffusion equation with variable coefficient. Firstly, based on a second-order scheme, applying the finite difference method, we derived the Crank-Nicolson scheme of the problem and rewrote the scheme in matrix form. Subsequently, we proved that the scheme is unconditionally stable and convergent with the accuracy of $\mathscr{O}(\tau^2 + h^2)$. Finally, some numerical results were given to show the stability, consistency and convergence of our computational approach. This technique could be extend to two-dimensional or three-dimensional problems with complex regions. In the future, we would like to investigate finite difference method for the two-sided FDE and fractional problem in high dimensions.

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Table 2: Comparison of two schemes for Parabolic case and different α.

$\tau = h$	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order
1/16	2.0543E-03		1.1090E-03		9.1087E-04	
1/32	1.0494E-03	0.97	5.3530E-04	1.05	4.0959E-04	1.15
1/64	5.3232E-04	0.98	2.6351E-04	1.02	1.9463E-04	1.07
1/128	2.6841E-04	0.99	1.3074E-04	1.01	9.4653E-05	1.04
1/256	1.3483E-04	0.99	6.5102E-05	1.01	4.6541E-05	1.02
$\tau = h$	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order
1/16	3.4579E-04		2.3554E-04		1.9359E-04	
1/32	8.2875E-05	2.06	6.3634E-05	1.89	4.9412E-05	1.97
1/64	2.0893E-05	1.99	1.7417E-05	1.87	1.3611E-05	1.86
1/128	5.2973E-06	1.98	4.6838E-06	1.89	3.7857E-06	1.85
1/256	1.3386E-06	1.98	1.2351E-06	1.92	1.0280E-06	1.88

Table 3: Comparison of two schemes for Exponential case and different α.

$\tau = h$	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order
1/16	1.9275E-03		1.4961E-03		1.3157E-03	
1/32	9.5627E-04	1.01	7.1739E-04	1.06	5.9150E-04	1.15
1/64	4.7579E-04	1.01	3.5108E-04	1.03	2.8011E-04	1.08
1/128	2.3729E-04	1.00	1.7364E-04	1.02	1.3629E-04	1.04
1/256	1.1849E-04	1.00	8.6351E-05	1.01	6.7237E-05	1.02
$\tau = h$	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order
1/16	5.8993E-04		4.6139E-04		3.5110E-04	
1/32	1.4721E-04	2.00	1.1679E-04	1.98	8.9358E-05	1.97
1/64	3.6797E-05	2.00	2.9482E-05	1.99	2.2713E-05	1.98
1/128	9.2014E-06	2.00	7.4238E-06	1.99	5.7642E-06	1.98
1/256	2.3008E-06	2.00	1.8658E-06	1.99	1.4625E-06	1.98

Table 4: Comparison of two schemes for Asymptotic case and different α.

$\tau = h$	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order
1/16	1.7374E-03		1.4290E-03		1.2552E-03	
1/32	8.6134E-04	1.01	6.8055E-04	1.07	5.6392E-04	1.15
1/64	8.6134E-04	1.01	3.3195E-04	1.04	2.6705E-04	1.08
1/128	2.1393E-04	1.00	1.6392E-04	1.02	1.2995E-04	1.04
1/256	1.0685E-04	1.00	8.1447E-05	1.01	6.4104E-05	1.02
$\tau = h$	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order	$\|\|E(h,\tau)\|\|_2$	Order
1/16	5.2882E-04		4.2085E-04		3.2388E-04	
1/32	1.3252E-04	2.00	1.0670E-04	1.98	8.2693E-05	1.97
1/64	3.3145E-05	2.00	2.6953E-05	1.99	2.1060E-05	1.97
1/128	8.2885E-06	2.00	6.7896E-06	1.99	5.3511E-06	1.98
1/256	2.0724E-06	2.00	1.7069E-06	1.99	1.3593E-06	1.98

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