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Upper bounds for E-J matrices

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Abstract: In a recent paper [5] Lashkaripour and Foroutannia obtained the norm of a Hausdorff matrix, considered as a bounded linear operator from $\ell_p(w)$ to $\ell_p(v)$, where $\ell_p(w)$ and $\ell_p(v)$ are weighted ℓ_p -spaces, and $p \geq 1$. As a corollary to this result they obtain a new proof for a Hausdorff matrix, with nonnegative entries, to be a bounded operator on ℓ_p for $p > 1$. In this paper these results are extended to the Endl- Jakimovski (E-J) generalized Hausdorff matrices.

Keywords: nonnegative decreasing sequences, E-J generalized Hausdorff matrices, upper bounds.

1. Introduction

Let $p \geq 1$, ℓ_p denote the linear space of all sequences $x = \{x_n\}$ satisfying

$$\|x\|_p := \Big(\sum_{n=0}^{\infty} |x_n|^p\Big)^{1/p}.$$

Let $w = \{w_n\}$ be a sequence with positive entries. For $p \geq 1$ define the weighted $\ell_p(w)$ space by

$$\ell_p(w) = \Big\{x : \sum_{n=0}^{\infty} w_n |x_n|^p < \infty\Big\},$$

with norm $\|\cdot\|_{p,w}$ defined by

$$\|x\|_{w,p} = \Big(\sum_{n=0}^{\infty} w_n |x_n|^p\Big)^{1/p}.$$

If w is a decreasing sequence with $\lim_n w_n = 0$, and $\sum_{n=0}^{\infty} w_n = \infty$, then the Lorentz space $d(w,p)$ is defined as follows:

$$d(w,p) = \Big\{x : \sum_{n=0}^{\infty} w_n x_n^{*p} < \infty\Big\},$$

where $\{x_n^*\}$ denotes the decreasing rearrangement of $\{x_n\}$.

The E-J generalized Hausdorff matrices were defined independently by Endl [1] and Jakimovski [3]. They are lower triangular matrices with entries

$$h_{nk}^{(\alpha)} = \{\ n+\alpha n-k\Delta^{n-k}\mu_k. \quad 0 \leq k \leq n, 0, \quad k > n,$$

where $\{\mu_n\}$ is any real or complex sequence, Δ is the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^{n+1}\mu_k = \Delta(\Delta^n \mu_k)$, and α is any real nonnegative number. The special case $\alpha = 0$ yields the ordinary Hausdorff matrices.

An infinite matrix is said to be conservative if it is a selfmap of c, the space of convergent sequences. An E-J matrix is conservative if and only if

$$\int_0^1 |d\mu(x)| < \infty,$$

where μ is a function of bounded variation over $[0,1]$. It is also the case that the μ_n have the representation

$$\mu_n = \int_0^1 x^{n+\alpha} d\mu(x).$$

A conservative E-J matrix has all nonnegative entries if and only if $\mu(x)$ is nonnegative and nondecreasing over [0, 1].

2. Upper Bounds For E-J Matrices

In the following theorem it will be assumed that $\{v_n\}$ and $\{w_n\}$ are nonnegative decreasing sequences with $v_0 = 1$.

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Theorem 1.*Let $H^{(\alpha)}(\mu)$ be a conservative E-J matrix with nonnegative entries, $p > 1$. Then $H^{(\alpha)}(\mu)$ maps $\ell_p(v)$ into $\ell_p(w)$ and* $\left(\inf \frac{w_n}{v_n}\right)^{1/p} \int_0^1 x^{-1/p} d\mu(x)$
$\leq \|H^\alpha(\mu)\|_{v.w.p}$
$\leq \left(\sup \frac{w_n}{v_n}\right)^{1/p} \int_0^1 x^{-1/p} d\mu(x)$. *Therefore $H^{(\alpha)}(\mu)$ maps $\ell_p(w)$ into itself and*

$$\|H^{(\alpha)}(\mu)\|_{w,p} = \int_0^1 x^{-1/p} d\mu(x).$$

For any sequence $\{s_n\}$, define

$$t_n = \sum_{k=0}^{n} h_{nk}^{(\alpha)} s_k.$$

Lemma 1.*If $s_n \geq 0$ and $p > 1$, then* $\sum t_n^p \leq \left(\int_0^1 x^{-1/p} d\mu(x)\right)^p \sum s_n^p$
$:= \|H^{(\alpha)}(\mu)\|^p \sum s_n^p$.

*Proof.*Define $\mathrm{e}_n = e_n(x) = \sum_{k=0}^n n + \alpha n - k x^{k+\alpha}(1-x)^{n-k} s_k$
$= \sum_{k=0}^n n + \alpha n - k x^{k+\alpha} y^{n-k} s_k$, where $0 \leq x \leq 1$ and $y = 1 - x$. Then, by Hólder's inequality,
$\sum_{n=0}^\infty e_n^p \leq \sum_{n=0}^\infty \sum_{k=0}^n n + \alpha n - k x^{k+\alpha} y^{n-k} s_k^p$
$= \sum_{k=0}^\infty x^{k+\alpha} s_k^p \sum_{n=k}^\infty n + \alpha n - k y^{n-k}$
$= \sum_{k=0}^\infty x^{k+\alpha} s_k^p \sum_{j=0}^\infty j + k + \alpha j y^j$
$= \sum_{k=0}^\infty x^{k+\alpha} s_k^p (1-y)^{-1-\alpha-k} = \sum_{k=0}^\infty s_k^p (1-y)^{-1}$
$= (1-y)^{-1} \sum_{k=0}^\infty s_k^p = x^{-1} \sum_{k=0}^\infty s_k^p$. But $\mathrm{t}_n = \int_0^1 \sum_{k=0}^n n + \alpha n - k x^{k+\alpha} (1-x)^{n-k} s_k d\mu(x)$
$= \int_0^1 e_n(x) d\mu(x)$. Using (1) - (3) and Minkowski's inequality $\left(\sum_{n=0}^\infty t_n^p\right)^{1/p} \leq \int_0^1 \left(\sum_{n=0}^\infty t_n^p\right)^{1/p} d\mu(x)$
$\leq \|H^{(\alpha)}(\mu)\| \left\{ \sum_{n=0}^\infty e_n^p \right\}^{1/p}$.

The special case of Lemma 1 for $\alpha = 0$ is the principal part of Theorem 216 of [2] To prove Theorem 1, since $\{s_n\}$ is a decreasing sequence, applying Lemma 1 gives $\|H^{(\alpha)} s\|_{w,p}^p$
$= \sum_{n=0}^\infty w_n \left(\sum_{k=0}^n n + \alpha n - k \int_0^1 x^{k+\alpha} (1-x)^{n-k} d\mu(x) s_k \right)^p$
$\leq \left(\int_0^1 x^{-1/p} d\mu(x) \right)^p \sum_{k=0}^\infty w_k |s_k^p|$
$= \left(\int_0^1 x^{-1/p} d\mu(x) \right)^p \sum_{k=0}^\infty \frac{w_k}{v_k} v_k |s_k^p|$
$\leq \sup_k \frac{w_k}{v_k} \left(\int_0^1 x^{-1/p} d\mu(x) \right)^p \|s\|_{v,p}^p$. Hence

$$\|H^{(\alpha)} s\|_{w,p} \leq \left(\sup_k \frac{w_k}{v_k} \right)^{1/p} \int_0^1 x^{-1/p} d\mu(x) \|s\|_{v,p}^p,$$

and so

$$\|H^{(\alpha)} s\|_{v,w,p}^p \leq \left(\sup_k \frac{w_k}{v_k} \right)^{1/p} \int_0^1 x^{-1/p} d\mu(x).$$

It remains to prove the left-hand inequality. Choose $0 < \delta < 1/p$ and $s_n = (n+1)^{-1/p-\delta}$. For any postive $\epsilon, < \epsilon < 1$, choose α, N, and δ so that

$$\left(1 + \frac{1}{\alpha}\right)^{-2p} > 1 - \epsilon,$$

$$\int_{\alpha/n}^1 x^{-1/p} d\mu(x) > (1-\epsilon) \int_0^1 x^{-1/p} d\mu(x), \quad n \geq N, .$$

and

$$\sum_{n=N}^\infty w_n s_n^p > (1-\epsilon) \sum_{n=0}^\infty w_n s_n^p.$$

Since $\{s_n\} \in \ell_p$ and $0 < v_n \leq 1$, it is clear that $\{s_n\} \in \ell_p(v)$. Also, $(\mathrm{H}^{(\alpha)}s)_n = \sum_{k=0}^{n} n+\alpha n-k\Big(\int_0^1 x^{k+\alpha}(1-x)^{n-k}d\mu(x)\Big)s_k$
$\geq (1-\epsilon)^2 s_n \int_0^1 x^{-1/p}d\mu(x), \quad n \geq N$. Hence

$$w_n^{1/p}(H^{(\alpha)}s)_n \geq (1-\epsilon)^2 w_n^{1/p} s_n \int_0^1 x^{-1/p}d\mu(x).$$

Therefore $\|H^{(\alpha)}s\|_{w,p}^p \geq \sum_{n=N}^{\infty} w_n(Hs)_n^p$
$\geq (1-\epsilon)^{2p}\Big(\int_0^1 x^{-1/p}d\mu(x)\Big)^p \sum_{n=N}^{\infty} w_n s_n^p$
$\geq (1-\epsilon)^{2p+1}\Big(\int_0^1 x^{-1/p}d\mu(x)\Big)^p \sum_{n=0}^{\infty} w_n s_n^p$
$= (1-\epsilon)^{2p+1}\Big(\int_0^1 x^{-1/p}d\mu(x)\Big)^p \sum_{n=0}^{\infty} \frac{w_n}{v_n} v_n s_n^p$
$\geq \inf \frac{w_n}{v_n}(1-\epsilon)^{2p+1}\Big(\int_0^1 x^{-1/p}d\mu(x)\Big)^p \|s\|_{v,p}^p$. The special case of Theorem 1 for $\alpha = 0$ is Corollary 2.3 of [5]. Corollary 2.3 of [5] was extended to the E-J matrices in [4]

Corollary 1.*If $H^{(\alpha)}(\mu)$ is a nonnegative E-J matrix bounded on ℓ_p for $p > 1$, then*

$$\|H^{(\alpha)}\|_p = \int_0^1 x^{-1/p}d\mu(x). \tag{1}$$

The special case of Corollary 1 for $\alpha = 0$ is Corollary 2.3 of [5] Although not mentioned in [5], Theorem 2.1 of that paper provides an alternate proof of the fact that the ℓ_p norm of a nonnegative Hausdorff matrix is given by equation (4) with $\alpha = 0$. Unfortunately, (4) does not give the correct norm, even for ℓ_2, if $H^{(\alpha)}(\mu)$ has negative entries. (See, e.g. [6].)

Theorem 2.*Let $p > 1$ and $H^{(\alpha)}(\mu)$ be an E-J generalized Hausdorff matrix satisfying the condition that, for all subsets M, N of natural numbers, having m, n elements, respectively,*

$$\sum_{i\in M}\sum_{j\in N} \leq \sum_{i=0}^{m}\sum_{j=0}^{n} h_{ij}^{(\alpha)}.$$

Then $H^{(\alpha)}(\mu)$ maps $d(w,p)$ into itself and satisfies

$$\|H^{(\alpha)}(\mu)\|_{d(w,p)} = \int_0^1 x^{-1/p}d\mu(x).$$

*Proof.*From Propositions 2.1 and 2.2 of [5] it is sufficient to consider nonnegative decreasing sequences. For such sequences we have proved that

$$\|H^{(\alpha)}(\mu)s\|_{d(w,p)} = \|H^{(\alpha)}(\mu)s\|_{w,p},$$

and the result follows from Theorem 1.

Theorem 2.2 of [5] is the special case of Theorm 2 for $\alpha = 0$.

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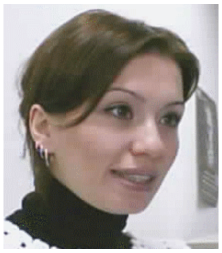

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B. E. Rhoades is a leading world-known figure in mathematics and is Professor Emeritus at Indiana University, USA. He obtained his Ph.D. from Lehigh University in 1958. He has received a number of honors and awards. He is a member of several mathematical organizations, and is on the editorial boards of 14 mathematical research journals. He is an active researcher in analysis and fixed point theory. He has published more than 360 research articles in reputed mathematical journals. In addition to his individual studies he carries out joint studies with other mathematicians and provides technical support to many researchers all over the world.