Applied Mathematics & Information Sciences
An International Journal

On construction of symmetric compactly supported biorthogonal multiwavelets with short sequences

Yi-Gang Cen[1,2], *Li-Hui Cen*[3] *and Xiao-Fang Chen*[3]

[1] Institute of Information Science, Beijing Jiaotong University,Beijing, P. R. China
[2] Beijing Key Laboratory of Advanced Information Science and Network Technology, Beijing, China
[3] School of Information Science and Engineering, Central South University, Hunan, Changsha, P. R. China

Received: Jan. 13, 2012; Revised April 28, 2012; Accepted 12 March 2012
Published online: 1 Sep. 2012

Abstract: A novel approach for constructing symmetric compactly supported biorthogonal multiwavelets with short sequences is proposed in this paper. For some symmetric types of biorthogonal multiwavelet systems, starting from the symmetric properties of scaling and wavelet functions, parameterized symmetric forms of polyphase matrices can be derived. Furthermore, according to the matrix equations of the perfect reconstruction condition, the parameters of polyphase matrices can be reduced, which finally leads to our proposed algorithm for the construction of symmetric compactly supported biorthogonal multiwavelets with short sequences.

Keywords: Construction of multiwavelet, compactly supported biorthogonal multiwavelet, symmetry, short sequence, high-pass filter sequence.

1. Introduction

Multiwavelets have several advantages over scalar wavelets because it is possible to construct multiwavelet bases possessing several properties at the same time, such as orthogonality, symmetry, short support and a high number of vanishing moments. Since the multiwavelet was firstly studied by Goodman *et. al.* [1], the construction of multiwavelets has been a hot issue. Many approaches have been proposed such as fractal interpolation method [2,3], polyphase matrix extension [4,5], lifting scheme [6,7] *etc.*. However, the involved computations are rather complex, and there is no simple and direct construction approach available so far.

This paper presents an algorithm to the construction of *symmetric compactly supported biorthogonal multiwavelets with short sequences* (SCSBMSS) based on the corresponding multiscaling vectors. Firstly, we will discuss how the symmetry of multiscaling and multiwavelet functions affects the corresponding polyphase matrices. As a result, the symmetric forms of polyphase matrices will be derived under different symmetric conditions. Moreover, because the construction of SCSBMSS can be boiled down to find high-pass filter sequences based on the low-pass filter sequences such that they satisfy the *perfect recostruction* (PR) condition. We will show how to solve the matrix equations of the PR condition, which finally leads to our proposed construction algorithm.

Let $\Phi(x) = (\phi_1(x), \phi_2(x), ..., \phi_p(x))^T$ and $\tilde{\Phi}(x) = (\tilde{\phi}_1(x), \tilde{\phi}_2(x), ..., \tilde{\phi}_p(x))^T$ be a pair of biorthogonal multiscaling vectors satisfying the matrix dilation equations $\Phi(x) = \sqrt{2}\sum_{k\in Z} H_k\Phi(2x-k)$ and $\tilde{\Phi}(x) = \sqrt{2}\sum_{k\in Z} \tilde{H}_k \tilde{\Phi}(2x-k)$, respectively. Here, H_k and $\tilde{H}_k$ are finite two-scale matrix coefficients, whose entries are real-valued numbers. The corresponding biorthogonal multiwavelet and dual multiwavelet vectors are $\Psi = (\psi_1, \psi_2, ..., \psi_p)^T$ and $\tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2, ..., \tilde{\psi}_p)^T$, which satisfy the dilation equations $\Psi(x) = \sqrt{2}\sum_{k\in Z} G_k\Phi(2x-k)$ and $\tilde{\Psi}(x) = \sqrt{2}\sum_{k\in Z} \tilde{G}_k \tilde{\Phi}(2x-k)$ with finite real-valued matrix coefficients G_k and $\tilde{G}_k$. We say that $\Phi(x)$ is a *multiscaling vector function with short sequence* if the length of the coefficient sequence $\{H_k\}_{k\in Z}$ is less than or equal to 4. $\tilde{\Phi}(x)$, $\Psi(x)$ and $\tilde{\Psi}(x)$ can be defined analogously. Define two $p\times p$ matrices $H_e(z)$ and $H_o(z)$ as $H_e(z) = 2^{-1/2}\sum_{k\in Z} H_{2k}z^k$ and $H_o(z) = 2^{-1/2}\sum_{k\in Z} H_{2k+1}z^k$, where $z = e^{-j\omega}$. Then the *polyphase matrix* of $\Phi(x)$ is defined as $H(z) = (H_e(z)$

* Corresponding author: e-mail: xiaofangchen@csu.edu.cn

$H_o(z))_{p\times 2p}$. Likewise, $\tilde{H}(z) = (\tilde{H}_e(z)\ \tilde{H}_o(z))_{p\times 2p}$, $G(z) = (G_e(z)\ G_o(z))_{p\times 2p}$ and $\tilde{G}(z) = (\tilde{G}_e(z)\ \tilde{G}_o(z))_{p\times 2p}$ are the polyphase matrices of $\tilde{\Phi}(x)$, $\Psi(x)$ and $\tilde{\Psi}(x)$, respectively. Let $R[z]$ be *the ring of univariate Laurent polynomials* over the complex field. It can be verified that the polyphase matrices $H(z)$, $\tilde{H}(z)$, $G(z)$ and $\tilde{G}(z)$ are all matrices over $R[z]$.

For a given biorthogonal multiwavelet system, the PR condition means that the following matrix equation must be met:

$$\begin{pmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{pmatrix} \begin{pmatrix} \tilde{H}_e(z)^* & \tilde{G}_e(z)^* \\ \tilde{H}_o(z)^* & \tilde{G}_o(z)^* \end{pmatrix} = I_{2p}, \tag{1}$$

i.e.,

$$H_e(z)\tilde{H}_e(z)^* + H_o(z)\tilde{H}_o(z)^* = I_p, \tag{2}$$

$$H_e(z)\tilde{G}_e(z)^* + H_o(z)\tilde{G}_o(z)^* = O_p, \tag{3}$$

$$G_e(z)\tilde{H}_e(z)^* + G_o(z)\tilde{H}_o(z)^* = O_p, \tag{4}$$

$$G_e(z)\tilde{G}_e(z)^* + G_o(z)\tilde{G}_o(z)^* = I_p, \tag{5}$$

where I_{2p} denotes the $2p \times 2p$ identity matrix. Our construction problem is: for a given pair of compactly supported biorthogonal multiscaling vectors $\Phi(x)$ and $\tilde{\Phi}(x)$ with polyphase matrices satisfying $H(z)\tilde{H}(z)^* = I_p$, how to find the corresponding $G(z)$ and $\tilde{G}(z)$ such that they satisfy (1). In this paper, only the multiwavelet systems with short sequences are considered, i.e., each of $\Phi(x)$, $\tilde{\Phi}(x)$, $\Psi(x)$ and $\tilde{\Psi}(x)$ only has at most 4 nonzero matrix coefficients.

2. Symmetric forms of polyphase matrices

In this section, we will study the symmetric properties of biorthogonal multiwavelets with short sequences so that parameterized symmetric forms of polyphase matrices can be derived for future development.

For the multiscaling and multiwavelet functions, they are called *symmetric* (*antisymmetric*) if they satisfy

$$\phi_i(x) = s_i\phi_i(2a_i - x), \psi_i(x) = t_i\psi_i(2b_i - x), \tag{6}$$

$$\tilde{\phi}_i(x) = \tilde{s}_i\tilde{\phi}_i(2\tilde{a}_i - x), \tilde{\psi}_i(x) = \tilde{t}_i\tilde{\psi}_i(2\tilde{b}_i - x), \tag{7}$$

where s_i, t_i, $\tilde{s}_i$, $\tilde{t}_i \in \{-1, 1\}$, a_i, b_i, $\tilde{a}_i$ and $\tilde{b}_i \in R$, $1 \le i \le p$, i.e., s_i, t_i, $\tilde{s}_i$, $\tilde{t}_i$ determine whether a function is symmetric or antisymmetric and a_i, b_i, $\tilde{a}_i$ and $\tilde{b}_i$ are the symmetric (antisymmetric) centers. Define the *two-scale matrix symbols* as

$$\begin{cases} P(z) = \sqrt{2}\sum_{k\in Z} H_k z^k,\ Q(z) = \sqrt{2}\sum_{k\in Z} G_k z^k; \\ \tilde{P}(z) = \sqrt{2}\sum_{k\in Z} \tilde{H}_k z^k,\ \tilde{Q}(z) = \sqrt{2}\sum_{k\in Z} \tilde{G}_k z^k. \end{cases} \tag{8}$$

According to (8) and take into account the Fourier transforms of (6) and (7), we have

$$P(z) = SA(z^2)P(z^{-1})SA(z^{-1}), \tag{9}$$

$$\tilde{P}(z) = \tilde{S}\tilde{A}(z^2)\tilde{P}(z^{-1})\tilde{S}\tilde{A}(z^{-1}), \tag{10}$$

$$Q(z) = TB(z^2)Q(z^{-1})SB(z^{-1}), \tag{11}$$

$$\tilde{Q}(z) = \tilde{T}\tilde{B}(z^2)\tilde{Q}(z^{-1})\tilde{S}\tilde{B}(z^{-1}), \tag{12}$$

where S and $A(z)$ are diagonal matrices with diagonal entries s_i and z^{2a_i}, respectively. T, $\tilde{S}$, $\tilde{T}$, $B(z)$, $\tilde{A}(z)$ and $\tilde{B}(z)$ are defined analogously.

Here, only the multiwavelet systems with multiplicity $p = 2$ are considered. Thus, there are only two components in each of multiscaling and multiwavelet vectors. Note that $Q(z) = G_e(z^2) + zG_o(z^2)$ ($P(z)$ also satisfies a similar equation), and take into account (11):

$$\begin{aligned} Q(z) &= G_e(z^2) + zG_o(z^2) \\ &= T\begin{pmatrix} z^{4b_1} & 0 \\ 0 & z^{4b_2} \end{pmatrix} Q(z^{-1}) \begin{pmatrix} z^{-2a_1} & 0 \\ 0 & z^{-2a_2} \end{pmatrix} S. \end{aligned} \tag{13}$$

Because the bases remain essentially the same when we replace any scaling or wavelet function by its integer translate, without a loss of generality, we can assume that a_i, $b_i \in [0, 1)$. Note that $P(z)$ and $Q(z)$ are required to be Laurent polynomial matrices in z, which ensures that the scaling and wavelet functions are compactly supported. Thus, according to (13), the values of a_i, $i = 1, 2$ can only be either 0 or 1/2 and b_i be 0, 1/4, 1/2 or 3/4. In the following, scaling vector functions with *same symmetric centers* (i.e., the components of scaling vectors have same symmetric centers) and *different symmetric centers* will be discussed, respectively. For the orthogonal case, multiwavelets with different symmetric centers have been studied in [9]. We say a multiscaling vector $\Phi(x) = (\phi_1(x)\ \phi_2(x))$ *is symmetric/antisymmetric about* a_1 *and* a_2, if $\phi_1(x)$ is symmetric about a_1 and $\phi_2(x)$ is antisymmetric about a_2. If there are no wavelets symmetric about 1/4 or 3/4, then, several useful symmetric forms of polyphase matrices can be derived, as follows.

Theorem 1. Suppose that all multiscaling and multiwavelet vectors $\Phi(x)$, $\Psi(x)$, $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$ are symmetric/antisymmetric about 0 (this symmetric form is denoted as **Type I**), i.e., $a_1 = a_2 = b_1 = b_2 = 0$, then the corresponding polyphase matrix of $\Psi(x)$ has the following form.

$$\begin{pmatrix} G_e(z) & G_o(z) \end{pmatrix} = \begin{pmatrix} c_1 & 0 & c_3(1+z^{-1}) & c_4(1-z^{-1}) \\ 0 & d_2 & d_3(1-z^{-1}) & d_4(1+z^{-1}) \end{pmatrix}, \tag{14}$$

where c_i and d_i, $1 \le i \le 4$, are constants. Also, $(\tilde{G}_e(z)\ \tilde{G}_o(z))$ has a similar form of (14).

Proof. By substituting $a_1 = a_2 = b_1 = b_2 = 0$ into (13), we have

$$\begin{cases} G_e(z) = T_0 G_e(z^{-1}) S_0 \\ G_o(z) = z^{-1} T_0 G_o(z^{-1}) S_0, \end{cases} \tag{15}$$

where the subscripts of T_0 and S_0 denote the symmetric centers of scaling and multiwavelet functions, respectively. In this case, $T_0 = S_0 = diag(1, -1)$.

Let

$$G_e(z) = \begin{pmatrix} G_e^{11}(z) & G_e^{12}(z) \\ G_e^{21}(z) & G_e^{22}(z) \end{pmatrix}, \tag{16}$$

by substituting (16) into (15), we have

$$\begin{cases} G_e^{11}(z) = \ \ G_e^{11}(z^{-1}), G_e^{12}(z) = -G_e^{12}(z^{-1}), \\ G_e^{21}(z) = -G_e^{21}(z^{-1}), G_e^{22}(z) = G_e^{22}(z^{-1}). \end{cases} \quad (17)$$

Note that only the biorthogonal multiwavelets with short sequences are considered here, i.e., the coefficient number of the multiwavelet is less than or equal to 4. Thus, each of the entries of $G_e(z)$ and $G_o(z)$ can only be a first-order polynomial in z (or z^{-1}) or a constant. Further, by (15), $G_e(z)$ can be chosen as

$$G_e(z) = \begin{pmatrix} c_1 + c_1' z & c_2 + c_2' z \\ d_1 + d_1' z & d_2 + d_2' z \end{pmatrix}. \quad (18)$$

$G_o(z)$ can only be chosen as

$$G_o(z) = \begin{pmatrix} c_3 + c_3' z^{-1} & c_4 + c_4' z^{-1} \\ d_3 + d_3' z^{-1} & d_4 + d_4' z^{-1} \end{pmatrix},$$

where c_i, c_i', d_i, d_i', $1 \le i \le 4$, are constants. By substituting (18) into (17), we get

$$G_e^{11}(z) = c_1, G_e^{12}(z) = 0, G_e^{21}(z) = 0, G_e^{22}(z) = d_2. \quad (19)$$

Analogously, we have $G_o^{11}(z) = c_3(1+z^{-1})$, $G_o^{12}(z) = c_4(1 - z^{-1})$, $G_o^{21}(z) = d_3(1 - z^{-1})$, $G_o^{22}(z) = d_4(1 + z^{-1})$. Thus, (14) holds. Also, we can prove that $(\tilde{G}_e(z)\ \tilde{G}_o(z))$ has a similar form of (14).

Besides the above symmetric form of Type I, in this paper, another two symmetric types are given, as follows.

Type II. Suppose that all the multiscaling and multiwavelet vectors are symmetric/antisymmetric about 1/2. According to (13), we have

$$\begin{cases} G_e(z) = T_{1/2} G_o(z^{-1}) S_{1/2} \\ G_o(z) = T_{1/2} G_e(z^{-1}) S_{1/2}, \end{cases} \quad (20)$$

where, $T_{1/2} = S_{1/2} = diag(1, -1)$.

Because all the entries of $G_e(z)$ and $G_o(z)$ can only be the first-order polynomials in z (or z^{-1}) or constants, we can assume that

$$G_e(z) = \begin{pmatrix} c_1 + d_1 z & c_2 + d_2 z \\ c_3 + d_3 z & c_4 + d_4 z \end{pmatrix}. \quad (21)$$

By (20),

$$G_o(z) = \begin{pmatrix} c_1 + d_1 z^{-1} & -(c_2 + d_2 z^{-1}) \\ -(c_3 + d_3 z^{-1}) & c_4 + d_4 z^{-1} \end{pmatrix}. \quad (22)$$

Thus, $G(z)$ has the following symmetric form:

$$G(z) = \big(G_e(z)\ G_o(z)\big) = \begin{pmatrix} c_1 + d_1 z & c_2 + d_2 z & c_1 + d_1 z^{-1} & -(c_2 + d_2 z^{-1}) \\ c_3 + d_3 z & c_4 + d_4 z & -(c_3 + d_3 z^{-1}) & c_4 + d_4 z^{-1} \end{pmatrix}.$$

Also, $(\tilde{G}_e(z)\ \tilde{G}_o(z))$ has a similar form.

Type III. Here, we will discuss a sort of biorthogonal multiscaling vector functions with different symmetric centers, i.e., the first and second components of both $\Phi(x)$ and $\tilde{\Phi}(x)$ are respectively symmetric about 0 and 1/2, while $\Psi(x)$ and $\tilde{\Psi}(x)$ are symmetric/antisymmetric about 0. In this case, we have the following theorem.

Theorem 2. Suppose that $\Phi(x)$, $\tilde{\Phi}(x)$, $\Psi(x)$ and $\tilde{\Psi}(x)$ satisfy the above mentioned symmetric conditions of Type III. Then $G(z)U = (G_e(z)\ G_o(z))U$ has the form of

$$G(z)U = \begin{pmatrix} c_1 & c_2(1 + z^{-1}) & c_3(1 + z^{-1}) & c_4(1 - z^{-1}) \\ 0 & d_2(1 - z^{-1}) & d_3(1 - z^{-1}) & d_4(1 + z^{-1}) \end{pmatrix}, \quad (23)$$

and the corresponding multiwavelet functions obtained by $(G_e(z)\ G_o(z))$ satisfy the symmetric conditions of Type III. Here,

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

is an orthogonal matrix and $U^{-1} = U^T = U = U^*$. c_i and d_i, $1 \le i \le 4$ are constants. Also, $\tilde{G}(z)U = (\tilde{G}_e(z)\ \tilde{G}_o(z))U$ has a similar form of (23).

Proof. By substituting $a_1 = 0$, $a_2 = 1/2$, $b_1 = b_2 = 0$ into (13) and note that $S = diag(1, 1), T = diag(1, -1)$, we have

$$G(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} G(z^{-1}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-1} \\ 0 & 0 & z^{-1} & 0 \\ 0 & z^{-1} & 0 & 0 \end{pmatrix}.$$

Denote $G(z)U$ as $R(z)$, then

$$R(z) = G(z)U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} G(z^{-1})UU \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-1} \\ 0 & 0 & z^{-1} & 0 \\ 0 & z^{-1} & 0 & 0 \end{pmatrix} U.$$

Note that

$$U \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-1} \\ 0 & 0 & z^{-1} & 0 \\ 0 & z^{-1} & 0 & 0 \end{pmatrix} U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-1} & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & -z^{-1} \end{pmatrix},$$

thus

$$R(z) = diag(1, -1) R(z^{-1}) diag(1, z^{-1}, z^{-1}, -z^{-1}),$$

i.e.,

$$\begin{cases} R_e(z) = diag(1, -1) R_e(z^{-1}) diag(1, z^{-1}), \\ R_o(z) = diag(1, -1) R_o(z^{-1}) diag(z^{-1}, -z^{-1}). \end{cases} \quad (24)$$

Let

$$R_e(z) = \begin{pmatrix} R_e^{11}(z) & R_e^{12}(z) \\ R_e^{21}(z) & R_e^{22}(z) \end{pmatrix}, R_o(z) = \begin{pmatrix} R_o^{11}(z) & R_o^{12}(z) \\ R_o^{21}(z) & R_o^{22}(z) \end{pmatrix}, \quad (25)$$

by substituting (25) into (24), we have

$$\begin{cases} R_e^{11}(z) = R_e^{11}(z^{-1}) \\ R_e^{12}(z) = z^{-1} R_e^{12}(z^{-1}) \\ R_e^{21}(z) = -R_e^{21}(z^{-1}) \\ R_e^{22}(z) = -z^{-1} R_e^{22}(z^{-1}), \end{cases} \begin{cases} R_o^{11}(z) = z^{-1} R_o^{11}(z^{-1}) \\ R_o^{12}(z) = -z^{-1} R_o^{12}(z^{-1}) \\ R_o^{21}(z) = -z^{-1} R_o^{21}(z^{-1}) \\ R_o^{22}(z) = z^{-1} R_o^{22}(z^{-1}). \end{cases} \quad (26)$$

Because each of the entries of $R_e(z)$ and $R_o(z)$ can only be a first-order polynomial in z (or z^{-1}) or a constant. By (24), $R_e(z)$ and $R_o(z)$ must have the following forms:

$$\begin{cases} R_e(z) = \begin{pmatrix} c_1 + c_1' z^{-1} & c_2 + c_2' z^{-1} \\ d_1 + d_1' z^{-1} & d_2 + d_2' z^{-1} \end{pmatrix}, \\ R_o(z) = \begin{pmatrix} c_3 + c_3' z^{-1} & c_4 + c_4' z^{-1} \\ d_3 + d_3' z^{-1} & d_4 + d_4' z^{-1} \end{pmatrix}. \end{cases} \quad (27)$$

By substituting (27) into (26), we have

$$\begin{cases} R_e^{11}(z) = c_1 \\ R_e^{12}(z) = c_2(1+z^{-1}) \\ R_e^{21}(z) = 0 \\ R_e^{22}(z) = d_2(1-z^{-1}), \end{cases} \quad \begin{cases} R_o^{11}(z) = c_3(1+z^{-1}) \\ R_o^{12}(z) = c_4(1-z^{-1}) \\ R_o^{21}(z) = d_3(1-z^{-1}) \\ R_o^{22}(z) = d_4(1+z^{-1}). \end{cases}$$

Therefore, (23) holds. Analogously, $(\tilde{R}_e(z)\ \tilde{R}_o(z)) = (\tilde{G}_e(z)\ \tilde{G}_o(z))U$ has a similar form of (23).

Denote $H'(z) = H(z)U$ and $\tilde{H}'(z)^* = U\tilde{H}(z)^*$. Thus, $H'(z)$, $\tilde{H}'(z)^*$, $R(z)$ and $\tilde{R}(z)^*$ satisfy the PR condition, which is equivalent to the fact that $H(z)$, $\tilde{H}(z)^*$, $G(z)$ and $\tilde{G}(z)^*$ satisfy the PR condition. For a given pair of $H'(z)$ and $\tilde{H}'(z)^*$, if the solution matrices $R(z)$ and $\tilde{R}(z)^*$ (they are in the form of (23)) can be found, then the multiwavelet vector functions $\Psi(x)$ and $\tilde{\Psi}(x)$ obtained by the polyphase matrices $G(z) = R(z)U$ and $\tilde{G}(z) = \tilde{R}(z)U$ satisfy the symmetric conditions of Type III.

According to the above discussions, the construction of SCSBMSS of the above three symmetric types from their corresponding multiscaling functions can be converted into how to determine the parameters c_i and d_i, $1 \le i \le 4$. Thus, for a given pair of low-pass filter sequences $\{H_k\}$ and $\{\tilde{H}_k\}$, the corresponding polyphase matrices $(G_e(z)\ G_o(z))$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))$ of high-pass filter sequences can be obtained by substituting the corresponding symmetric forms of them into (1). Then $(G_e(z)\ G_o(z))$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))$ can be solved from the matrix equation. The construction algorithm for the above three symmetric types is presented as follows.

Step 1. Compute the polyphase matrices $(H_e(z)\ H_o(z))$ and $(\tilde{H}_e(z)\ \tilde{H}_o(z))$ ($(H_e(z)\ H_o(z))U$ and $(\tilde{H}_e(z)\ \tilde{H}_o(z))U$ for Type III) by the low-pass filter matrix coefficients.

Step 2. Substitute the corresponding symmetric forms of $(G_e(z)\ G_o(z))$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))$ ($(G_e(z)\ G_o(z))U$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))U$ for Type III) into (1) according to the symmetric centers of scaling functions. By solving (1), the relations of c_i and d_i, $1 \le i \le 4$, can be obtained to reduce the number of parameters.

Step 3. Select proper parameter values to obtain the polyphase matrices of the corresponding multiwavelets.

3. Examples

Example 1. Consider the biorthogonal sets of scaling vector functions and their corresponding multiwavelets presented in [5]. Here, $\Phi(x)$, $\tilde{\Phi}(x)$, $\Psi(x)$ and $\tilde{\Psi}(x)$ are symmetric/antisymmetric about 0. This satisfies the symmetric Type I. All the scaling and multiwavelet functions have a support in $[-1, 1]$. The scaling coefficients were given in [5] as follows.

$$H_{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ -1 & -\frac{2}{5} \end{pmatrix}, H_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, H_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{5} \\ 1 & -\frac{2}{5} \end{pmatrix};$$

$$\tilde{H}_{-1} = \begin{pmatrix} \frac{1}{2} & \frac{5}{4} \\ \frac{-7}{16} & \frac{-35}{32} \end{pmatrix}, \tilde{H}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \tilde{H}_1 = \begin{pmatrix} \frac{1}{2} & \frac{-5}{4} \\ \frac{7}{16} & \frac{-35}{32} \end{pmatrix}.$$

Solution. From the definitions of polyphase matrices described in Section I, we have,

$$H_e(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{pmatrix},$$

$$H_o(z) = \begin{pmatrix} \frac{\sqrt{2}(z^{-1}+1)}{4} & \frac{\sqrt{2}(z^{-1}-1)}{10} \\ \frac{\sqrt{2}(1-z^{-1})}{2} & \frac{-\sqrt{2}(z^{-1}+1)}{5} \end{pmatrix},$$

$$\tilde{H}_e(z)^* = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{pmatrix},$$

$$\tilde{H}_o(z)^* = \begin{pmatrix} \frac{\sqrt{2}(z+1)}{4} & \frac{7\sqrt{2}(1-z)}{32} \\ \frac{5\sqrt{2}(z-1)}{8} & \frac{-35\sqrt{2}(z+1)}{64} \end{pmatrix}.$$

Because the symmetric centers of the scaling functions are 0. We can use (14) for computation. By substituting (14) into (4), we have

$$\begin{pmatrix} G_e(z)\ G_o(z) \end{pmatrix} \begin{pmatrix} \tilde{H}_e(z)^* \\ \tilde{H}_o(z)^* \end{pmatrix}$$
$$= \begin{pmatrix} c_1 & 0 & c_3(1+z^{-1}) & c_4(1-z^{-1}) \\ 0 & d_2 & d_3(1-z^{-1}) & d_4(1+z^{-1}) \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}(z+1)}{4} & \frac{7\sqrt{2}(1-z)}{32} \\ \frac{5\sqrt{2}(z-1)}{8} & \frac{-35\sqrt{2}(z+1)}{64} \end{pmatrix} = O_2.$$

The solutions of the above matrix equation are

$$\begin{cases} c_3 = -\frac{5}{2}c_4; c_1 = 5c_4 \\ d_3 = -\frac{5}{2}d_4; d_2 = \frac{35}{4}d_4. \end{cases} \quad (28)$$

Thus,

$$\begin{pmatrix} G_e(z)\ G_o(z) \end{pmatrix}$$
$$= \begin{pmatrix} 5c_4 & 0 & \frac{-5(1+z^{-1})}{2}c_4 & c_4(1-z^{-1}) \\ 0 & \frac{35}{4}d_4 & \frac{5(z^{-1}-1)}{2}d_4 & d_4(1+z^{-1}) \end{pmatrix} \quad (29)$$

Analogously, we can obtain $\tilde{G}_e(z)$ and $\tilde{G}_o(z)$ by solving the matrix equation $(\tilde{G}_e(z)\ \tilde{G}_o(z))(H_e(z)\ H_o(z))^* = O_2$. The resultant matrix is

$$\begin{pmatrix} \tilde{G}_e(z)\ \tilde{G}_o(z) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{4}{5}\tilde{c}_4 & 0 & \frac{-2(1+z^{-1})}{5}\tilde{c}_4 & \tilde{c}_4(1-z^{-1}) \\ 0 & \frac{16}{5}\tilde{d}_4 & \frac{2(z^{-1}-1)}{5}\tilde{d}_4 & \tilde{d}_4(1+z^{-1}) \end{pmatrix}. \quad (30)$$

Finally, by substituting (29) and (30) into (5), we have $c_4 = \frac{1}{8\tilde{c}_4}$, $d_4 = \frac{1}{32\tilde{d}_4}$. Thus,

$$\left(G_e(z)\ G_o(z)\right) = \begin{pmatrix} \frac{5}{8\tilde{c}_4} & 0 & \frac{-5(1+z^{-1})}{16\tilde{c}_4} & \frac{(1-z^{-1})}{8\tilde{c}_4} \\ 0 & \frac{35}{128\tilde{d}_4} & \frac{5(z^{-1}-1)}{64\tilde{d}_4} & \frac{(1+z^{-1})}{32\tilde{d}_4} \end{pmatrix},$$

$$\begin{pmatrix} \tilde{G}_e(z)^* \\ \tilde{G}_o(z)^* \end{pmatrix} = \begin{pmatrix} \frac{4}{5}\tilde{c}_4 & 0 \\ 0 & \frac{16}{5}\tilde{d}_4 \\ \frac{-2(1+z)}{5}\tilde{c}_4 & \frac{-2(1-z)}{5}\tilde{d}_4 \\ \tilde{c}_4(1-z) & \tilde{d}_2(1+z) \end{pmatrix}.$$

It can be seen that there are only two parameters in $(G_e(z)\ G_o(z))$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*$. In fact, as long as $\tilde{c}_4 \neq 0$ and $\tilde{d}_4 \neq 0$, the above two matrices are a solution of the construction problem. Especially, let $\tilde{c}_4 = -\frac{\sqrt{2}}{8}$ and $\tilde{d}_4 = \frac{35\sqrt{2}}{128}$. Then the resultant matrices $(G_e(z)\ G_o(z))$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))$ coincide with the results as documented in [5]. The graphs of multiscaling and multiwavelet functions of this example are shown in Figure 1 and Figure 2.

Example 2. Here, we will reconstruct the biorthogonal multiwavelet presented in [8], which satisfies the symmetric Type II. Both the scaling and wavelet functions are symmetric/antisymmetric about 1/2. The scaling coefficients were given in [8] as follows.

$$H_0 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, H_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \tilde{H}_{-1} = \begin{pmatrix} 0 & \frac{1}{8} \\ 0 & -\frac{1}{8} \end{pmatrix},$$

$$\tilde{H}_0 = \begin{pmatrix} 1 & \frac{1}{8} \\ -1 & \frac{1}{8} \end{pmatrix}, \tilde{H}_1 = \begin{pmatrix} 1 & -\frac{1}{8} \\ 1 & \frac{1}{8} \end{pmatrix}, \tilde{H}_2 = \begin{pmatrix} 0 & -\frac{1}{8} \\ 0 & -\frac{1}{8} \end{pmatrix}.$$

Solution. The polyphase matrices of the scaling coefficients are

$$H_e(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, H_o(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix};$$

$$\tilde{H}_e(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1-z}{8\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1-z}{8\sqrt{2}} \end{pmatrix}, \tilde{H}_o(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{z^{-1}-1}{8\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1-z^{-1}}{8\sqrt{2}} \end{pmatrix}.$$

According to (4), (21) and (22), we have

$$\left(G_e(z)\ \ G_o(z)\right) \begin{pmatrix} \tilde{H}_e(z)^* \\ \tilde{H}_o(z)^* \end{pmatrix}$$

$$= \begin{pmatrix} c_1 + d_1 z & c_2 + d_2 z & c_1 + d_1 z^{-1} & -(c_2 + d_2 z^{-1}) \\ c_3 + d_3 z & c_4 + d_4 z & -(c_3 + d_3 z^{-1}) & c_4 + d_4 z^{-1} \end{pmatrix}.$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1-z^{-1}}{8\sqrt{2}} & \frac{1-z^{-1}}{8\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{z-1}{8\sqrt{2}} & \frac{1-z}{8\sqrt{2}} \end{pmatrix} = O_2.$$

The above matrix equation equals to four equations with 8 parameters. The solutions of the matrix equation are $c_1 = -\frac{1}{8}c_2$, $d_1 = \frac{1}{8}c_2$, $d_2 = 0$; $c_3 = \frac{1}{8}c_4$, $d_3 = -\frac{1}{8}c_4$, $d_4 = 0$. Thus,

$$\left(G_e(z)\ G_o(z)\right) = \begin{pmatrix} \frac{c_2(z-1)}{8} & c_2 & \frac{c_2(z^{-1}-1)}{8} & -c_2 \\ \frac{c_4(1-z)}{8} & c_4 & \frac{c_4(z^{-1}-1)}{8} & c_4 \end{pmatrix}.$$

Analogously, $\tilde{G}_e(z)$ and $\tilde{G}_o(z)$ can be obtained by solving (3). The resultant matrix is

$$\left(\tilde{G}_e(z)\ \tilde{G}_o(z)\right) = \begin{pmatrix} 0 & \tilde{c}_2 + \tilde{d}_2 z & 0 & -(\tilde{c}_2 + \tilde{d}_2 z^{-1}) \\ 0 & \tilde{c}_4 + \tilde{d}_4 z & 0 & \tilde{c}_4 + \tilde{d}_4 z^{-1} \end{pmatrix}.$$

Finally, by solving (5), we have $\tilde{d}_2 = 0, \tilde{d}_4 = 0, \tilde{c}_2 = \frac{1}{2c_2}, \tilde{c}_4 = \frac{1}{2c_4}$. Thus

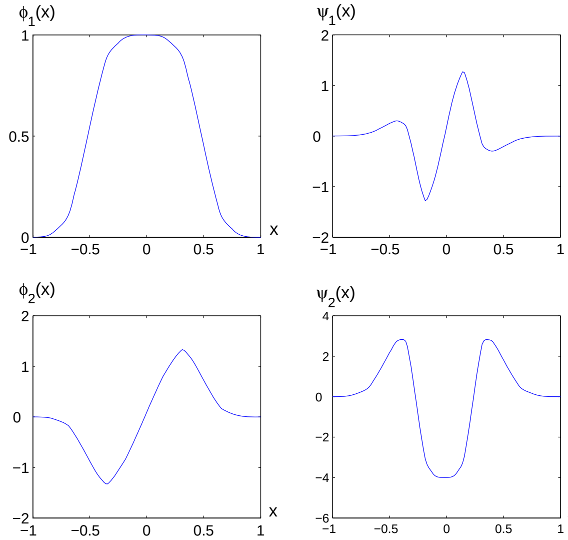

Figure 1 Graphs of scaling functions and their corresponding multiwavelets of Example 1: $\phi_1(x)$, $\psi_1(x)$, $\phi_2(x)$ and $\psi_2(x)$.

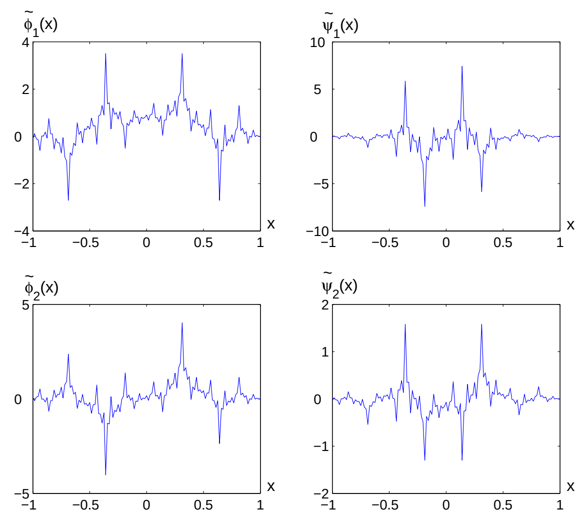

Figure 2 Graphs of the dual scaling functions and their corresponding dual multiwavelets of Example 1: $\tilde{\phi}_1(x)$, $\tilde{\psi}_1(x)$, $\tilde{\phi}_2(x)$ and $\tilde{\psi}_2(x)$.

$$\left(\tilde{G}_e(z)\ \tilde{G}_o(z)\right) = \begin{pmatrix} 0 & \frac{1}{2c_2} & 0 & -\frac{1}{2c_2} \\ 0 & \frac{1}{2c_4} & 0 & \frac{1}{2c_4} \end{pmatrix}.$$

Especially, when $c_2 = -\frac{1}{\sqrt{2}}$ and $c_4 = \frac{1}{\sqrt{2}}$, we can obtain the so-called *2/4 SABMF* biorthogonal multiwavelet as documented in [8]. The graphs of multiscaling and multiwavelet functions of this example are shown in Figure 3 and Figure 4.

Example 3. The biorthogonal multiwavelet constructed in [3] accords with the symmetric form of Type III. The

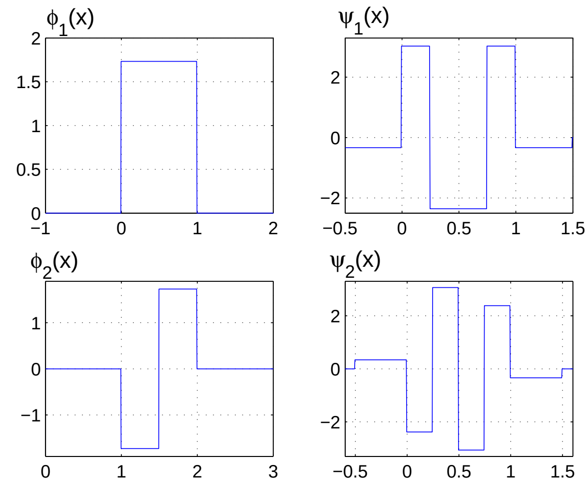

Figure 3 Graphs of the scaling functions and their corresponding multiwavelets of Example 2: $\phi_1(x)$, $\psi_1(x)$, $\phi_2(x)$ and $\psi_2(x)$.

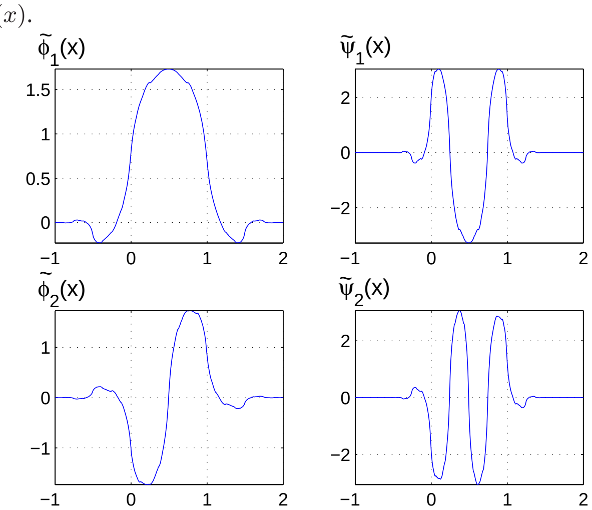

Figure 4 Graphs of the dual scaling functions and their corresponding dual multiwavelets of Example 2: $\tilde{\phi}_1(x)$, $\tilde{\psi}_1(x)$, $\tilde{\phi}_2(x)$ and $\tilde{\psi}_2(x)$.

components of the scaling vectors have different symmetric centers, i.e., $\phi_1(x)$ and $\phi_2(x)$ are symmetric about 0 and 1/2, respectively (likewise, $\tilde{\phi}_1(x)$ and $\tilde{\phi}_2(x)$ are also symmetric about 0 and 1/2, respectively). The matrix coefficients of scaling vector functions were given in [3] as follows.

$$H_{-2} = \begin{pmatrix} 0 & -\frac{1}{6\sqrt{3}} \\ 0 & 0 \end{pmatrix}, H_{-1} = \begin{pmatrix} -\frac{1}{6} & \frac{5}{6\sqrt{3}} \\ 0 & 0 \end{pmatrix}, H_0 = \begin{pmatrix} 1 & \frac{5}{6\sqrt{3}} \\ 0 & \frac{2}{3} \end{pmatrix},$$

$$H_1 = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{6\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{2}{3} \end{pmatrix}; \tilde{H}_{-2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{H}_{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 \end{pmatrix},$$

$$\tilde{H}_0 = \begin{pmatrix} 1 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} \end{pmatrix},$$

$$\tilde{H}_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{2}{\sqrt{3}} & \frac{1}{2} \end{pmatrix}.$$

Solution. Compute $H'(z) = H(z)U$ and $\tilde{H}'(z)^* = U\tilde{H}(z)^*$, as follows.

$$H'(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{(1+z^{-1})}{3\sqrt{3}} & \frac{-(1+z^{-1})}{6\sqrt{2}} & \frac{(1-z^{-1})}{2\sqrt{3}} \\ 0 & \frac{2}{3} & \frac{\sqrt{6}}{3} & 0 \end{pmatrix},$$

$$\tilde{H}'(z)^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}(1+z)}{4} & \frac{1}{2} \\ \frac{-(1+z)}{2\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{\sqrt{3}(1-z)}{4} & 0 \end{pmatrix}. \tag{31}$$

According to (23), let $R(z) = G(z)U$, then

$$\begin{pmatrix} R_e(z) & R_o(z) \end{pmatrix} \begin{pmatrix} \tilde{H}'_e(z)^* \\ \tilde{H}'_o(z)^* \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & c_2(1+z^{-1}) & c_3(1+z^{-1}) & c_4(1-z^{-1}) \\ 0 & d_2(1-z^{-1}) & d_3(1-z^{-1}) & d_4(1+z^{-1}) \end{pmatrix}.$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}(1+z)}{4} & \frac{1}{2} \\ \frac{-(1+z)}{2\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{\sqrt{3}(1-z)}{4} & 0 \end{pmatrix} = O_2.$$

By solving this matrix equation, we have $c_1 = 6c_3, c_2 = -\frac{4}{\sqrt{6}}c_3, c_4 = -\sqrt{6}c_3$; $d_2 = -\frac{4}{\sqrt{6}}d_3, d_4 = -\sqrt{6}d_3$. Thus, $\begin{pmatrix} R_e(z) & R_o(z) \end{pmatrix} =$

$$\begin{pmatrix} 6c_3 & \frac{-4(1+z^{-1})}{\sqrt{6}}c_3 & c_3(1+z^{-1}) & \sqrt{6}c_3(z^{-1}-1) \\ 0 & \frac{4(z^{-1}-1)}{\sqrt{6}}d_3 & d_3(1-z^{-1}) & -\sqrt{6}d_3(1+z^{-1}) \end{pmatrix}. \tag{32}$$

Analogously, we can obtain $\tilde{R}_e(z)$ and $\tilde{R}_o(z)$ by solving the matrix equation $(\tilde{R}_e(z)\ \tilde{R}_o(z))(H'_e(z)\ H'_o(z))^* = O_2$. The resultant matrix is
$\begin{pmatrix} \tilde{R}_e(z) & \tilde{R}_o(z) \end{pmatrix} =$

$$\begin{pmatrix} 2\tilde{c}_3 & \frac{-\sqrt{6}(1+z^{-1})}{2}\tilde{c}_3 & \tilde{c}_3(1+z^{-1}) & \frac{\sqrt{6}(z^{-1}-1)}{2}\tilde{c}_3 \\ 0 & \frac{\sqrt{6}(z^{-1}-1)}{2}\tilde{d}_3 & \tilde{d}_3(1-z^{-1}) & \frac{-\sqrt{6}(1+z^{-1})}{2}\tilde{d}_3 \end{pmatrix}. \tag{33}$$

Finally, by substituting (32) and (33) into $(R_e(z)\ R_o(z)) \cdot (\tilde{R}_e(z)\ \tilde{R}_o(z))^* = I_2$, we have

$$c_3 = \frac{1}{24\tilde{c}_3}, d_3 = \frac{1}{12\tilde{d}_3}. \tag{34}$$

Substitute (34) into (32) and compute $G(z) = R(z)U$, $\tilde{G}(z)^* = U\tilde{R}(z)^*$, the resultant polyphase matrices are
$(G_e(z)\ G_o(z))$

$$= \begin{pmatrix} \frac{1}{4\tilde{c}_3} & \frac{(z^{-1}-5)}{24\sqrt{3}\tilde{c}_3} & \frac{(1+z^{-1})}{24\tilde{c}_3} & \frac{(1-5z^{-1})}{24\sqrt{3}\tilde{c}_3} \\ 0 & \frac{-(z^{-1}+5)}{12\sqrt{3}\tilde{d}_3} & \frac{(1-z^{-1})}{12\tilde{d}_3} & \frac{(1+5z^{-1})}{12\sqrt{3}\tilde{d}_3} \end{pmatrix},$$

$(\tilde{G}_e(z)\ \tilde{G}_o(z))$

$$= \begin{pmatrix} 2\tilde{c}_3 & -\sqrt{3}\tilde{c}_3 & \tilde{c}_3(1+z^{-1}) & -\sqrt{3}\tilde{c}_3 z^{-1} \\ 0 & -\sqrt{3}\tilde{d}_3 & \tilde{d}_3(1-z^{-1}) & \sqrt{3}\tilde{d}_3 z^{-1} \end{pmatrix}.$$

Especially, when $\tilde{c}_3 = -\frac{\sqrt{2}}{4}$ and $\tilde{d}_3 = \frac{1}{2}$, we can obtain the biorthogonal multiwavelet as documented in [3]. The graphs of multiscaling and multiwavelet functions of this example are shown in Figure 5 and Figure 6.

Remark. Besides the three symmetric types discussed in Section 2, symmetric forms of other symmetric types can also be derived. For example, suppose the multiscaling and multiwavelet vector functions are symmetric/antisymmetric about 0 and 1/2, respectively. According to (13), we have

$$\begin{cases} G_e(z) = zT_{1/2}G_e(z^{-1})S_0 \\ G_o(z) = T_{1/2}G_o(z^{-1})S_0. \end{cases}$$

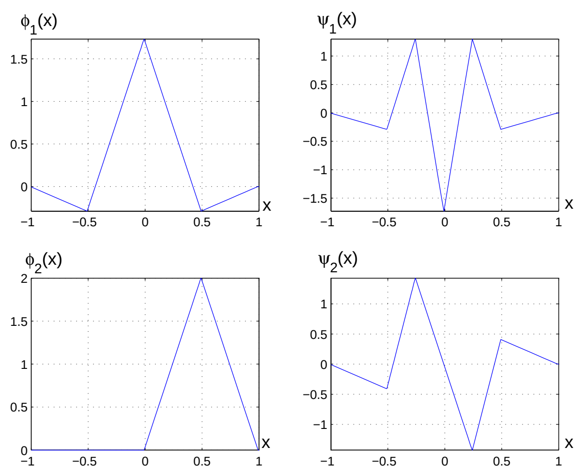

Figure 5 Graphs of the scaling functions and their corresponding multiwavelets of Example 3: $\phi_1(x)$, $\psi_1(x)$, $\phi_2(x)$ and $\psi_2(x)$.

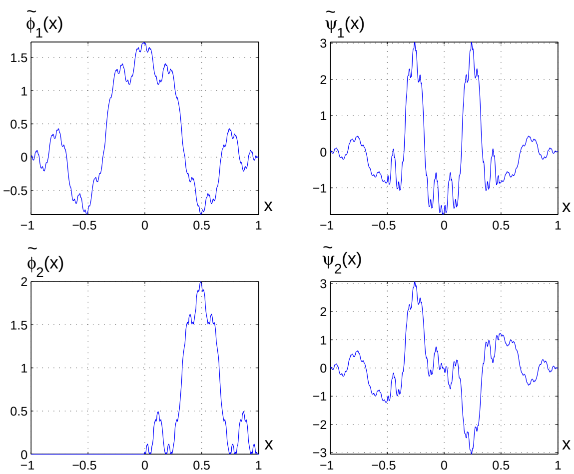

Figure 6 Graphs of the dual scaling functions and their corresponding dual multiwavelets of Example 3: $\tilde{\phi}_1(x)$, $\tilde{\psi}_1(x)$, $\tilde{\phi}_2(x)$ and $\tilde{\psi}_2(x)$.

For such a biorthogonal multiwavelet system with short sequences, the polyphase matrix has the form of

$$\begin{pmatrix} G_e(z) \; G_o(z) \end{pmatrix} = \begin{pmatrix} c_1(1+z) & c_2(1-z) & c_3 & 0 \\ d_1(1-z) & d_2(1+z) & 0 & d_4 \end{pmatrix}. \quad (35)$$

Also, $(\tilde{G}_e(z)\; \tilde{G}_o(z))$ is in a similar form.

For the scaling vectors given in Example 1, we can prove that there are no corresponding biorthogonal multiwavelets with short sequences such that the polyphase matrices $G(z)$ and $\tilde{G}(z)$ satisfy the symmetric form of (35). But we can verify that the following high-pass matrix sequences satisfy the symmetric properties described in this Remark (i.e., the multiscaling and multiwavelet vectors are symmetric/antisymmetric about 0 and 1/2 respectively).

$$G(-1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, G(0) = \sqrt{2}\begin{pmatrix} -\frac{1}{2} & -\frac{7}{8} \\ \frac{5}{4} & \frac{35}{16} \end{pmatrix},$$

$$G(1) = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, G(2) = \sqrt{2}\begin{pmatrix} -\frac{1}{2} & \frac{7}{8} \\ -\frac{5}{4} & \frac{35}{16} \end{pmatrix};$$

$$\tilde{G}(-1) = \sqrt{2}\begin{pmatrix} \frac{3}{32} & \frac{15}{64} \\ \frac{-3}{80} & \frac{-3}{32} \end{pmatrix}, \tilde{G}(0) = \sqrt{2}\begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{10} & \frac{1}{10} \end{pmatrix},$$

$$\tilde{G}(1) = \sqrt{2}\begin{pmatrix} \frac{10}{32} & 0 \\ 0 & \frac{10}{32} \end{pmatrix}, \tilde{G}(2) = \sqrt{2}\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{10} & \frac{1}{10} \end{pmatrix},$$

$$\tilde{G}(3) = \sqrt{2}\begin{pmatrix} \frac{3}{32} & -\frac{15}{64} \\ \frac{3}{80} & -\frac{3}{32} \end{pmatrix}.$$

The above matrix coefficients can not be obtained by the approach proposed in this paper because the coefficient number of the dual-multiwavelet exceeds 4. We will further study the construction of biorthogonal multiwavelets with arbitrary sequences by other approaches.

4. Conclusion

In this paper, a simple and direct approach for the construction of SCSBMSS based on the multiscaling vectors was proposed. By studying the symmetric properties of the multiscaling and multiwavelet vectors, parameterized symmetric forms of the polyphase matrices could be obtained. Thus, for a given pair of scaling vectors, by substituting the corresponding symmetric forms of polyphase matrices into matrix equations of the PR condition, the parameters could be greatly reduced, which finally led to the solution of the multiwavelet construction problem. Examples showed that our proposed approach was direct and useful for the construction of SCSBMSS.

Acknowledgement

This work was supported by National Natural Science Foundation of China (No. 60802045, 61104078, 61073079); the Scientific Research Foundation for Returned Overseas Chinese Scholars ([2009] 1001) and the Fundamental Research Funds for the Central Universities of China (2011JBM223, 201012200159, 201012200110);the Beijing Natural Science Foundation (A study of the general approach for signal sparse representation and Compressed Sensing based on the multiwavelet analysis, No. 4113075); new Teacher Foundation of State Education Ministry (20100162120019); the Major State Basic Research Development Program of China (2011CB302203-4). Foundation of Key Laboratory of System Control and Information Processing, (Grant No. SCIP2011009)

References

[1] T.N.T.Goodman, S.L.Lee, W.S.Tang, Wavelet in wandering subspace. *Trans. Amer. Math. Soc.* 338 (1993), 639–654.
[2] G.Donovan, J.S.Geronimo, D.P.Hardin, P.R.Massopust, Construction of orthogonal wavelets using fractal interpolation functions. *SIAM J. Math. Anal.* 27 (1996), 1158–1192.
[3] D.P.Hardin, S.A.Marasovich, Biorthogonal multiwavelets on [-1,1]. *Appl. Comput. Harmon. Anal.* 7 (1999), 34-53.
[4] W.Lawton, S.L.Lee, Z.Shen, An algorithm for matrix extension and wavelet construction. *Math. Comp.* 65 (1996), 723–737.
[5] S.S.Goh, V.B.Yap, Matrix extension and biorthogonal multiwavelet construction. *Linear Alg. Appl.* 269 (1998), 139–157.
[6] S.S.Goh, Q.T.Jiang, T.Xia, Construction of biorthogonal multiwavelets using the lifting scheme. *Appl. Comput. Harmon. Anal.* 9 (2000), 336–352.
[7] R.Turcajova, Construction of symmetric biorthogonal multiwavelets by lifting, *Proceeding of SPIE, Wavelet Applications in Signal and Image Processing VII, Denver, Colorado, USA, July* (1999), 443–454.
[8] H.H.Tan, L.X.Shen, J.Y.Tham, New biorthogonal multiwavelets for image compression. *Signal Process.* 79 (2000), 45–65.
[9] L.H.Cui, Some properties and construction of multiwavelets related to different symmetric centers. *Math. Comput. Simulat.* 70 (2005), 69–89.

Yi-Gang Cen received the Ph.D. degree in Control Theory and Control Engineering from the Huazhong University of Science and Technology, Wuhan, China, in 2006. From 2006 to 2007, he was with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. Since 2010, he has been an Associate Professor with the School of Computer Science and Information Technology, Beijing Jiaotong University, China. His research interest includes wavelet analysis, compressed sensing, signal processing and image processing.

Li-Hui Cen received the Ph.D. degree in Control Theory and Control Engineering from the Shanghai Jiaotong University, Shanghai, China, in 2009. Now she is a Lecture with the School of Information Science and Engineering, Central South University, Hunan, Changsha, China. Her research interest includes control of open channels, signal processing,predict control *etc.*.

Xiao-Fang Chen received the Ph.D. degree in Control Theory and Control Engineering from Central South University, Changsha, China, in 2004. Now he is a Lecture in the School of Information Science and Engineering, Central South University, Hunan, Changsha, China. His research interest includes complex system modeling, optimal control.