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Algebraic Aspects of Evolution Partial Differential Equations Arising in Financial Mathematics

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In the modelling of various financial instruments, risks, prices of commodities, etc., the end result is frequently an evolution partial differential equation. A remarkable number of these have rich algebraic structures. This richness facilitates the process of resolution of problems in an algorithmic fashion rather than the apparently 'seat-of-the-pants' methods which abound in the literature. We illustrate the algebraic resolution of these problems with a number of examples.

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1 Philosophical Preliminaries

It is deeply rooted within the human psyche to ask the question 'Why is it so?'.

From ancient times the process of modelling of observed phenomena has been one of mankind's favoured intellectual pastimes and has led to the development *inter alia* of Mathematics and its applications as we know them today. The central theme of modelling is to extract the essence of the phenomenon under consideration so that its gross behaviour can be faithfully predicted without undue effort. The process of modelling comes with certain ancillary criteria. The most important one of these is the degree of detail/accuracy

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*permanent address: School of Mathematical Sciences, University of KwaZulu-Natal, Durban, Republic of South Africa required from the predictions implied by the model. To take an old and familiar example one could consider that of Thomas Malthus [31] in which he proposed that population, $p(t)$, followed the differential equation

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = rp(t), \tag{1.1}$$

where r is a positive constant – somewhat to the distress of Malthus as it would seem from his comments – and so a finite Earth must inevitably collapse under the weight of an infinite morass of wretched humanity.

It is generally considered that Malthus was optimistic in one sense or another with the interpretation of the sense being perhaps more in the mind of the interpreter than in the necessities of reality.

We began this paper with the cautionary example to emphasise that we are discussing the modelling of a situation – one hesitates to call it the real world – in which one considers the possibilities of making an investment. In recent decades there has been an expansion in the concept of financial instruments which in a very real sense beggars one's understanding. the financial pages of the weekly of my country. In the early Sixties of the last century the idea of options was considered in the investment section of weekly journals as a minor matter and to be regarded as somewhat risky and also somewhat expensive.

The situation has changed somewhat since then! In the late Sixties and early Seventies there was some serious mathematical modelling of the problem of the wisdom or otherwise of exercising an option. Generally this was perceived to be concerned with options to purchase shares, which at that time was a minor activity of the Stock Exchange, but already in one of the fundamental papers of that period the authors observed that the mathematical considerations could be applied to any situation in which one wished to make now a financial decision about something in the future. A minor activity suddenly became a major industry in itself.

The basis for the modelling of the type of problem under consideration here is the realisation that many factors in the financial markets as widely defined behave stochastically, usually about some trendline. The stochastic variation of the variables in the models considered here is assumed to be Brownian. The advantage is that the integrodifferential equations which arise can be replaced by partial differential equations and this makes for a subsequent easing of the analysis. The basic concepts are very similar to those to be found in Statistical Physics and were introduced into Econometrics by writers such as Samuelson, Modigliani, Merton, Black and Scholes.

About 140 years ago an important contribution to the resolution of differential equations was introduced by the Norwegian mathematician, Sophus Lie, who considered the implications of invariance under transformation [25–30]. The class of transformation considered was continuous and so calculations could be made in the vicinity of the identity, i.e., at the infinitesimal level. This made everything linear! Over the intervening years the concepts introduced by Lie have become cornerstones of many aspects of the exact sciences.

If the process of modelling a situation leads to a differential equation, this is a constraint upon the free ranging of the variables related by the equation. If the equation admits transformations which leave it invariant, each one of these transformations imposes a further constraint. The greater the number of constraints the less freely can the variables range over their space. This makes the process of solution of the equation easier.

2 Some Mathematical Preliminaries

In the case of continuous transformations the transformation can be written as a deformation from the identities

$$\bar{t} = t + \varepsilon\tau, \quad \bar{x} = x + \varepsilon\xi, \quad \bar{u} = u + \varepsilon\eta,$$

where ε is an infinitesimal parameter, and can be considered to be generated by the differential operator

$$\Gamma = \tau\partial_t + \xi\partial_x + \eta\partial_u.$$

If the differential equation is invariant under the transformation generated by Γ, the differential operator is called a symmetry of the differential equation. As a general principle the greater the number of symmetries the greater the ease of solution of the equation.

If a differential equation possesses more than one symmetry, the differential operators constitute a Lie Algebra under the operation of taking the Lie Bracket, namely

$$[\Gamma_1, \Gamma_2]_{LB} = \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1.$$

A Lie Algebra is a representation of a Lie Group. Some common algebras are $so(3)$, $so(2,1)$ and $sl(2,R)$. A systematic nomenclature was established by Mubarakzyanov [33–36]. An important aspect of the possession of the algebra is that differential equations may look different, but possess the same algebra and so be related.

The coefficient functions in the differential operator Γ, τ, ξ and η, have not been given specific arguments. In this talk we confine our attention to symmetries in which the coefficient functions depend only upon the independent and dependent variables. Such symmetries are called point symmetries and the transformations which they engender are point transformations, i.e., the mapping is from the space of independent and dependent variables to the same. Other dependencies can include derivatives and integrals. In confining our attention to point symmetries we implicitly impose a constraint on the class of problems which can be successfully treated. The payoff is that the calculation of the symmetries can be done in an algorithmic fashion and so the whole process can be left to the tender mercies of the computer. There have been quite a few programs written for the calculation of symmetries and we have found three to be of particular use, Alan Head's LIE [19], Clara Nucci's REDUCE-based code [38, 39] and Stelios Dimas' Sym which is a Mathematica add-on [1, 9, 10]. In the assessment of any code to be used for the computation of symmetries there are two essential criteria which must be successfully addressed. The first is completeness, *ie* symmetries of the class being determined must not be omitted. The second is the avoidance of spurious results. Some codes have been found to be wanting in one or other or both of these respects. For a simple equation it is not difficult to check the results, but in the case of a complex and unwieldy system one must be able to rely upon the output of the machine.

The differential equations which we consider are of the form

$$u_t = f\left(t,\, x,\, u,\, u_x,\, u_{xx}\right) \tag{2.1}$$

in which u and x may be multivariables. It is usual for t to denote time. The meanings of the other variables depend upon the specific situation being modelled. Generally the equation is liberally decorated with parameters. The derivation of (2.1) is to be found in stochastic processes, usually leading to an Itô equation [23] which can then be associated with a Fokker-Planck [15, 41] or a Feynman-Kac [14, 24] partial differential equation. It is this equation which we seek to solve using symmetries.

A standard problem is to determine the solution of (2.1) which satisfies a terminal condition, i.e., $u(T, x)$ has some specific form. A typical example is $u(T, x) = U$.

3 The Black-Scholes-Merton Equation

The Black-Scholes-Merton Equation [3, 4, 32] is probably the best known of the equations arising in the Mathematics of Finance. Although it was presented almost forty years ago and much has developed since then, it is still used in various forms. The equation is

$$u_t + \tfrac{1}{2}\sigma^2 x^2 u_{xx} + r x u_x - r u = 0, \tag{3.1}$$

where σ is the variance of the underlying stochastic process, x is the value of the asset and r is the yield on a 'safe' investment. The Lie point symmetries of (3.1) are [1]

$$\begin{aligned}
\Gamma_1 &= x\partial_x \\
\Gamma_2 &= 2tx\partial_x + \left\{t - \frac{2}{\sigma^2}(rt - \log x)\right\} u\partial_u \\
\Gamma_3 &= u\partial_u \\
\Gamma_4 &= \partial_t
\end{aligned}$$

[1] This analysis was firstly undertaken by Gasizov and Ibragimov [16]. Unfortunately the problem which they solved was that of (3.1) subject to an initial condition being a delta function. This is not the type of initial condition which one would wish to encounter in a financial situation.

$$\Gamma_5 = 8t\partial_t + 4x\log x\partial_x + \left\{4tr + \sigma^2 t + 2\log x + \frac{4r}{\sigma^2}(rt - \log x)\right\} u\partial_u$$

$$\Gamma_6 = 8t^2\partial_t + 8tx\log x\partial_x + \left\{-4t + 4t^2 r + \sigma^2 t^2 + 4t\log x + \frac{4}{\sigma^2}(rt - \log x)^2\right\} u\partial_u$$

$$\Gamma_\infty = f(t,x)\partial_u,$$

where Γ_∞ is an infinite-dimensional subalgebra with $f(t,x)$ being any solution of (3.1). The algebra of the finite-dimensional subalgebra of symmetries is $sl(2,R) \oplus_s W_3$, where W_3 is the three-dimensional Heisenberg-Weyl algebra familiar from the quantum mechanics of the simple harmonic oscillator. In terms of the notation of Mubarakzyanov the algebra is $\{A_{3,8} \oplus_s A_{3,3}\} \oplus_s \infty A_1$.

The problem as stated is not just (3.1) but also contains the terminal condition, $u(T,x) = U$. Since for the purposes of symmetry analysis the dependent variable is treated as an independent variable, the terminal condition is in fact two conditions, namely $t = T$ and $u = U$ and both have to hold for all admitted values of the other variable, x. To determine which combinations of the symmetries above are consistent with these conditions we write a general symmetry as

$$\Gamma = \sum_{i=1}^{i=6} \alpha_i \Gamma_i, \tag{3.2}$$

where the coefficients are constants to be determined. The application of Γ to the terminal condition gives

$$\alpha_4 + 8T\alpha_5 + 8T^2\alpha_6 = 0 \tag{3.3}$$

in the case of $t = T$ and

$$\alpha_2\left\{T - \frac{2}{\sigma^2}(rT - \log x)\right\}U + \alpha_3 U + \alpha_5\left\{4Tr + \sigma^2 T + 2\log x + \frac{4r}{\sigma^2}(rT - \log x)\right\}U$$
$$+\,\alpha_6\left\{-4T + 4T^2 r + \sigma^2 T^2 + 4T\log x + \frac{4}{\sigma^2}(rT - \log x)^2\right\}U \tag{3.4}$$

in the case of $u = U$. We solve (3.3) and (3.4) for the α_i bearing in mind that the equations have to hold for all valid x. We obtain

$$\begin{array}{ll} \alpha_1 \text{ is arbitrary} & \alpha_2 = (2r - \sigma^2)\alpha_5 \\ \alpha_5 \text{ is arbitrary} & \alpha_3 = -8rT\alpha_5 \\ \alpha_6 = 0 & \alpha_4 = -8T\alpha_5 \end{array} \tag{3.5}$$

so that there are two symmetries,

$$\Lambda_1 = x\partial_x \quad \text{and}$$
$$\Lambda_2 = 8(t-T)\partial_t + (4rt - 2\sigma^2 t + 4\log x)x\partial_x + 8r(t-T)u\partial_u,$$

compatible with the terminal condition.

Since $[\Lambda_1,\, \Lambda_2]_{LB} = 4\Lambda_1$, reduction of (3.1) by the normal subgroup, Λ_1, is to be preferred on theoretical grounds as well as ergonomic grounds. The invariants are found by the solution of the associated Lagrange's system,

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}u}{0},$$

are t and u. We introduce the similarity variables, $y = t$ and $v = u$, into (3.1) to obtain the ordinary differential equation

$$v' - rv = 0 \quad \longrightarrow \quad v = K\mathrm{e}^{ry} \quad \longrightarrow \quad u = K\mathrm{e}^{rt}.$$

We use the terminal condition to evaluate the constant of integration and arrive at the solution of the terminal problem for (3.1),

$$u(t,x) = U\exp[r(t-T)]. \tag{3.6}$$

As the solution of this problem is unique, there is no need to make use of the second symmetry.

The Black-Scholes-Merton Equation is rather typical of a wide class of equations which arise in Financial Mathematics. Naturally the details vary with the precise nature of the problem being modelled, but the overall picture is quite similar. The derived $(1+1)$ partial differential evolution equation is linear, possesses the same number of symmetries and algebra as the Black-Scholes-Merton Equation and is solved along the lines indicated above. Admittedly the calculations are sometimes more complex!

4 Some Generalisations of the Class of Black-Scholes-Merton Equation

From a mathematical point of view the Black-Scholes-Merton Equation is a rather simple instance of an evolution of partial differential equation endowed with the algebra $\{A_{3,8} \oplus_s A_{3,3}\} \oplus_s \infty A_1$. An obvious generalisation is

$$u_t + \tfrac{1}{2}\sigma(t)^2 x^2 u_{xx} + r(t)xu_x - s(t)u = 0, \tag{4.1}$$

i.e., allowance is made for the variation of the parameters in time. In financial terms this makes sense although it does require a certain amount of daring to predict the temporal variation of some of the quantities involved. Nevertheless the solution of (4.1) subject to a terminal condition as above is conceptually no more difficult than the solution of (3.1). It may come as no surprise that the practical resolution of the equation in terms of a closed-form solution could be a nontrivial exercise. Nevertheless the knowledge of the existence of a unique solution is of some comfort when one has to resort to a numerical simulation.

Another example of an equation well-endowed with symmetry is presented by Henderson and Hobson [21, 22] who model the problem of how to deal with claims on nontraded assets. It can happen that there is another traded asset which is correlated to the nontraded asset and this traded asset can be used as a proxy for hedging purposes. A major difference from the Black-Scholes-Merton Equation is that the equation as it is presented is quite nonlinear being

$$2\sigma^2 uu_t + \sigma^2\eta^2 u\left(u_x - u_{xx}\right) + \left(\sigma\eta r u_x - \mu u\right)^2 = 0. \tag{4.2}$$

The 'terminal' condition is $u(0,x) = \exp[-\lambda\gamma\exp[x]]$. For $r \neq 1$ the equation possesses the Lie point symmetries

$$\begin{aligned}
\Gamma_1 &= u\partial_u\\
\Gamma_2 &= \partial_x\\
\Gamma_3 &= t\partial_x + \left\{t\left(\frac{\mu r}{\eta\sigma\left(r^2-1\right)} - \frac{1}{2\left(r^2-1\right)}\right) + x\frac{1}{\eta^2\left(r^2-1\right)}\right\}u\partial_u\\
\Gamma_4 &= \partial_t\\
\Gamma_5 &= t\partial_t + \tfrac{1}{2}x\partial_x + \frac{1}{8\eta\sigma^2\left(r^2-1\right)}\left\{\left(\eta^3\sigma^2 + 4\eta\mu^2 - 4\eta^2\sigma\mu r\right)t + \left(4\sigma\mu r - 2\eta\sigma^2\right)x\right\}u\partial_u\\
\Gamma_6 &= t^2\partial_t + tx\partial_x + \frac{1}{8\eta^2\sigma^2\left(r^2-1\right)}\left\{\left(\eta^4\sigma^2 + 4\eta^2\mu^2 - 4\eta^3\sigma\mu r\right)t^2 + 4\eta^2\sigma^2 t\right.\\
&\qquad \left. + \left(8\eta\sigma\mu r - 4\eta^2\sigma^2\right)tx + 4\sigma^2 x^2\right\}u\partial_u\\
\Gamma_\infty &= u^{r^2} f(t,x)\partial_u,
\end{aligned}$$

where $f(t,x)$ is a solution of the linear equation

$$2\sigma^2 f_t - \eta^2\sigma^2 f_{xx} + \left(\eta^2\sigma^2 - 2\eta\sigma\mu r\right)f_x + \left(1-r^2\right)\mu^2 f = 0. \tag{4.3}$$

Again at the algebra is $\{A_{3,8} \oplus_s A_{3,3}\} \oplus_s \infty A_1$. Equation (4.2) can be linearised by means of a point transformation. In fact the coefficient function in Γ_∞ essentially tells us that the transformation should be

$$w(t,x) = u(t,x)^{1-r^2}.$$

Not surprisingly the linearised equation is (4.3) with $f(t,x)$ replaced by $w(t,x)$.

Unfortunately the symmetries are not compatible with the given terminal condition and so they cannot be used to determine the solution. Naturally one can use the linearised version of the problem to ease the burden of the numerical computations. Another approach would be to persuade one's financial advisers to select a more user-friendly terminal condition.

One should emphasise that it was not the nonlinearity of (4.2) which caused the symmetry-based analysis to come to grief. Rather it was the nature of the functional dependence of the terminal condition. Heath [20] discussed the equation

$$2u_t + 2au_x + b^2 u_{xx} - u_x^2 + 2\nu(x) = 0. \tag{4.4}$$

For a general function $\nu(x)$ the equation possesses the point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= \partial_u \\ \Gamma_\infty &= b^2 f(t,x)\exp[u/b^2]\partial_u, \end{aligned}$$

where $f(t,x)$ is a solution of the linear equation

$$2b^2 f_t + 2ab^2 u_x + b^4 u_{xx} + 2\nu(x)u = 0.$$

Consequently in general (4.4) can be treated as a linear equation with all of the benefits which that brings.

There are some forms of $\nu(x)$ for which additional symmetry exists. Specifically, if $\nu(x)$ is a quadratic polynomial in x, (4.4) possesses the same algebra as the Black-Scholes-Merton Equation [37].

5 Lesser Algebraic Structures

If in (4.4) $\nu(x)$ is a combination of a quadratic polynomial in x and an inverse square term, say,

$$\nu(x) = a_0 + a_1 x + a_2 x^2 + \frac{a_3}{(a_1 + 2a_3 x)^2},$$

the number of Lie point symmetries is reduced by two and the algebra is now $\{A_{3,8} \oplus_s A_1\} \oplus_s \infty A_1$, i.e., the three-dimensional Heisenberg-Weyl subalgebra is lost and all that is left is the one-dimensional subalgebra of ∂_u. The persistence of the $sl(2,R)$ subalgebra is not surprising due to the close relationship of the inverse square term with the well-known Ermakov-Pinney equation [13,40] in Mechanics – not to mention a host of other places. Nevertheless all is not lost as we can see in the example of the Cox-Ingersoll-Ross Equation [6,8,12,18,42,43,46],

$$u_t + \tfrac{1}{2}\sigma^2 x u_{xx} - (\kappa - \lambda x)u_x - xu = 0. \tag{5.1}$$

Independently of the values of the parameters in (5.1) the equation possesses the symmetries [11]

$$\begin{aligned} \Gamma_1 &= u\partial_u \\ \Gamma_{2\pm} &= \exp[\pm\beta t]\left\{\pm\partial_t + \beta x\partial_x - \frac{1}{\sigma^2}(-\beta \pm \lambda)(\kappa \pm \beta x)u\partial_u\right\} \end{aligned}$$

$$\Gamma_3 = \partial_t$$
$$\Gamma_\infty = f(t,x)\partial_u,$$

where $\beta^2 = 2\sigma^2 + \lambda^2$, $f(t,x)$ is any solution of (5.1) and the finite subalgebra is $A_{3,8} \oplus A_1$. The change in the algebraic structure from that of the Black-Scholes-Merton Equation signals the absence of a point transformation to bring (5.1) the form of the classical heat equation.

Although the finite subalgebra of (5.1) is of lower dimension, nevertheless we can consider the possibility that the symmetries are consistent with a terminal condition[2] of the form $u(T,x) = 0$. When we apply the general symmetry to the terminal condition, we find that there does indeed exist a symmetry of (5.1) compatible with the condition. The coefficients in the general symmetry are related according to

$$\begin{aligned}
\alpha_1 &= -\frac{2(\beta-\lambda)\kappa}{\sigma^2}\exp[\beta T]\alpha_{2+} \\
\alpha_{2-} &= \frac{\beta-\lambda}{\beta+\lambda}\exp[2\beta T]\alpha_{2+} \\
\alpha_3 &= -\frac{2\lambda}{\beta+\lambda}\exp[\beta T]\alpha_{2+}.
\end{aligned}$$

One can then reduce (5.1) to an ordinary differential equation and proceed with the process of solution.

Admittedly one obtains only a single symmetry compatible with the terminal condition by comparison with the two obtained in the case of the Black-Scholes-Merton Equation, but flippantly one could observe that the problem of choice has been obviated.

6 Commodities and Other Algebraic Structures

Hitherto we have been concerned with the algebra $\{A_{3,8} \oplus_s A_{3,3}\} \oplus_s \infty A_1$ and its lesser relation $\{A_{3,8} \oplus_s A_1\} \oplus_s \infty A_1$ as potentially viable algebras to enable an algebraic solution to the problem of an evolution partial differential equation coupled to a condition. A sequence of loss of symmetry is observed [37] in the finite subalgebra even for linear equations. Equations related to the classical heat equation by means of a point transformation have the six symmetries of $A_{3,8} \oplus_s A_{3,3}$. With increasing complexity of the equation in a real rather than in a transformational sense the $A_{3,3}$ becomes A_1. Then the $A_{3,8}$ also reduces to A_1. For a linear equation the final reduction is the loss of the latter A_1 when the equation becomes seriously nonautonomous.

There is a class of equations for which this progressive loss of symmetry is in a sense reversed.

[2]A terminal value of zero in the case of a nonlinear equation is equivalent to a constant value in the corresponding linear equation.

The pricing of a commodity is a matter of interest to almost every person who has to make a sale or a purchase. The interest is almost independent of the magnitude of the sale/purchase since the resources of the person concerned are quite critical in that respect. There exists a considerable literature on the interaction of different effects upon the pricing of commodities [2, 5, 7, 17, 44] and several of the models have been subjected to symmetry analysis [45] with some algebraically interesting results. The striking part is that the loss of symmetry from the maximal possible has not followed the pattern mentioned above. In fact it is reminiscent of what happens with linear (and by extension linearisable) ordinary differential equations. There the first symmetries to disappear are in the $sl(2,R)$ subalgebra.

In their original presentation the equations of Schwartz and his coworkers are quite complicated and obscure the algebraic points which we wish to make. Consequently we make use of the elemental form given in Sophocleous *et al.* [45]. The one-factor model can be transformed into the heat equation and so does not present any new concerns. Consequently we begin with the two-factor model.

After a sequence of elementary point transformations the two-factor model may be written as

$$u_{xx} + u_{yy} - 2(ay + f)u_x - 2yu_y - 2u_t = 0 \tag{6.1}$$

and the three-factor model as

$$u_{xx} + u_{yy} + u_{zz} - 2(ay + bz + f)u_x - 2(cy + dz)u_y - 2zu_z - 2u_t = 0. \tag{6.2}$$

The lowercase letters, a through f, are constants in the models under consideration.

The process of generalisation is quite transparent and for the n-factor model one can write the transformed equation as

$$\nabla^2 u(t,\mathbf{x}) - 2\mathbf{p}\cdot\nabla u(t,\mathbf{x}) - 2u(t,\mathbf{x})_t = 0, \tag{6.3}$$

where the elements of the vector $\mathbf{p}$ are given by

$$p_1 = \sum_{j=2}^{j=n} c_{1j}x_j + k, \quad p_i = \sum_{j=i}^{j=n} c_{ij}x_j,\ i = 2,\, n-1, \quad \text{and} \quad p_n = x_n,$$

where c_{ij} and k are constants. The major criterion is that the additional stochastic process in each case be of the same type.

The Lie point symmetries of (6.1) are

$$\begin{aligned}
\Delta_1 &= \partial_t \\
\Delta_2 &= \partial_x \\
\Delta_3 &= t(a^2+1)\partial_x + a\partial_y - (x - ay - ft)u\partial_u \\
\Delta_4 &= \exp[t](a\partial_x + \partial_y)
\end{aligned}$$

$$
\begin{aligned}
\Delta_5 &= \exp[-t](a\partial_x - \partial_y - 2yu\partial_u \\
\Delta_6 &= u\partial_u \\
\Delta_\infty &= f(t,x,y)\partial_u,
\end{aligned}
$$

in which the number of symmetries has not increased over the equation for the one-factor model despite the presence of the extra factor. The algebraic structure is $\{A_1 \oplus_s \{2A_1 \oplus_s 3A_1\} \oplus_s \infty A_1$.

There is an increase in the number of symmetries associated with (6.2) so that it is evident that the one-factor model is exceptional. The symmetries of (6.2) are

$$
\begin{aligned}
\Sigma_1 &= \partial_t \\
\Sigma_2 &= \partial_x \\
\Sigma_3 &= \left[(ad-bc) + a^2 + c^2\right] t\partial_x + [a + (ad-bc)d]\partial_y - c(ad-bc)\partial_z \\
&\quad + c[-cx + ay - (ad-bc)z + cft]u\partial_u \\
\Sigma_4 &= \exp[t]\left\{[b + ad - bc]\partial_x + d\partial_y + (1-c)\partial_z\right\} \\
\Sigma_5 &= \exp[-t]\left\{[b - ad + bc]\partial_x + d\partial_y - (1+c)\partial_z - 2(1+c)zu\partial_u\right\} \\
\Sigma_6 &= \exp[ct][a\partial_x + c\partial_y] \\
\Sigma_7 &= \exp[-ct]\left\{\left[a\left(1 - c^2 + d^2\right) - 2bcd\right]\partial_x + c\left(c^2 - d^2 - 1\right)\partial_y + 2c^2 d\partial_z \right. \\
&\quad \left. +2c^2\left[\left(c^2 - 1\right)y + d(c+1)z\right]u\partial_u\right\} \\
\Sigma_8 &= u\partial_u \\
\sigma_\infty &= f(t,x,y,z)\partial_u
\end{aligned}
$$

and the algebra is $\{A_1 \oplus_s \{3A_1 \oplus_s 4A_1\} \oplus_s \infty A_1$.

The algebraic structure for the n-factor model has been established in [45]. The number of Lie point symmetries is $2n+3$, where n is the number of independent variables in the equation. For $n > 1$ the algebra of the model equation is $\{A_1 \oplus_s \{nA_1 \oplus_s (n+1)A_1\} \oplus_s (\infty A_1)$.

There are two interesting features which we wish to mention. We use the three-factor model as the vehicle for our discussion. If one considers the terminal problem $u(0,x,y,z) = \exp[x]$, which is typical of the type of problem encountered in dealing with commodities, and applies the usual combination, $\Gamma = \sum_{i=1}^{8} \alpha_i \Gamma_i$, to this condition, it is a simple calculation to show that the only nonzero coefficients are constrained according to

$$\alpha_8 = \alpha_2 + \alpha_4(b + ad - bc) + \alpha_6 a.$$

There are three symmetries consistent with the terminal condition. They are

$$
\begin{aligned}
\Lambda_1 &= u\partial_u + \partial_x \\
\Lambda_2 &= (b + ad - bc)u\partial_u + \exp[t]\left\{(b - ad + bc)\partial_x + d\partial_d + (1-c)\partial_z\right\}
\end{aligned}
$$

$$\Lambda_3 = au\partial_u + \exp[ct](a\partial_x + c\partial_y.$$

These three symmetries constitute an abelian algebra. Consequently they may be used simultaneously to reduce the $(1 + 3)$ partial evolution equation to an ordinary differential equation. Moreover the independent variable in this equation is time. An interesting feature is that the structure of the transformation for the reduction gives precisely the structure of the solution for the more generally expressed problem found in the financial literature thereby emphasising the relationship between the existence of an underlying symmetry and the ability to solve the equation.

The abelian subalgebra of Γ_2, Γ_4 and Γ_6 plays another role. If one reduces (6.2) using Γ_6, the invariants obtained from the solution of the associated Lagrange's system,

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{c} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}u}{0},$$

are t, $cx - ay$, z and u. One makes the reduction by setting

$$u(t, x, y, z) = v(t, p, z),$$

where $p = cx - ay$. Equation (6.2) becomes

$$\left(a^2 + c^2\right) \frac{\partial^2 v}{\partial p^2} + \frac{\partial^2 v}{\partial z^2} - 2[(bc - ad)z + fc]\frac{\partial v}{\partial p} - 2z\frac{\partial v}{\partial z} - 2\frac{\partial v}{\partial t} = 0,$$

which is just the two-factor equation.

The multifactor equations are nested!

7 A Parting Thrust

If the variance in the Black-Scholes-Merton Equation is taken to be proportional to the second derivative with respect to the stock price, one obtains the more than moderately nonlinear equation

$$2V_t + 2(r - q)SV_S + \sigma^2 S^2 \left(V_{SS}\right)^3 - 2rV = 0. \tag{7.1}$$

This equation possesses five Lie point symmetries, namely

$$\begin{aligned}
\Gamma_1 &= \exp[rt]\partial_V \\
\Gamma_2 &= S\exp[qt]\partial_V \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= \exp[(2r - 4q)t]\left\{\partial_t + (r - q)S\partial_S + rV\partial_V\right\} \\
\Gamma_5 &= S\partial_S + 2V\partial_V
\end{aligned}$$

in the case that $r \neq 2q$ and

$$\begin{aligned}
\Gamma_1 &= \exp[2qt]\partial_V \\
\Gamma_2 &= S\exp[qt]\partial_V \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= t\partial_t + qtS\partial_S + (2qt - \tfrac{1}{2})V\partial_V \\
\Gamma_5 &= S\partial_S + 2V\partial_V
\end{aligned}$$

in the case that $r = 2q$ with the algebra $\{A_1 \oplus A_2\} \oplus_s 2A_1$ in both instances.

The algebra can definitely be described as 'interesting'. The structure is similar in form to that of the Black-Scholes-Merton Equation and this prompts one to look more carefully at the symmetries before embarking upon the solution of the terminal problem $V(T, S) = G(S)$. The coefficient functions of the symmetries Γ_1 and Γ_2 satisfy (7.1) and as solution symmetries are of no use in giving a symmetry which is compatible with any other conditions. One can only hope for some joy with the remaining three symmetries. We consider the latter case. The application of $\Gamma = \alpha_3\Gamma_3 + \alpha_4\Gamma_4 + \alpha_5\Gamma_5$ to the terminal condition gives the two relations

$$\begin{aligned}
&\alpha_3 = -\alpha_4 T \qquad \text{and} \\
&\alpha_4(2qT - \tfrac{1}{2})G(S) + 2\alpha_5 G(S) - \alpha_4 qTSG'(S) - \alpha_5 SG'(S) = 0,
\end{aligned}$$

which leaves us with something of a quandary. There are three parameters, two conditions and one unspecified function. Naturally it would help if the function were specified, but, if one is interested in possible functions, $G(S)$, for which one can be assured that a solution both exists and is unique, this option is not open. There are two options. Either α_4 is zero and α_5 is arbitrary and hence $G(S) = KS^2$, where K is a constant or α_4 is arbitrary and α_5 is zero in which case $G(S) = KS^{2-m}$, where $m = (2qT)^{-1}$.

When the former applies, the single symmetry is

$$\Gamma = S\partial_S + 2V\partial_V$$

from which it follows that the reduction is given by

$$V = S^2 f(t),$$

where $f(t)$ satisfies the first-order equation

$$\frac{\dot{f}}{f^3} = -4\sigma^2 \quad \longrightarrow \quad V(t, S) = \frac{KS^2}{\sqrt{1 - 8\sigma^2 K^2 (T - t)}}.$$

In the latter situation the symmetry is

$$\Gamma = (t - T)\partial_t + qtS\partial_S + (2qt - \tfrac{1}{2})V\partial_V$$

from which it follows that the invariants are

$$x = S\exp[-qt](t-T)^{-qt} \quad \text{and} \quad z = V\exp[-2qt](t-T)^{-n}, \tag{7.2}$$

where $n = 2qT - \frac{1}{2}4$. The reduced equation is of the form

$$x^2 y''^3 - xy' + by = 0, \tag{7.3}$$

where $b = (4qT-1)/(2qT)$, the solution of which is not immediately obvious. One does note that (7.3) does possess the self-similar symmetry, $\Gamma = x\partial_x + 2y\partial_y$, so that it can be rendered autonomous or reduced to a first-order equation. Neither option has yielded any results so far!

8 Conclusion

We have seen by way of several examples from a variety of differential equations which arise in the analysis of models of financial scenarii how symmetry analysis can be of benefit to the resolution of the partial differential equations which are the product of the modelling process. For some strange reason symmetry analysis has not penetrated Financial Mathematics as it has in other exact sciences such as Physics. A similar reluctance has been observed in the area of Mathematical Bioscience and it can only be a matter for speculation as to why this is so. One sees in the literature various marvellous *Ansätze* proposed by experienced practitioners in the field which lead to the solution of these evolution partial differential equations. One has the impression of the masters of the trade using the mysteries which they have garnered over the decades. Pity the neophyte who has not learnt in what proportions to use 'eye of newt' and 'toe of frog'! In the analysis of differential equations one has a variety of techniques at one's disposal. It seems strange not to use the most fundamental technique of them all which is the probing of the symmetries hidden in the equation.

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