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On The Estimation of $R = P(Y > X)$ for a Class of Lifetime Distributions by Transformation Method

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Abstract: The problem of estimating the $R = P(Y > X)$ v́is-a-v́is point and interval estimation are considered for a class of lifetime distributions. In order to obtain these estimators, the major role is played by the transformation methods.

Keywords: Class of lifetime distributions, Stress-Strength reliability, Transformation method, MLE, UMVUE, Confidence interval.

1 Introduction

A lot of work has been done in the literature to deal with various inferential problems related to $R = P(Y > X)$, which represents the reliability of an item of random strength Y subject to a random stress X. For a brief review, one may refer to Church and Harris (1970) [6], Enis and Geisser (1971)[9], Downton (1973)[8], Tong (1974, 1975)[12,13], Kelly, Kelly and Schucany (1976)[10], Sathe and Shah (1981)[11], Chao (1982)[5], Awad and Gharraf (1986)[1], Chaturvedi and Surinder (1999)[4], Chaturvedi and Sharma (2007)[3].

The purpose of present paper is many-fold. We considered a class of lifetime distributions proposed by Chaturvedi and Rani (1997)[2], which covers many lifetime distributions as specific cases. In section 2, the MLE and UMVUE of 'R' is derived. In section 3, we construct the confidence interval for 'R'. In order to derive the MLE, UMVUE and confidence interval for 'R', the major role is played by the transformation method.

2 MLE and UMVUE of $R = P(Y > X)$ for a Class of lifetime distributions

Chaturvedi and Rani (1997)[2] has defined the following class of distributions

$$f(x;\theta,a,b,c) = \frac{cx^{ac-1}}{\theta^{ab}\Gamma a} exp(-x^c/\theta^b); \quad x,\theta,a,b,c > 0, \tag{1}$$

where θ is assumed to be unknown and a, b, c are known constants. On considering different values for a, b, c, the pdf's of different continuous distributions, such as one-parameter exponential distribution, gamma distribution, generalized gamma distribution, Weibull distribution, Half-Normal distribution, Rayleigh distribution, Chi-distribution and Maxwell's failure distribution etc., can be obtained. Let the random variables X and Y follows the class of lifetime distributions given at (1).

Theorem 1: The MLE of $R = P(Y > X)$ is given by

$$\tilde{R} = \left(\frac{a_1\bar{\eta}}{a_2\bar{\varepsilon}+a_1\bar{\eta}}\right)^{a_1} \frac{1}{B(a_1,a_2)} {}_2F_1\left(a_1, 1-a_1; a_1; \frac{a_1\bar{\eta}}{a_2\bar{\varepsilon}+a_1\bar{\eta}}\right), \tag{2}$$

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where, $\bar{\varepsilon} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c_1} = \bar{T}_X$(say) and, $\bar{\eta} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c_2} = \bar{T}_Y$(say).

Proof: Let us consider the transformation $x^c = \varepsilon$ in (1), we get

$$f(\varepsilon;a,\lambda) = \frac{\varepsilon^{a-1}}{\lambda^a \Gamma a} exp(-\varepsilon^c/\lambda); \quad \varepsilon, a, \lambda > 0, \tag{3}$$

which follows gamma distribution, where $\lambda = \theta^b$.
Now, let us consider ε and η two independent random variables which follows gamma distribution with parameters (a_1, λ_1) and (a_2, λ_2) respectively, where $\varepsilon = x^{c_1}$ and $\eta = y^{c_2}$.
Thus, for $R = P(\eta > \varepsilon)$, we have

$$R = P\left(\frac{\eta}{\varepsilon} > 1\right),$$

or,

$$P\left(\frac{\eta/\lambda_2}{\varepsilon/\lambda_1} + 1 > \frac{\lambda_1}{\lambda_2} + 1\right),$$

or,

$$P\left(\frac{\varepsilon/\lambda_1}{\eta/\lambda_2 + \varepsilon/\lambda_1} < \frac{\lambda_2}{\lambda_1 + \lambda_2}\right).$$

Here, random variable $z = \frac{\varepsilon/\lambda_1}{\eta/\lambda_2 + \varepsilon/\lambda_1}$, has a beta distribution with pdf

$$f(z, a_1, a_2) = [B(a_1, a_2)]^{-1} z^{a_1 - 1}(1 - z)^{a_2 - 1},$$

or,

$$R = I_{\left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)}(a_1, a_2), \tag{4}$$

On using the relationship between incomplete beta distribution and hypergeometric series, we get

$$R = \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)\right). \tag{5}$$

On replacing λ_1 and λ_2 by their respective MLE's i.e , $\tilde{\lambda}_1 = \frac{\bar{\varepsilon}}{a_1}$ and, $\tilde{\lambda}_2 = \frac{\bar{\eta}}{a_2}$ in (5), we get the MLE of $R = P(\eta > \varepsilon)$, where ε and η are independent gamma distributions. The reliability for the random strength Y and random stress X i.e. $R = P(Y > X)$ is given by (6) and MLE of R is given by (7) respectively.

$$R = \left(\frac{\theta_2^{b_2}}{\theta_1^{b_1} + \theta_2^{b_2}}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \left(\frac{\theta_2^{b_2}}{\theta_1^{b_1} + \theta_2^{b_2}}\right)\right), \tag{6}$$

$$\tilde{R} = \left(\frac{a_1\bar{\eta}}{a_2\bar{\varepsilon} + a_1\bar{\eta}}\right)^{a_1} \frac{1}{B(a_1, a_2)} {}_2F_1\left(a_1, 1 - a_1; a_1; \left(\frac{a_1\bar{\eta}}{a_2\bar{\varepsilon} + a_1\bar{\eta}}\right)\right), \tag{7}$$

where $\bar{\varepsilon} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c_1} = \bar{T}_X$(say) and, $\bar{\eta} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c_2} = \bar{T}_Y$(say)
Hence, the theorem follows.

Corollary 1.

1.For $a = b = c = 1$,

$$\tilde{R} = \frac{\bar{T}_Y}{T_X + T_Y},$$

where, $\overline{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j$ and, $\overline{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i$,
which is the MLE of exponential distribution.

2.For $b = c = 1$ and 'a' as positive integer,

$$\tilde{R} = \left(\frac{a_1\overline{Y}}{a_2X + a_1Y}\right)^{a_1} \frac{1}{B(a_1,a_2)} {}_2F_1\left(a_1, 1-a_1; a_1; \frac{a_1\overline{Y}}{a_2X + a_1Y}\right),$$

which is the MLE of gamma distribution.

3.For $b = c = \alpha$,

$$\tilde{R} = \left(\frac{a_1\overline{T}_Y}{a_2T_X + a_1T_Y}\right)^{a_1} \frac{1}{B(a_1,a_2)} {}_2F_1\left(a_1, 1-a_1; a_1; \frac{a_1\overline{T}_Y}{a_2T_X + a_1T_Y}\right),$$

where, $\overline{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{\alpha}$ and, $\overline{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{\alpha}$,
which is the MLE of generalised gamma distribution.

4.For $a = 1$ and $b = c$,

$$\tilde{R} = \frac{\overline{T}_Y}{T_X + T_Y},$$

where, $\overline{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c}$ and, $\overline{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c}$,
which is the MLE of Weibul distribution.

5.For $a = 1/2, b = c = 2$,

$$\tilde{R} = \left(\frac{\overline{T}_Y}{T_X + T_Y}\right)^{1/2} \frac{1}{\pi} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{\overline{T}_Y}{T_X + T_Y}\right),$$

where, $\overline{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{2}$ and, $\overline{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{2}$,
which is the MLE of Half-Normal distribution.

6.For $a = b = 1$ and $c = 2$,

$$\tilde{R} = \frac{\overline{T}_Y}{T_X + T_Y},$$

where, $\overline{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{2}$ and, $\overline{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{2}$,
which is the MLE of Rayleigh distribution.

7.For $a = \alpha/2, b = 1$ and $c = 2$,

$$\tilde{R} = \left(\frac{\overline{T}_Y}{T_X + T_Y}\right)^{\alpha/2} \frac{1}{B\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)} {}_2F_1\left(\frac{\alpha}{2}, 1-\frac{\alpha}{2}; \frac{\alpha}{2}; \frac{\overline{T}_Y}{T_X + T_Y}\right),$$

where, $\overline{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{2}$ and, $\overline{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{2}$,
which is the MLE of Chi distribution.

8.For $a = 3/2, b = 1$ and $c = 2$,

$$\tilde{R} = \left(\frac{\overline{T}_Y}{T_X + T_Y}\right)^{3/2} \frac{8}{\pi} {}_2F_1\left(\frac{3}{2}, \frac{-1}{2}; \frac{3}{2}; \frac{\overline{T}_Y}{T_X + T_Y}\right),$$

where, $\overline{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{2}$ and, $\overline{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{2}$,
which is the MLE of Maxwell's failure distribution.

Theorem 2: The UMVUE of $R=P(Y>X)$ is given by

$$\hat{R}=\begin{cases}\dfrac{B[(n_2-1)a_2+i+1,a_1+j]}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\sum_{i=0}^{\infty}\dfrac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times\sum_{j=0}^{\infty}(-1)^j\binom{(n_1-1)a_1-1}{j}\left(\dfrac{T_Y}{T_X}\right)^{a_1+j} & ;\text{if } T_Y<T_X \\ \text{where } 0\le i\le a_1-1<\infty \text{ and } 0\le j\le (n_1-1)a_1-1<\infty \\ \\ \dfrac{B[(n_1-1)a_1,a_1+j]}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\sum_{i=0}^{\infty}\dfrac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times\sum_{j=0}^{\infty}(-1)^j\binom{(n_2-1)a_2+i}{j}\left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_X<T_Y \\ \text{where } 0\le i\le a_2-1<\infty \text{ and } 0\le j\le (n_2-1)a_2+i<\infty\end{cases} \quad (8)$$

where, $T_X=\sum_{i=1}^{n_1}X_i^{c_1}$ and, $T_Y=\sum_{j=1}^{n_2}Y_j^{c_2}$.

Proof: Let us consider the transformation $x^c=\varepsilon$ in (1), we get

$$f(\varepsilon;a,b,c,\theta)=\frac{\varepsilon^{a-1}}{\lambda^a\Gamma a}exp(-\varepsilon/\lambda)$$

which is gamma distribution, where $\lambda=\theta^b$.
Now to obtain $P(\eta>\varepsilon)$, we have to obtain the UMVUE of $f(\varepsilon;a,\lambda)$ i.e. $\hat{f}(\varepsilon;a,\lambda)$ and $f(\eta;a,\lambda)$ i.e. $\hat{f}(\eta;a,\lambda)$ which is given by

$$f(\varepsilon;a_1,\lambda)=\frac{1}{B(a_1,(n_1-1)a_1)}\left\{\frac{\varepsilon^{a_1-1}}{(n_1\bar{\varepsilon})_1^a}\right\}\left\{1-\frac{\varepsilon}{n_1\bar{\varepsilon}}\right\}^{(n_1-1)a_1-1};\quad \text{if } 0<\varepsilon<n_1\bar{\varepsilon} \quad (9)$$

Replacing ε by η and n_1 by n_2 in (9), we get the UMVUE of $f(\eta;a_2,\lambda)$. Now, let us consider ε and η be two independent random variables follows gamma distribution with parameters (a_1,λ_1) and (a_2,λ_2) respectively, where $\varepsilon=x^{c_1}$ and $\eta=y^{c_2}$.

$$\hat{R}=P(\eta>\varepsilon)=\int_0^{\infty}\int_{\varepsilon}^{\infty}\hat{f}(\eta;a_2,\lambda_2)\hat{f}(\varepsilon;a_1,\lambda_1)d\eta d\varepsilon$$

$$=\frac{1}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\int_0^{n_1\bar{\varepsilon}}\int_{\varepsilon}^{n_2\bar{\eta}}\frac{\eta^{a_2-1}}{(n_2\bar{\eta})^{a_2}}\left(1-\frac{\eta}{n_2\bar{\eta}}\right)^{(n_2-1)a_2-1} \times\frac{\varepsilon^{a_1-1}}{(n_1\bar{\varepsilon})^{a_1}}\left(1-\frac{\varepsilon}{n_2\bar{\varepsilon}}\right)^{(n_1-1)a_1-1}d\eta d\varepsilon$$

let, $1-\dfrac{\eta}{n_2\bar{\eta}}=z$

$$=\frac{1}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\int_0^{n_1\bar{\varepsilon}}\int_{\varepsilon}^{(1-\frac{\varepsilon}{n_2\bar{\eta}})}z^{(n_2-1)a_2-1}(1-z)^{a_2-1} \times\frac{\varepsilon^{a_1-1}}{(n_1\bar{\varepsilon})^{a_1}}\left(1-\frac{\varepsilon}{n_2\bar{\varepsilon}}\right)^{(n_1-1)a_1-1}dzd\varepsilon$$

$$=\frac{1}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\sum_{i=0}^{\infty}\frac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \times\int_0^{min(n_1\bar{\varepsilon},n_2\bar{\eta})}\left(1-\frac{\varepsilon}{n_1\bar{\varepsilon}}\right)^{(n_1-1)a_1-1}\frac{\varepsilon^{a_1-1}}{(n_1\bar{\varepsilon})^{a_1}}\left(1-\frac{\varepsilon}{n_2\bar{\eta}}\right)^{(n_1-1)a_1-1}dzd\varepsilon$$

Now, consider the case when $n_1\overline{\varepsilon} > n_2\overline{\eta}$, in this situation,

let, $1 - \frac{\varepsilon}{n_2\overline{\eta}} = z$

$$= \frac{1}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \int_0^1 z^{(n_2-1)a_2+i} \left(1-\left(\frac{n_2\overline{\eta}}{n_1\overline{\varepsilon}}\right)(1-z)\right)^{(n_1-1)a_1-1} \frac{[(n_2\overline{\eta})(1-z)]^{(a_1-1)}}{(n_1\overline{\varepsilon})^{a_1}}(n_2\overline{\eta})dz$$

$$= \frac{1}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \int_0^1 \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \left(\frac{n_2\overline{\eta}}{n_1\overline{\varepsilon}}\right)^{a_1+j} (1-z)^{a_1+j-1} z^{(n_2-1)a_2+i} dz$$

$$= \frac{B[(n_2-1)a_2+i+1,a_1+j]}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)a_1-1}{j} \left(\frac{n_2\overline{\eta}}{n_1\overline{\varepsilon}}\right)^{a_1+j} ;\text{if } n_2\overline{\eta} < n_1\overline{\varepsilon} \quad (10)$$

Similarly, we can tackle the case when $n_2\overline{\eta} > n_1\overline{\varepsilon}$, we get

$$= \frac{B[(n_1-1)a_1,a_1+j]}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)a_2+i} \binom{a_2-1}{i} \times \sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)a_2+i}{j} \left(\frac{n_1\overline{\varepsilon}}{n_2\overline{\eta}}\right)^{j} ;\text{if } n_2\overline{\eta} > n_1\overline{\varepsilon} \quad (11)$$

Now, to obtain the UMVUE for the class of distribution,
substituting $n_1\overline{\varepsilon} = \sum_{i=1}^{n_1} X_i^{c_1} = T_X$ and, $n_2\overline{\eta} = \sum_{j=1}^{n_2} Y_j^{c_2} = T_Y$ in (10) and (11),
and the Theorem follows.

Corollary 2.

1.For $a = b = c = 1$,

$$\hat{R} = \begin{cases} \frac{\Gamma n_1 \Gamma n_2}{\Gamma(n_1-j-1)\Gamma(n_2+j+1)} \sum_{j=0}^{n_1-2} (-1)^j \left(\frac{T_Y}{T_X}\right)^{1+j} & ;\text{if } T_Y < T_X \\ \frac{\Gamma n_1 \Gamma n_2}{\Gamma(n_2-j)\Gamma(n_1+j)} \sum_{j=0}^{n_2-1} (-1)^j \left(\frac{T_X}{T_Y}\right)^{j} & ;\text{if } T_Y > T_X \end{cases}$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j$ and, $T_X = \sum_{i=1}^{n_1} X_i$,
which is the UMVUE of exponential distribution.

2.For $b=c=1$ and 'a' as positive integer,

$$\hat{R}=\begin{cases} \frac{B[(n_2-1)a_2+i+1,a_1+j]}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\sum_{i=0}^{a_2-1}\frac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times\sum_{j=0}^{(n_1-1)a_1-1}(-1)^j\binom{(n_1-1)a_1-1}{j}\left(\frac{T_Y}{T_X}\right)^{a_1+j} & ;\text{if } T_Y<T_X \\ \\ \frac{B[(n_1-1)a_1,a_1+j]}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\sum_{i=0}^{a_2-1}\frac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times\sum_{j=0}^{(n_2-1)a_2+i}(-1)^j\binom{(n_2-1)a_2+i}{j}\left(\frac{T_X}{T_Y}\right)^{j} & ;\text{if } T_X<T_Y \end{cases}$$

where, $T_Y=\sum_{j=1}^{n_2}Y_j$ and, $T_X=\sum_{i=1}^{n_1}X_i$,
which is the UMVUE of gamma distribution.

3.For $b=c=\alpha$,

$$\hat{R}=\begin{cases} \frac{B[(n_2-1)a_2+i+1,a_1+j]}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\sum_{i=0}^{a_2-1}\frac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times\sum_{j=0}^{(n_1-1)a_1-1}(-1)^j\binom{(n_1-1)a_1-1}{j}\left(\frac{T_Y}{T_X}\right)^{a_1+j} & ;\text{if } T_Y<T_X \\ \\ \frac{B[(n_1-1)a_1,a_1+j]}{B[a_1,(n_1-1)a_1]B[a_2,(n_2-1)a_2]}\sum_{i=0}^{a_2-1}\frac{(-1)^i}{(n_2-1)a_2+i}\binom{a_2-1}{i} \\ \times\sum_{j=0}^{(n_2-1)a_2+i}(-1)^j\binom{(n_2-1)a_2+i}{j}\left(\frac{T_X}{T_Y}\right)^{j} & ;\text{if } T_X<T_Y \end{cases}$$

where, $T_Y=\sum_{j=1}^{n_2}Y_j^{\alpha}$ and, $T_X=\sum_{i=1}^{n_1}X_i^{\alpha}$,
which is the UMVUE of generalised gamma distribution.

4.For $a=1$ and $b=c$,

$$\hat{R}=\begin{cases} \frac{\Gamma n_1\Gamma n_2}{\Gamma(n_1-j-1)\Gamma(n_2+j+1)}\sum_{j=0}^{n_1-2}(-1)^j\left(\frac{T_Y}{T_X}\right)^{1+j} & ;\text{if } T_Y<T_X \\ \\ \frac{\Gamma n_1\Gamma n_2}{\Gamma(n_2-j)\Gamma(n_1+j)}\sum_{j=0}^{n_2-1}(-1)^j\left(\frac{T_X}{T_Y}\right)^{j} & ;\text{if } T_Y>T_X \end{cases}$$

where, $T_Y=\sum_{j=1}^{n_2}Y_j^{c}$ and, $T_X=\sum_{i=1}^{n_1}X_i^{c}$,
which is the UMVUE of Weibull distribution.

5.For $a=b=1$ and $c=2$,

$$\hat{R}=\begin{cases} \frac{\Gamma n_1\Gamma n_2}{\Gamma(n_1-j-1)\Gamma(n_2+j+1)}\sum_{j=0}^{n_1-2}(-1)^j\left(\frac{T_Y}{T_X}\right)^{1+j} & ;\text{if } T_Y<T_X \\ \\ \frac{\Gamma n_1\Gamma n_2}{\Gamma(n_2-j)\Gamma(n_1+j)}\sum_{j=0}^{n_2-1}(-1)^j\left(\frac{T_X}{T_Y}\right)^{j} & ;\text{if } T_Y>T_X \end{cases}$$

where, $T_Y=\sum_{j=1}^{n_2}Y_j^{2}$ and, $T_X=\sum_{i=1}^{n_1}X_i^{2}$,
which is the UMVUE of Rayleigh distribution.

6.For $a = \alpha/2, b = 1$ and $c = 2$,

$$\hat{R} = \begin{cases} \dfrac{B\left[\frac{(n_2-1)\alpha}{2}+i+1, \frac{\alpha}{2}+j\right]}{B\left[\frac{\alpha}{2}, \frac{(n_1-1)\alpha}{2}\right] B\left[\frac{\alpha}{2}, \frac{(n_2-1)\alpha}{2}\right]} \sum\limits_{i=0}^{\frac{\alpha}{2}-1} \dfrac{(-1)^i}{\frac{(n_2-1)\alpha}{2}+i} \binom{\frac{\alpha}{2}-1}{i} & \\ \times \sum\limits_{j=0}^{\frac{(n_1-1)\alpha}{2}-1} (-1)^j \binom{\frac{(n_1-1)\alpha}{2}-1}{j} \left(\dfrac{T_Y}{T_X}\right)^{\frac{\alpha}{2}+j} & ;\text{if } T_Y < T_X \\ \dfrac{B\left[\frac{(n_1-1)\alpha}{2}, \frac{\alpha}{2}+j\right]}{B\left[\frac{\alpha}{2}, \frac{(n_1-1)\alpha}{2}\right] B\left[\frac{\alpha}{2}, \frac{(n_2-1)\alpha}{2}\right]} \sum\limits_{i=0}^{\frac{\alpha}{2}-1} \dfrac{(-1)^i}{\frac{(n_2-1)\alpha}{2}+i} \binom{\frac{\alpha}{2}-1}{i} & \\ \times \sum\limits_{j=0}^{\frac{(n_1-1)\alpha}{2}+i} (-1)^j \binom{\frac{(n_2-1)\alpha}{2}+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j} & ;\text{if } T_X < T_Y \end{cases}$$

where, $T_Y = \sum\limits_{j=1}^{n_2} Y_j^2$ and, $T_X = \sum\limits_{i=1}^{n_1} X_i^2$,
which is the UMVUE of Chi distribution.

3 Interval estimation of $R = P(Y > X)$

Theorem 3: The Confidence interval for $R = P(Y > X)$ is given by

$$P\left(I_{\left(\frac{(a_1\overline{T}_Y/a_2\overline{T}_X)F_{1-\gamma_2}}{(a_1T_Y/a_2T_X)F_{1-\gamma_2}+1}\right)}(a_1,a_2) < R < I_{\left(\frac{(a_1\overline{T}_Y/a_2\overline{T}_X)F_{\gamma_1}}{(a_1T_Y/a_2T_X)F_{\gamma_1}+1}\right)}(a_1,a_2) \right) = 1-\gamma \tag{12}$$

where, $\overline{T}_X = \dfrac{1}{n_1}\sum\limits_{i=1}^{n_1} X_i^{c_1}$ and, $\overline{T}_Y = \dfrac{1}{n_2}\sum\limits_{j=1}^{n_2} Y_j^{c_2}$.

Proof: Let us consider ε and η be two independent random variables follows gamma distribution with parameters (a_1, λ_1) and(a_2, λ_2) respectively, where $\varepsilon = x^{c_1}$ and $\eta = y^{c_2}$.
Denote -

$$\lambda = \frac{\lambda_2}{\lambda_1}; \tilde{\lambda} = \frac{a_1\overline{\eta}}{a_2\overline{\varepsilon}} \tag{13}$$

Now, it is well known that,

$$\frac{2n_1\overline{\varepsilon}}{\lambda_1} \sim Gamma(n_1a_1, 2) \equiv \chi^2_{2n_1a_1}$$

Similarly,

$$\frac{2n_2\overline{\eta}}{\lambda_2} \sim Gamma(n_2a_2, 2) \equiv \chi^2_{2n_2a_2}$$

where, χ^2_ν is the p.d.f. of the Chi-Square distribution with ν degree of freedom. Hence,

$$\frac{\tilde{\lambda}}{\lambda} = \frac{2n_2\overline{\eta}/n_2\lambda_2a_2}{2n_1\overline{\varepsilon}/n_1\lambda_1a_1}$$

$$= \frac{\chi^2_{2n_2a_2}/2n_2a_2}{\chi^2_{2n_1a_1}/2n_1a_1} \sim F\left(2n_1a_1, 2n_2, a_2\right) \tag{14}$$

where $F(\nu_1, \nu_2)$ denotes the Snedecor's F-distribution with ν_1 and ν_2 degree of freedom.
Analogously,

$$\frac{\tilde{\lambda}}{\lambda} \sim F(2n_1a_1, 2n_2a_2)$$

For any δ denote by $F_\delta = F_\delta(2n_1a_1, 2n_2a_2)$, the $1-\delta$ quantile (i.e. the δ cut off points) of $F(2n_1a_1, 2n_2a_2)$ distribution. Also, the $1-\delta$ quantile of $F(2n_2a_2, 2n_1a_1)$ distribution is related to F_δ as follows.

$$F_\delta(2n_2a_2, 2n_1a_1) = [F_{1-\delta}(2n_1a_1, 2n_2a_2)]^{-1}$$

Let, γ_1 and γ_2 be non-negative numbers such that $\gamma_1 + \gamma_2 = \gamma$. Then,

$$P\left[\tilde{\lambda}F_{1-\gamma_2} < \lambda < \tilde{\lambda}F_{\gamma_1}\right] = 1-\gamma \tag{15}$$

Recall now that $R = I_{\frac{\lambda}{\lambda+1}}(a_1, a_2)$. Since, $I_z(a,b)$ is an increasing function of z for any a, b. So, is $I_{\frac{\lambda}{\lambda+1}}(a_1, a_2)$ as a function of λ. Hence (15) implies that,

$$P\left(I_{\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1}}(a_1, a_2) < R < I_{\frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}}(a_1, a_2)\right) = 1-\gamma \tag{16}$$

The Confidence interval (16) has originally derived by Constantine et al (1986)[7].

Now, in (16) replacing $\tilde{\lambda} = \frac{a_1\overline{\eta}}{a_2\overline{\varepsilon}}$ and $\lambda_1 = \theta_1^{b_1}$, $\lambda_2 = \theta_2^{b_2}$

Thus, $R = I_{\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)}(a_1, a_2) = I_{\left(\frac{\theta_2^{b_2}}{\theta_1^{b_1}+\theta_2^{b_2}}\right)}(a_1, a_2)$ and the confidence interval for R is given as,

$$P\left(I_{\left(\frac{(a_1\overline{\eta}/a_2\overline{\varepsilon})F_{1-\gamma_2}}{(a_1\overline{\eta}/a_2\overline{\varepsilon})F_{1-\gamma_2}+1}\right)}(a_1, a_2) < R < I_{\left(\frac{(a_1\overline{\eta}/a_2\overline{\varepsilon})F_{\gamma_1}}{(a_1\overline{\eta}/a_2\overline{\varepsilon})F_{\gamma_1}+1}\right)}(a_1, a_2)\right) = 1-\gamma \tag{17}$$

where, $\overline{\varepsilon} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c_1} = \overline{T}_X$(say) and, $\overline{\eta} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c_2} = \overline{T}_Y$(say).
and the Theorem follows.

Corollary 3.

1.For $a = b = c = 1$ in (17), we get confidence interval for one parameter exponential distribution.

$$P\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1} < R < \frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right) = 1-\gamma,$$

Where, $\tilde{\lambda} = \frac{\overline{Y}}{\overline{X}}$ and, $R = \frac{\theta_2}{\theta_1+\theta_2}$,
which coincides with interval derived by Enis and Geiser.
2.For $b = c = 1$ and 'a' as positive integer,

$$P\left(I_{\frac{(a_1/a_2)\tilde{\lambda}F_{1-\gamma_2}}{(a_1/a_2)\tilde{\lambda}F_{1-\gamma_2}+1}}(a_1, a_2) < R < I_{\frac{(a_1/a_2)\tilde{\lambda}F_{\gamma_1}}{(a_1/a_2)\tilde{\lambda}F_{\gamma_1}+1}}(a_1, a_2)\right) = 1-\gamma,$$

Where, $\tilde{\lambda} = \frac{\overline{Y}}{\overline{X}}$, $\overline{Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j$, $\overline{X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i$ and, $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(a_1, a_2)$,
which is the Confidence interval of gamma distribution.

3.For $b = c = \alpha$,

$$P\left(I_{\frac{(a_1/a_2)\tilde{\lambda}F_{1-\gamma_2}}{(a_1/a_2)\tilde{\lambda}F_{1-\gamma_2}+1}}(a_1,a_2) < R < I_{\frac{(a_1/a_2)\tilde{\lambda}F_{\gamma_1}}{(a_1/a_2)\tilde{\lambda}F_{\gamma_1}+1}}(a_1,a_2)\right) = 1-\gamma,$$

Where, $\tilde{\lambda} = \frac{\overline{T_Y}}{\overline{T_X}}$, $\overline{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{\alpha}$, $\overline{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{\alpha}$ and, $R = I_{\frac{\theta_2^{\alpha}}{\theta_1^{\alpha}+\theta_2^{\alpha}}}(a_1,a_2)$,

which is the Confidence interval of generalised gamma distribution.

4.For a = 1 and b = c,

$$P\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1} < R < \frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right) = 1-\gamma,$$

Where, $\tilde{\lambda} = \frac{\overline{T_Y}}{\overline{T_X}}$, $\overline{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{c}$, $\overline{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{c}$ and, $R = I_{\frac{\theta_2^{c}}{\theta_1^{c}+\theta_2^{c}}}(a_1,a_2)$,

which is the Confidence interval of Weibull distribution.

5.For a = 1/2, b = c = 2,

$$P\left(I_{\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1}\right)}\left(\frac{1}{2},\frac{1}{2}\right) < R < I_{\left(\frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right)}\left(\frac{1}{2},\frac{1}{2}\right)\right) = 1-\gamma,$$

Where, $\tilde{\lambda} = \frac{\overline{T_Y}}{\overline{T_X}}$, $\overline{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{2}$, $\overline{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{2}$ and, $R = I_{\frac{\theta_2^{2}}{\theta_1^{2}+\theta_2^{2}}}\left(\frac{1}{2},\frac{1}{2}\right)$,

which is the Confidence interval of Half-Normal distribution.

6.For $a = b = 1$ and $c = 2$,

$$P\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1} < R < \frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right) = 1-\gamma,$$

Where, $\tilde{\lambda} = \frac{\overline{T_Y}}{\overline{T_X}}$, $\overline{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{2}$, $\overline{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{2}$ and, $R = \frac{\theta_2}{\theta_1+\theta_2}$,

which is the Confidence interval of Rayleigh distribution.

7.For $a = \alpha/2, b = 1$ and $c = 2$,

$$P\left(I_{\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1}\right)}\left(\frac{\alpha}{2},\frac{\alpha}{2}\right) < R < I_{\left(\frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right)}\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)\right) = 1-\gamma,$$

where, $\tilde{\lambda} = \frac{\overline{T_Y}}{\overline{T_X}}$, $\overline{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{2}$, $\overline{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{2}$ and, $R = I_{\frac{\theta_2}{\theta_2+\theta_1}}\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)$,

which is the Confidence interval of Chi distribution.

8.For $a = 3/2, b = 1$ and $c = 2$,

$$P\left(I_{\left(\frac{\tilde{\lambda}F_{1-\gamma_2}}{\tilde{\lambda}F_{1-\gamma_2}+1}\right)}\left(\frac{3}{2},\frac{3}{2}\right) < R < I_{\left(\frac{\tilde{\lambda}F_{\gamma_1}}{\tilde{\lambda}F_{\gamma_1}+1}\right)}\left(\frac{3}{2},\frac{3}{2}\right)\right) = 1-\gamma,$$

where, $\tilde{\lambda} = \frac{\bar{T_Y}}{T_X}$, $\bar{T_Y} = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^2$, $\bar{T_X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^2$ and, $R = I_{\frac{\theta_2}{\theta_2+\theta_1}}\left(\frac{3}{2}, \frac{3}{2}\right)$,
which is the Confidence interval of Maxwell's failure distribution.

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