

# Design Algorithm for Smoothly Connecting Curves by Fuzzy Membership Function

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**Abstract:** In this paper, we propose a technique using a fuzzy membership function to generate a curve to approximate a desired function. The concave preserving issues are discussed and proved for the approximation functions. The desired function can be widely approximated piece by piece by combining two parabolic functions in each segment. The combined function passes through two given points in common and has the given slopes at their two respective points. The smooth and concave properties of approximation functions are proved to be preserved for those properties of the desired function. This paper aims to prove that the concavity preserving can be achieved by combining two parabolic functions using the fuzzy membership functions and fuzzy inference rules. Two numerical results are used to demonstrate that the approximation function can approximate parts of the ellipse and sine functions.

**Keywords:** Approximation function, fuzzy membership function, smoothness function

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## 1 Introduction

Fitting function is to generate a function which is highly correlated to the sample data. In the field of numerical analysis, B-spline is a spline function that has minimal support with respect to a given degree, smoothness, and domain partition. Nikolai Lobachevsky investigated B-splines in the early nineteenth century. In 1946, Schoenberg [1] used B-splines for statistical data smoothing, and his paper started the modern theory of spline approximation. Gordon and Reisenfeld [2] formally introduced B-splines into computer-assisted designing.

Every spline function of a given degree, smoothness, and domain partition can be uniquely represented as a linear combination of B-splines with the same degree which can preserve the smoothness over the same partition.

Considering the stability and smoothness, the cubic polynomial functions are widely used to approximate the experimental data. The cubic polynomials are smooth, continuous and their order is only three. Hence, cubic polynomial functions are always used as the spline

functions to approximate the given data or functions. There are three kinds of cubic polynomial spline functions: the natural splines, parabolic runout splines and cubic runout splines [3]. Spline functions use some control points to increase the precision of an approximated curve, only affecting functions locally. When compared with Bezier curve, which affects the graph globally, spline functions apparently produce better results. Unfortunately, the drawback of the cubic polynomials is that it can not ensure concavity preserving stably in the intermediate points. We thus, based on fuzzy theory, propose an algorithm, using a series of functions generated by two parabolic functions with their respective membership function to ensure that the approximate function will keep concavity preserving, as shown in Fig. 2.

In recent years, the fuzzy theory has been used in many fields, for example, images and graphic processing [4,5]. The fuzzy model can be adjusted by the preprocessed data to achieve better performance [6]. However, up to now, no fuzzy model has been proposed as the cubic spline function to approximate functions in preserving the concavity of the initial graph function.

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In fuzzy systems, through the fuzzy inference operation, human linguistic knowledge can be easily encoded in the form of fuzzy rules, a set of fuzzy if-then rules, which generate output with respect to the given input. The encoded knowledge can be processed numerically [7]. Different types of fuzzy models have thus been designed according to different situations. Every fuzzy model consists of a set of fuzzy if-then rules with linguistic terms in their antecedent and/or consequences. According to the types of constituting fuzzy rules, fuzzy models can be grouped into three kinds: Sugeno fuzzy model, Tsukamoto fuzzy model, and Mamdani fuzzy model. The details of the three types of fuzzy rules may be referred to [8].

In Section 2, we introduce the definitions and the theorems that will be needed in Section 3. Then, in section 3, we define a new fuzzy function  $f(x)$ , which is generated by two parabolic functions with a common joint point and some properties of it. Following that, the concavity property of the fuzzy function  $f(x)$  will be discussed. Section 4 shows the results of the experiment, and the last section covers our conclusions.

## 2 Definitions and Preliminaries

We first introduce some definitions and Theorem 1 to 2 which come from reference [9] and [10]. Some changes are made in order to meet our needs.

**Definition 1 [9]** Let  $A$  and  $B$  be any sets. By the Cartesian product of  $A$  and  $B$ , denoted by  $A \times B$ , we mean the set of all ordered pairs by  $(x, y)$  such that  $x$  is an element of  $A$  and  $y$  is an element of  $B$ .

**Definition 2 [9]** Let  $A$  and  $B$  be any sets. A subset  $f$  of  $A \times B$  is called a function if and only if no distinct ordered pairs in  $f$  have the same first number. That is  $f$  is a function if and only if, whenever  $(x, y)$  and  $(x, z)$  are both in  $f$ , then  $y = z$ .

**Definition 3** Given  $m$  real values  $x_i$ , called knots, with  $x_0 \leq x_1 \leq \dots \leq x_{m-1}$ , data pairs are points  $[x_i, y_i]$  for  $i \in (0, m-1)$  and  $y_i = f(x_i)$ .

**Definition 4 [10]** Let  $A$  real-valued function  $f$  defined on an interval  $I$  is strictly concave up if  $f(\lambda x_s + (1-\lambda)x_t) < \lambda f(x_s) + (1-\lambda)f(x_t)$  whenever  $x_s, x_t \in I$  and  $0 < \lambda < 1$ .

**Theorem 1 [10]** Let  $f(x)$  be a real-valued function defined on an interval  $I$  is strictly concave up if and only if

$$(x_t - x_s)f(x) < (x_t - x)f(x_s) + (x - x_s)f(x_t) \quad \text{for all } x_s, x_t, x \in I \text{ such that } x_s < x < x_t.$$

**Theorem 2 [10]** Let  $f(x)$  be a real-valued function which is continuous on an interval  $I$  and which has a derivative at each interior point of  $I$ . Then  $f(x)$  is strictly concave up on the interval  $I$  if and only if  $f'(x)$  is strictly increasing on the interior of  $I$ .

## 3 Mathematical Theories

First, we propose two formulas of quadratic functions that will be used in Section 3.2.

### 3.1 Formulas related to the fuzzy functions

**Proposition 1** Given a quadratic function  $k_{(m_i, i, i+1)}(x)$  passing through points  $P_i(x_i, y_i)$  and  $P_{i+1}(x_{i+1}, y_{i+1})$ , the slope at  $P_i(x_i, y_i)$  is  $m_i$ , then

$$k_{(m_i, i, i+1)}(x) = (x - x_{i+1}) \left[ (x - x_i) \frac{\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - m_i}{x_{i+1} - x_i} + \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \right] + y_{i+1}.$$

**Proof.** Since function  $k_{(m_i, i, i+1)}(x)$  passing through the points  $P(x_i, y_i)$  and  $P_i(x_{i+1}, y_{i+1})$ , let

$$k_{(m_i, i, i+1)}(x) = (x - x_{i+1})[(x - x_i)q_{iR} + r_{iR}] + s_{iR}$$

$$k_{(m_i, i, i+1)}(x_{i+1}) = (x_i - x_{i+1})[(x_{i+1} - x_i)q_{iR} + r_{iR}] + s_{iR}.$$

We can get  $s_{iR} = y_{i+1}$ .

Since  $k_{(m_i, i, i+1)}(x)$  passing through the point  $P_i(x_i, y_i)$ , we get  $k_{(m_i, i, i+1)}(x_i) = (x_i - x_{i+1})[(x_i - x_i)q_{iR} + r_{iR}] + y_{i+1}$ ,

$$\text{hence, } r_{iR} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}.$$

Then we can get

$$k_{(m_i, i, i+1)}(x) = (x - x_{i+1})[(x - x_i)q_{iR} + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}] + y_{i+1}$$

$$= (x - x_{i+1})(x - x_i)q_{iR} + (x - x_{i+1})\frac{y_{i+1} - y_i}{x_{i+1} - x_i} + y_{i+1}.$$

Since  $k'(x_i) = m_i$ , and

$$(k_{(m_i, i, i+1)}(x))' = (x - x_{i+1})'(x - x_i)q_{iR} + (x - x_{i+1})(x - x_i)'q_{iR} + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}.$$

$$\text{Hence, } m_i = (k_{(m_i, i, i+1)}(x))' |_{x=x_i} = (x_i - x_{i+1})q_{iR} + \frac{y_{i+1} - y_i}{x_{i+1} - x_i},$$

$$\text{so } q_{iR} = \frac{\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - m_i}{x_{i+1} - x_i}. \quad \square$$

Similarly, we can prove the following proposition.

**Proposition 2** Given a quadratic function  $k_{(m_i, i, i+1)}(x)$  passing through points  $P_i(x_i, y_i)$  and  $P_{i+1}(x_{i+1}, y_{i+1})$ , the slope at  $P_{i+1}(x_{i+1}, y_{i+1})$  is  $m_{i+1}$ , then

$$g_{(m_i, i, i+1)}(x) = (x - x_i) \left[ (x - x_{i+1}) \frac{m_{i+1} - \frac{y_{i+1} - y_i}{x_{i+1} - x_i}}{x_{i+1} - x_i} + \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \right] + y_i.$$

### 3.2 Fuzzy functions

Sugeno Fuzzy model is used to approach the fuzzy problem. Given paired data set  $\{(x_i, y_i) \mid x \in [1, N]\}$  and the sequence  $(x_i, y_i)$ . Given data set  $A_{m_i(i, i+1)} = \{x \mid x \in [x_i, x_{i+1}]\}$  and the parabola function  $k_{(m_i, i, i+1)}(x)$  passes the point  $P(x_i, y_i)$  and  $P(x_{i+1}, y_{i+1})$  with the slope  $m_i$  of the tangent line at the point  $P(x_i, y_i)$ . The data set  $A_{m_{(i+1)}(i, i+1)}$  is such that  $x$  in  $[x_i, x_{i+1}]$ ,  $g_{(m_{i+1}, i, i+1)}(x)$  is the

parabola function passing the points  $P(x_i, y_i)$  and  $P(x_{i+1}, y_{i+1})$ , with slope  $m_{i+1}$  at  $P(x_{i+1}, y_{i+1})$ . The fuzzy rules are as follows:

If  $x \in A_{m_i(i,i+1)}$ , then the crisp function is  $k_{(m_i,i,i+1)}(x)$ .

If  $x$  is  $A_{m_{i+1}(i,i+1)}$  then the crisp function is  $g_{(m_{i+1},i,i+1)}(x)$ .

Let  $d_{i,i+1} = x_{i+1} - x_i$ , the membership function of  $k_{(m_i,i,i+1)}(x)$  is defined as:

$$m_{k_{(m_i,i,i+1)}}(x) = \begin{cases} \left[ 1 - \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \right], & \text{if } x_i < x < x_i + \frac{d_{i,i+1}}{2} \\ \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2, & \text{if } x_i + \frac{d_{i,i+1}}{2} < x < x_{i+1}. \end{cases}$$

where  $x_i$  is the  $x$  coordinate of the control point,  $i = 0, 1, \dots, m - 1$ . The membership function of  $g_{(m_{i+1},i,i+1)}(x)$  is defined as:

$$m_{g_{(m_{i+1},i,i+1)}}(x) = \begin{cases} \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2, & \text{if } x_i < x < x_i + \frac{d_{i,i+1}}{2} \\ \left[ 1 - \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \right], & \text{if } x_i + \frac{d_{i,i+1}}{2} < x < x_{i+1}. \end{cases}$$

where  $x_i$  is the  $x$  coordinate of the control point,  $i = 0, 1, \dots, m - 1$ .

**Definition 5** The fuzzy function  $f_{(i,i+1)}(x)$  is defined as following:

$$f_{(i,i+1)}(x) = \begin{cases} \left[ 1 - \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \right] \cdot k_{(m_i,i,i+1)}(x) \\ \quad + \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \cdot g_{(m_{i+1},i,i+1)}(x), & \text{if } x_i < x < x_i + \frac{d_{i,i+1}}{2} \\ \frac{1}{2} \left( \frac{x-x_{i+1}}{\frac{d_{i,i+1}}{2}} \right)^2 \cdot k_{(m_i,i,i+1)}(x) \\ \quad + \left[ 1 - \frac{1}{2} \left( \frac{x-x_{i+1}}{\frac{d_{i,i+1}}{2}} \right)^2 \right] \cdot g_{(m_{i+1},i,i+1)}(x), & \text{if } x_i + \frac{d_{i,i+1}}{2} < x < x_{i+1}. \end{cases}$$

where  $x_i$  is the  $x$  coordinate of the control point,  $i = 0, 1, \dots, m - 1$ .

From Definition 5 of the fuzzy function  $f_{(i,i+1)}(x)$ , we can prove the following statements easily.

**Proposition 3** Let  $f_{(i,i+1)}(x)$  is a fuzzy function, then the following statements are true:

- (1)  $\lim_{x \rightarrow (x_i)^+} f_{(i,i+1)}(x) = y_i$ .
- (2)  $\lim_{x \rightarrow (x_{i+1})^-} f_{(i,i+1)}(x) = y_{i+1}$ .
- (3)  $\lim_{x \rightarrow (x_i)^+} f'_{(i,i+1)}(x) = m_i$ .
- (4)  $\lim_{x \rightarrow (x_{i+1})^-} f'_{(i,i+1)}(x) = m_{i+1}$ .
- (5)  $\lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^-} f_{(i,i+1)}(x) = \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^+} f_{(i,i+1)}(x)$ .
- (6)  $\lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^-} k'_{(m_i,i,i+1)}(x) = \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^+} k'_{(m_i,i,i+1)}(x)$ .
- (7)  $\lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^-} g'_{(m_{i+1},i,i+1)}(x) = \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^+} g'_{(m_{i+1},i,i+1)}(x)$ .

$$(8) \quad \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^-} f'_{(i,i+1)}(x) = \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^+} f'_{(i,i+1)}(x).$$

**Proof.** From the definition of fuzzy function  $f_{(i,i+1)}(x)$ , we can prove item (1) to item (7) easily. We just only prove item (8).

$$\begin{aligned} \text{Let } s &= x_i + \frac{d_{i,i+1}}{2}, \\ \lim_{x \rightarrow s^-} f'_{(i,i+1)}(x) &= \lim_{x \rightarrow s^-} \left\{ \left( \left[ 1 - \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \right] \cdot k_{(m_i,i,i+1)}(x) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \cdot g_{(m_{i+1},i,i+1)}(x) \right)' \right\} \\ &= \lim_{x \rightarrow s^-} \left\{ \left[ 1 - \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \right]' \cdot k_{(m_i,i,i+1)}(x) \right. \\ &\quad \left. + \left[ 1 - \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \right] \cdot [k_{(m_i,i,i+1)}(x)]' \right\} \\ &\quad + \lim_{x \rightarrow s^-} \left\{ \left[ \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \right]' \cdot g_{(m_{i+1},i,i+1)}(x) \right. \\ &\quad \left. + \frac{1}{2} \left( \frac{x-x_i}{\frac{d_{i,i+1}}{2}} \right)^2 \cdot [g_{(m_{i+1},i,i+1)}(x)]' \right\} \\ &= \frac{1}{2} [k'_{(m_i,i,i+1)}(s) + g'_{(m_{i+1},i,i+1)}(s)] \\ &\quad + \frac{1}{d_{i,i+1}} [g_{(m_{i+1},i,i+1)}(s) - k_{(m_i,i,i+1)}(s)]. \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow s^+} f'_{(i,i+1)}(x) &= \lim_{x \rightarrow s^+} \left\{ \frac{1}{2} \left( \frac{x-x_{i+1}}{\frac{d_{i,i+1}}{2}} \right)^2 \cdot k_{(m_i,i,i+1)}(x) \right. \\ &\quad \left. + \left[ 1 - \frac{1}{2} \left( \frac{x-x_{i+1}}{\frac{d_{i,i+1}}{2}} \right)^2 \right] \cdot [g_{(m_{i+1},i,i+1)}(x)]' \right\}' \\ &= \lim_{x \rightarrow s^+} \left\{ \left[ \frac{1}{2} \left( \frac{x-x_{i+1}}{\frac{d_{i,i+1}}{2}} \right)^2 \right]' \cdot k_{(m_i,i,i+1)}(x) \right. \\ &\quad \left. + \frac{1}{2} \left( \frac{x-x_{i+1}}{\frac{d_{i,i+1}}{2}} \right)^2 \cdot [k_{(m_i,i,i+1)}(x)]' \right\} \\ &\quad + \lim_{x \rightarrow s^+} \left\{ \left[ 1 - \frac{1}{2} \left( \frac{x-x_{i+1}}{\frac{d_{i,i+1}}{2}} \right)^2 \right]' \cdot g_{(m_{i+1},i,i+1)}(x) \right. \\ &\quad \left. + \left[ 1 - \frac{1}{2} \left( \frac{x-x_{i+1}}{\frac{d_{i,i+1}}{2}} \right)^2 \right] \cdot [g_{(m_{i+1},i,i+1)}(x)]' \right\} \\ &= \frac{1}{2} [k'_{(m_i,i,i+1)}(s) + g'_{(m_{i+1},i,i+1)}(s)] \\ &\quad + \frac{1}{d_{i,i+1}} [g_{(m_{i+1},i,i+1)}(s) - k_{(m_i,i,i+1)}(s)] \end{aligned}$$

□

Based on the above-mentioned propositions, we can define:

$$\begin{aligned}
 f_{(i,i+1)}(x_i) &= y_i, f_{(i,i+1)}(x_{i+1}) = y_{i+1}, f'_{(i,i+1)}(x_i) = m_i, \\
 f'_{(i,i+1)}(x_{i+1}) &= m_{i+1}, \\
 f_{(i,i+1)}\left(x_i + \frac{d_{i,i+1}}{2}\right) &= \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^-} f_{(i,i+1)}(x) \text{ and} \\
 f'_{(i,i+1)}\left(x_i + \frac{d_{i,i+1}}{2}\right) &= \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^-} f'_{(i,i+1)}(x).
 \end{aligned}$$

**Proposition 4** Let a function  $h(x)$  be differentiable in  $[x_i, x_{i+1}]$ , passing through points  $P_i(x_i, y_i)$  and  $P_{i+1}(x_{i+1}, y_{i+1})$ , the slopes are  $m_i$  and  $m_{i+1}$ , respectively. The function  $f_{(i,i+1)}(x)$  passing the same points and has the same slopes as  $h(x)$ . If  $h(x)$  is concave up then  $f_{(i,i+1)}(x)$  is concave up, too.

**Proof.** If  $x \in [x_i, x_{i+1}]$ ,  $d_{i,i+1} = x_{i+1} - x_i$  and  $x \in (x_i, x_i + \frac{d_{i,i+1}}{2})$ , then

$$\begin{aligned}
 f_{(i,i+1)}(x) &= \left[1 - \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2\right] \cdot k_{(m_i, i, i+1)}(x) \\
 &\quad + \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2 \cdot g_{(m_{i+1}, i, i+1)}(x).
 \end{aligned}$$

Since  $h_{(i,i+1)}(x)$  is concave up, then  $k_{(m_i, i, i+1)}(x)$  and  $g_{(m_{i+1}, i, i+1)}(x)$  will be concave up, too. Hence, if  $x \in (x_s, x_t)$ ,  $k_{(m_i, i, i+1)}(x) < \lambda_1 \cdot k_{(m_i, i, i+1)}(x_s) + (1 - \lambda_1) \cdot k_{(m_i, i, i+1)}(x_t)$ , and  $g_{(m_{i+1}, i, i+1)}(x) < \lambda_1 \cdot g_{(m_{i+1}, i, i+1)}(x_s) + (1 - \lambda_1) \cdot g_{(m_{i+1}, i, i+1)}(x_t)$ , where  $\lambda_1 = \frac{x_t - x}{x_t - x_s}$  and  $1 - \lambda_1 = \frac{x - x_s}{x_t - x_s}$ . Then we have

$$\begin{aligned}
 f_{(i,i+1)}(x) &= \left[1 - \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2\right] \cdot k_{(m_i, i, i+1)}(x) \\
 &\quad + \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2 \cdot g_{(m_{i+1}, i, i+1)}(x) \\
 &< \left[1 - \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2\right] \cdot [\lambda_1 \cdot k_{(m_i, i, i+1)}(x_s) \\
 &\quad + (1 - \lambda_1) \cdot k_{(m_i, i, i+1)}(x_t)] \\
 &\quad + \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2 \cdot [\lambda_1 \cdot g_{(m_{i+1}, i, i+1)}(x_s) \\
 &\quad + (1 - \lambda_1) \cdot g_{(m_{i+1}, i, i+1)}(x_t)] \\
 &= \lambda_1 \left\{ \left[1 - \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2\right] \cdot k_{(m_i, i, i+1)}(x_s) \right. \\
 &\quad \left. + \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2 \cdot g_{(m_{i+1}, i, i+1)}(x_s) \right\} \\
 &\quad + (1 - \lambda_1) \left\{ \left[1 - \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2\right] \cdot k_{(m_i, i, i+1)}(x_t) \right.
 \end{aligned}$$

$$\begin{aligned}
 &\quad \left. + \frac{1}{2} \left(\frac{x - x_i}{\frac{d_{i,i+1}}{2}}\right)^2 \cdot g_{(m_{i+1}, i, i+1)}(x_t) \right\} \\
 &= \lambda_1 \cdot f_{(i,i+1)}(x_s) + (1 - \lambda_1) \cdot f_{(i,i+1)}(x_t).
 \end{aligned}$$

Hence,  $f_{(i,i+1)}(x)$  is strictly concave up in the interval  $(x_i, x_i + \frac{d_{i,i+1}}{2})$ . Since  $f_{(i,i+1)}(x)$  is continuous, with derivative at each interior point, and strictly concave up in the interval  $(x_i, x_i + \frac{d_{i,i+1}}{2})$ . By Theorem 2,  $f'_{(i,i+1)}(x)$ , the derivative of  $f_{(i,i+1)}(x)$ , is increasing in the interval  $(x_i, x_i + \frac{d_{i,i+1}}{2})$ .

Similarly,  $f'_{(i,i+1)}(x)$  is increasing in the interval  $(x_i + \frac{d_{i,i+1}}{2}, x_{i+1})$ . Since  $f_{(i,i+1)}(x_i) = y_i, f_{(i,i+1)}(x_{i+1}) = y_{i+1}, f'_{(i,i+1)}(x_i) = m_i$  and  $f'_{(i,i+1)}(x_{i+1}) = m_{i+1}$ , then

$$\begin{aligned}
 f_{(i,i+1)}\left(x_i + \frac{d_{i,i+1}}{2}\right) &= \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^-} f_{(i,i+1)}(x) = \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^+} f_{(i,i+1)}(x).
 \end{aligned}$$

and

$$\begin{aligned}
 f'_{(i,i+1)}\left(x_i + \frac{d_{i,i+1}}{2}\right) &= \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^-} f'_{(i,i+1)}(x) = \lim_{x \rightarrow (x_i + \frac{d_{i,i+1}}{2})^+} f'_{(i,i+1)}(x).
 \end{aligned}$$

Then the function  $f'_{(i,i+1)}(x)$  is strictly increasing in the interval  $[x_i, x_{i+1}]$ . By Theorem 2,  $f_{(i,i+1)}(x)$  is concave up in the interval  $[x_i, x_{i+1}]$ . □

Similarly, if  $h(x)$  is concave down the interval  $[x_i, x_{i+1}]$ , we can prove that  $f_{(i,i+1)}(x)$  is concave down in the interval  $[x_i, x_{i+1}]$ , too.

Based on the above propositions, we can propose the following algorithm to generate the control points to approximate a given function. The algorithm is as follows:

**Algorithm Fuzzy\_function**( $h(x), h'(x), [x_L, x_R]$ )

Input: an original function  $h(x)$ , its derivatives  $h'(x)$  in the interval  $[x_L, x_R]$ .

$x_L$  is the left coordinate of the interval.

$x_R$  is the right coordinate of the interval.

$\delta$  is the selected defined error.

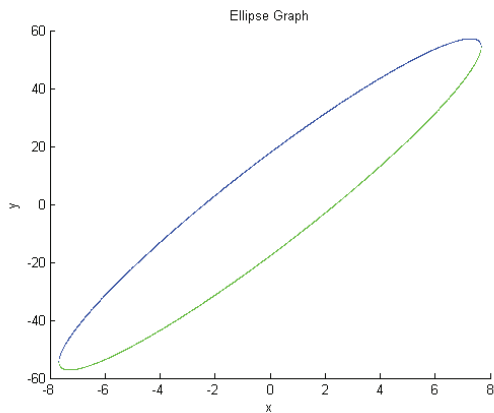
Output: the coordinates of the control points and their respected slopes of the function  $h(x)$ .

Let  $f(x)$  be the fuzzy function of the original function  $h(x)$ .

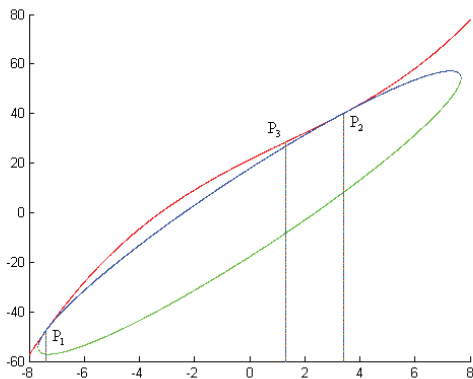
Define Max\_error =  $|f(x) - h(x)|$  in the interval  $[x_L, x_R]$ .

Procedure *Fuzzy\_function* ( $x_L, x_R$ )

1. Choose  $(x_L, h(x_L)), h'(x_L), (x_R, h(x_R))$  and  $h'(x_R)$ ,

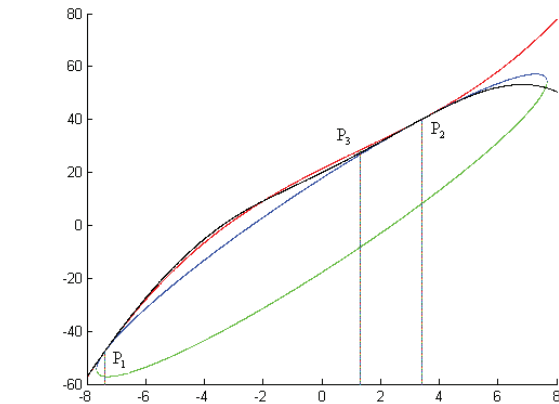


(a)

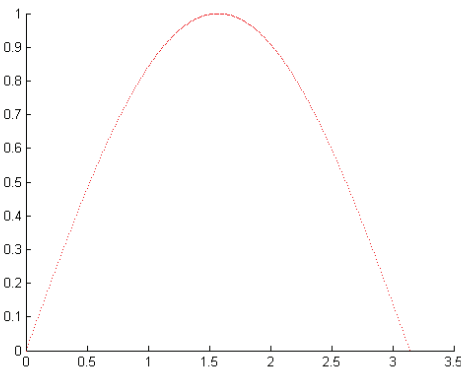


(b)

**Fig. 1:** The graph of the ellipse (a) without and (b) with cubic polynomial function



**Fig. 2:** The graph of the ellipse with cubic polynomial function (red color) and fuzzy function (black color)



**Fig. 3:** Example of a sine function in the interval  $[0, \pi]$

- generate the related fuzzy function  $f(x)$ .
- 2. Find the  $x$ -coordinate  $x_{Max}$  of the Max\_error between the original function  $h(x)$  and the fuzzy function  $f(x)$  in the interval.
- 3. Store  $(x_{Max}, h(x_{Max})$  and  $h'(x_{Max})$ .
- 4. If  $Max\_error > \delta$  then

```

    Fuzzy_function ( $x_L, x_{Max}$ )
else
    Fuzzy_function ( $x_{Max}, x_R$ )
End if
    
```

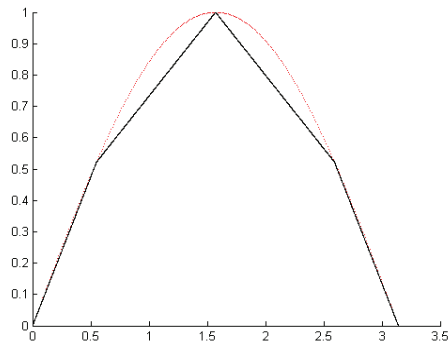
From the resultant of the four propositions as mentioned above, we find the approximate function  $\sum_0^{m-1} f_{(i,i+1)}(x)$  is a function which is smooth and concave preserving over the domain.

### 4 Results and Discussions

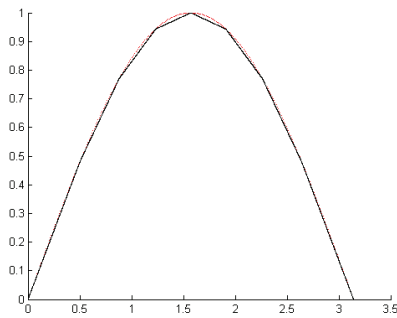
The following is an example illustrating the drawback of a cubic polynomial function for fitting function. A graph of ellipse is shown in Fig. 1(a). The function of the upper part of an ellipse is shown below:

$$y = \frac{25.5x + \sqrt{(25.5x)^2 - 4(1.8)(100x^2 - 570)}}{2(1.8)}$$

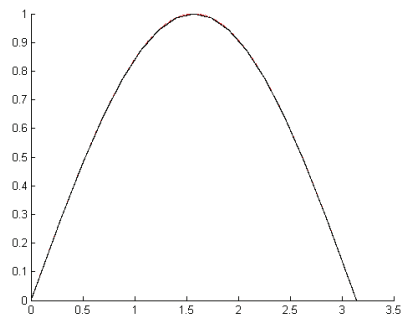
The upper part of an ellipse function is indicated in blue in Fig. 1(a). The cubic polynomial function generated by the algorithms of two points of the coordinates and their slopes to the initial function is used to approximate the function shown in Fig. 1(b). (In this example we choose  $P_1(-7.4, -47.7314)$ ,  $m_1=15.58374$ ,  $P_2(3.4, 40.0349)$ ,  $m_2=5.93620$ ). The coordinate of  $P_3(1.3, 26.74604)$  is the inflection point. The cubic polynomial function in the interval of  $[-7.4, 1.3]$  is concave down, and that in the interval of  $[1.3, 3.4]$  is concave up.



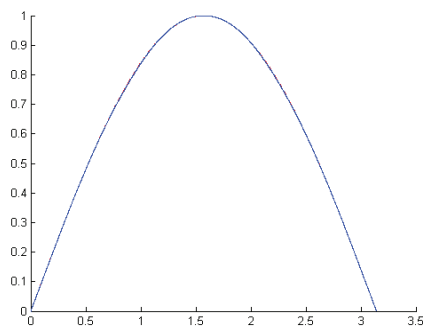
(a)



(b)



(c)



(d)

**Fig. 4:** A sine function approximated by (a) 4 (b) 8 (c) 16 lines and (d) 4 fuzzy functions

In Fig. 2, the black curve is drawn by the fuzzy function fitting algorithm proposed in this paper, which is concave down in the interval  $[-7.4, 3.4]$ .

Next, we use the fuzzy function fitting algorithm proposed in this paper to approximate the upper part of a sine function in the interval  $[0, \pi]$ . Fig. 3 is the sine function.

Fig. 4(a), 4(b), and 4(c) are the approximated function by 4, 8 and 16 lines. Fig. 4(d) is approximated by 4 fuzzy function curves of our fitting algorithms.

## 5 Conclusions and Future work

A fuzzy membership function is developed to interpolate a desired function or data pairs. The concave preserving issues are discussed and proved for the properties of the approximation functions. The desired function can be widely approximated piece by piece by two parabolic functions. Some numerical results demonstrate that the approximation functions preserve some properties and can approximate parts of the ellipse and sine functions. Choosing better approximation functions is our future work.

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