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# Numerical Study of Mixed Convection MHD Flow in Vertical Channels Using Differential Transformation Method

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**Abstract:** The problem of combined forced and free convection in a parallel-plate vertical channel is relevant in many industrial and engineering applications, such as heat exchangers, chemical processing equipment, fluid transport, and so on. A numerical analysis is performed within a combined forced and free convective magnetohydrodynamic (MHD) flow in a parallel-plate vertical channel. The MHD flow is assumed to be steady state, laminar and fully developed. The analysis takes account of the effects of both Joule heating and viscous dissipation, and is therefore relevant for MHD flow with high values of the dynamic viscosity as well as for high velocity flows. The non-linear governing equations for the velocity and temperature fields are solved using the differential transformation method. It is shown that the numerical results are in good agreement with the analytical solutions.

**Keywords:** MHD flow; Mixed convection; Differential transformation; Vertical channel

## 1 Introduction

The problem of combined forced and free convection in a parallel-plate vertical channel is relevant in many industrial and engineering applications, such as heat exchangers, chemical processing equipment, fluid transport, and so on. The earliest analyses of laminar and fully developed mixed convection in the parallel-plate vertical channel with uniform temperatures at the boundaries can be found in Tao [1]. Hamadah and Wirtz [2] showed that for mixed convection in a vertical channel subject to asymmetric heating conditions, the buoyancy force enhances the heat transfer near the hotter wall and causes a flow reversal near the cooler wall. Barletta [3] presented a perturbation-based method for analyzing the effects of viscous dissipation in laminar combined forced and free convection flows in a parallel-plate vertical channel. The characteristics of natural and mixed convection flows subject to magnetohydrodynamic (MHD) effects have attracted significant interest in recent years [4,5]. For example, Baker *et al.* [4] studied the mixed convection in a vertical plan channel with a horizontal magnetic field, in conditions of microgravity with a gravitational acceleration that oscillates in time with g-jitter effect. Sposito and Ciofalo [5] obtained analytical solutions for the temperature, velocity and electrical potential fields in the fully developed laminar flow of an electrically conducting fluid within a vertical channel under the simultaneous effects of a pressure driving head, buoyancy, and a MHD force, respectively. Setayesh and Sahai [6] studied the effect of various temperature-dependent transport properties on the developing MHD flow and heat transfer in a parallel-plate channel in which the walls were held at a constant and equal temperature. Umavathi and Malashetty [7] presented analytical and numerical solutions by a perturbation method for the temperature and velocity fields in a combined free and forced convective MHD flow in a vertical channel. It was shown that the viscous dissipation effect enhanced the flow reversal which occurred when the flow in the downward direction encountered that in the upward direction.

Differential transformation theory has been widely applied to the solution of general initial value problems in the mechanical engineering domain. For example, Chen and Ho [8,9] used differential transformation theory to

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solve general eigenvalue problems and to analyze the free vibration response of Timoshenko beams. Kuo [10] applied the differential transformation method (D.T.M) to investigate the velocity and temperature profiles in a free convection boundary-layer flow over a vertical plate.

In the present study, the D.T.M method is used to solve the temperature and velocity profiles in a combined forced and free convective MHD flow in a parallel-plate vertical channel. In performing the analysis, the effects of Joule heating and viscous dissipation are taken into consideration.

## 2 Differential Transformation Theory

This section reviews the basic principles of differential transformation theory. Assume that $y(t)$ is an analytic function in the time domain $T$. The differential transformation of$y$ at time $t=t_0$ in the $K$ domain is given by

$$Y(k;t_0)=W(k)\left(\frac{d^k}{dt^k}(s(t)y(t))\right)_{t=t_0}\ ,\ k\in K \tag{1}$$

where $k$ belongs to a set of non-negative integers which collectively define the $K$ domain; $W(k)$ is a weighting factor; and $s(t)$ is a kernel function corresponding to $y(t)$. Note that $W(k)$ and $s(t)$ are both non-zero and $s(t)$ is analytic in the time domain. The inverse differential transformation of $Y(k;t_0)$ is formulated as

$$y(t)=\frac{1}{s(t)}\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\frac{Y(k;t_0)}{W(k)}\ ,\ t\in T\ , \tag{2}$$

1.in which $W(k)=H^k/k!$ and $s(t)=1$. Note that $H$ is the time interval.
2.At time $t_0=0$, Eq. (**??**) becomes

$$Y(k)=\frac{H^k}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=0}\ ,\ k\in K\,. \tag{3}$$

3.From Eq. (2), the inverse differential transformation of $Y(k)$is obtained as

$$y(t)=\sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^k Y(k)\ ,\ t\in T\,. \tag{4}$$

4.Substituting Eq. (3) into Eq. (4) gives

$$y(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=0}\ ,\ t\in T\,. \tag{5}$$

5.Eq. (5) has the form of a Taylor series expansion. Therefore, the basic operational properties of the differential transformation method (D.T.M) can be summarized as follows:
6.(a) Linearity operation

$$T[\alpha p(t)+\beta q(t)]=\alpha P(k)+\beta Q(k), \tag{6}$$

7.where $T$ denotes differential transformation and $\alpha$ and $\beta$are any real number.
8.(b) Convolution operation

$$T[p(t)q(t)]=P(k)\otimes Q(k)=\sum_{\ell=0}^{k}P(\ell)Q(k-\ell),$$

$$T[p^m(t)]=kP(0)P^m(k)=\sum_{\ell=1}^{k}[(m+1)\ell]P(\ell)P^m(k-\ell),m\in N \tag{7}$$

9.where $\otimes$ denotes convolution.
10.(c) Differential operation

$$T\left[\frac{d^n p(t)}{dt^n}\right]=\frac{(k+n)!}{k!H^n}P(k+n), \tag{8}$$

11.where $n$ is the order of differentiation
12.(d) Differential transformation of $\sin(t)$ and $\cos(t)$ functions

$$T[\sin(\alpha t+\beta)]=\frac{(\alpha H)^k}{k!}\sin\left(\frac{\pi k}{2}+\beta\right),T[\cos(\alpha t+\beta)]=\frac{(\alpha H)^k}{k!}\cos\left(\frac{\pi k}{2}+\beta\right), \tag{9}$$

where $\alpha$ and $\beta$are any real number. [11-12].

## 3 Mathematical Formulation

### 3.1 *Governing Equations of Velocity and Temperature*

Consider the steady-laminar MHD flow of an incompressible fluid etween the two vertical walls of a parallel-plate channel (see Fig. 1). Applying the mass balance equation, momentum balance equation and energy balance equation, and taking the effects of Joule heating and viscous dissipation into account, the differential equations for the velocity in the x direction ($u$) and the temperature ($\overline{T}$) are obtained as [7]

$$\frac{d^4u}{dy^4}=\frac{\beta g}{\alpha C_p}\left(\frac{du}{dy}\right)^2+\frac{\sigma_e B^2}{\mu}\frac{d^2u}{dy^2}+\frac{\sigma_e B^2\beta g}{\alpha C_p\mu}u^2, \tag{10}$$

$$\alpha\frac{d^2\overline{T}}{dy^2}+\frac{\nu}{C_p}\left(\frac{du}{dy}\right)^2+\frac{\sigma_e B^2}{\rho C_p}u^2=0, \tag{11}$$

A no-slip condition is imposed on $u$at each of the channel walls, i.e.,

$$u(0)=u(L)=0, \tag{12}$$

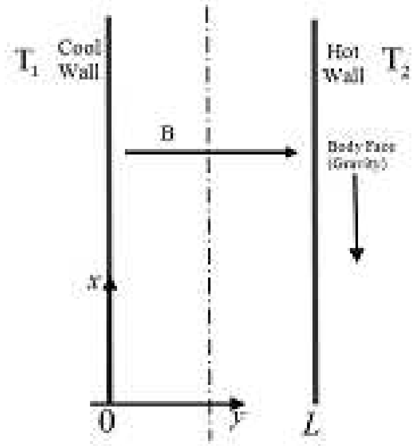


**Fig. 1:** Schematic diagram of vertical channel

and those induced by the boundary conditions on temperature, namely

$$\frac{d^2u}{dy^2} = \frac{A}{\mu} - \frac{\beta g(T_1 - T_0)}{\nu} \quad \text{at} \quad y = 0, \tag{13}$$

$$\frac{d^2u}{dy^2} = \frac{A}{\mu} - \frac{\beta g(T_2 - T_0)}{\nu} \quad \text{at} \quad y = L, \tag{14}$$

For analytical convenience, let the following dimensionless v variables be defined:

$$\bar{u} = \frac{u}{u_0}\ ,\ \ \bar{\theta} = \frac{\bar{T} - T_0}{\Delta T}\ ,\ \ \bar{y} = \frac{2y}{D}\ ,\ \ G_r = \frac{\beta g \Delta T D^3}{\nu^2}\ ,$$

$$R_T = \frac{T_2 - T_1}{\Delta T}\ ,\ R_e = \frac{u_0 D}{\nu}\ ,\ B_r = \frac{\mu u_0^2}{k \Delta T}\ ,\ \ \alpha = \frac{k}{\rho C_p}\ ,$$

$$H_m^2 = \frac{\sigma_e B^2 D^2}{\mu}\ ,\ \Xi = \frac{G_r}{R_e} = \frac{g\beta \Delta T D^2}{\gamma u_0}, \tag{15}$$

where $D = 2L$is the hydraulic diameter.

The reference velocity $u_0$and reference temperature $T_0$are defined respectively as

$$u_0 = -\frac{AD^2}{48u} \quad , \quad T_0 = \frac{T_1 + T_2}{2}, \tag{16}$$

Moreover, the reference temperature difference, $\Delta T$, is given by

$$\Delta T = T_2 - T_1 \quad \text{if} \quad T_1 < T_2$$

or

$$\Delta T = \frac{\nu^2}{C_p D^2} \quad \text{if} \quad T_1 = T_2, \tag{17}$$

For symmetric heating ($T_1 = T_2$), the temperature difference ratio $R_T$is equal to zero. By contrast, for asymmetric heating ($T_1 < T_2$),$R_T$is equal to one.

Substituting the dimensionless variables given in Eq. (15) into Eqs. (10), (11) and (12) yields the following normalized differential equation for the velocity profile in the $\bar{y}$ direction:

$$\frac{d^4\bar{u}}{d\bar{y}^4} = \frac{1}{4}\Xi B_r\left(\frac{d\bar{u}}{d\bar{y}}\right)^2 + \frac{1}{4}H_m^2\frac{d^2\bar{u}}{d\bar{y}^2} + \frac{1}{16}H_m^2 \Xi B_r \bar{u}^2, \tag{18}$$

$$\bar{u}(0) = 0\ ,\ \frac{d^2\bar{u}}{d\bar{y}^2} = -12 + \frac{R_T \Xi}{8} \quad \text{at} \quad \bar{y} = 0, \tag{19}$$

$$\bar{u}(1) = 0\ ,\ \frac{d^2\bar{u}}{d\bar{y}^2} = -12 - \frac{R_T \Xi}{8} \quad \text{at} \quad \bar{y} = 1, \tag{20}$$

Taking the effects of Joule heating and viscous dissipation into account, the dimensionless equation for the temperature profile in the $\bar{y}$ direction is obtained as

$$\frac{d^2\bar{\theta}}{d\bar{y}^2} + B_r\left(\frac{d\bar{u}}{d\bar{y}}\right)^2 + \frac{1}{4}H_m^2 B_r \bar{u}^2 = 0, \tag{21}$$

where the boundary conditions of the temperature field are given respectively by

$$\bar{\theta}(0) = -R_T/2 \quad \text{at} \quad \bar{y} = 0, \tag{22}$$

$$\bar{\theta}(1) = R_T/2 \quad \text{at} \quad \bar{y} = 1, \tag{23}$$

### *3.2 Application of Differential Transformation Method (D.T.M) to Solution of Velocity and Temperature Fields*

In this section, the differential transformation method (D.T.M) is used to solve the velocity and temperature fields within the vertical channel, together with their corresponding boundary conditions. The process is as follows:

$$T\left[\frac{d^4\bar{u}}{d\bar{y}^4}\right] = \frac{(k+1)(k+2)(k+3)(k+4)}{H^4}U(k+4),$$

$$T\left[\frac{1}{4}\Xi B_r\left(\frac{d\bar{u}}{d\bar{y}}\right)^2\right] = \frac{1}{4}\Xi B_r\sum_{\ell=0}^{k}(\ell+1)U(\ell+1)(k-\ell+1)U(k-\ell+1),$$

$$T\left[\frac{1}{4}H_m^2\frac{d^4\bar{u}}{d\bar{y}^4}\right] = \frac{1}{4}H_m^2\frac{(k+1)(k+2)}{H^2}U(k+2),$$

$$T\left[\frac{1}{16}H_m^2 \Xi B_r \bar{u}^2\right] = \frac{1}{16}H_m^2 \Xi B_r \sum_{\ell=0}^{k} U(\ell)U(k-\ell), \tag{24}$$

Equation (24) can be rewritten as

$$\begin{array}{l} \frac{(k+1)(k+2)(k+3)(k+4)}{H^4}U(k+4) = \\ \frac{1}{4}\Xi B_r \sum_{\ell=0}^{k}(\ell+1)U(\ell+1)(k-\ell+1)U(k-\ell+1) \\ +\frac{1}{4}H_m^2\frac{(k+1)(k+2)}{H^2}U(k+2) + \frac{1}{16}H_m^2 \Xi B_r \sum_{\ell=0}^{k} U(\ell)U(k-\ell) \end{array}, \tag{25}$$

In addition, the boundary condition given in Eq. (19) can be reformulated as

$$T\left[\bar{u}(\bar{y})\right] = U(k) = 0, \tag{26}$$

$$\begin{aligned} &T\left[\frac{d^2\bar{u}}{d\bar{y}^2}\right] = T\left[-12 + \frac{R_T\Xi}{8}\right] \Rightarrow \\ &(k+1)(k+2)U(k+2) = \left(-12 + \frac{R_T\Xi}{8}\right)\ \delta(\mathrm{k}) \end{aligned}, \tag{27}$$

Similarly, the boundary condition given in Eq. (20) can be reformulated as

$$T\left[\bar{u}(\bar{y})\right] = \sum_{k=0}^{m} U(k) = 0, \tag{28}$$

$$\begin{aligned} &T\left[\frac{d^2\bar{u}}{d\bar{y}^2}\right] = T\left[-12 - \frac{R_T\Xi}{8}\right] \Rightarrow \\ &\sum_{\mathrm{k}=0}^{\mathrm{m}} k(k-1)U(k) = \left(-12 - \frac{R_T\Xi}{8}\right)\delta(\mathrm{k}) \end{aligned}, \tag{29}$$

Applying the same procedure to the temperature profile (Eq. (21)) and corresponding boundary conditions (Eqs. (Eqs. (22) (23)), it can be shown that

$$T\left[\frac{d^2\bar{\theta}}{d\bar{y}^2}\right] = \frac{(k+1)(k+2)}{H^2}\Theta(k+2),$$

$$T\left[B_r\left(\frac{d\bar{u}}{d\bar{y}}\right)^2\right] = B_r\sum_{\ell=0}^{k}(\ell+1)U(\ell+1)(k-\ell+1)U(k-\ell+1),$$

$$T\left[\frac{1}{4}H_m^2B_r\bar{u}^2\right] = \frac{1}{4}H_m^2B_r\sum_{\ell=0}^{k}U(\ell)U(k-\ell), \tag{30}$$

Substituting Eq. (30) into Eq. (21) yields the following differential equation:

$$\begin{aligned} &\tfrac{(k+1)(k+2)}{H^2}\Theta(k+2) \\ &= -B_r\textstyle\sum_{\ell=0}^{k}(\ell+1)U(\ell+1)(k-\ell+1)U(k-\ell+1), \\ &-\tfrac{1}{4}H_m^2B_r\textstyle\sum_{\ell=0}^{k}U(\ell)U(k-\ell) \end{aligned} \tag{31}$$

The boundary condition for the temperature profile can be reformulated as

$$T\left[\bar{\theta}(\bar{y})\right] = T\left[-R_T/2\right] \Rightarrow \Theta(k) = -\frac{R_T}{2}\delta(\mathrm{k}), \tag{32}$$

$$T\left[\bar{\theta}(\bar{y})\right] = T\left[R_T/2\right] \Rightarrow \sum_{k=0}^{m}\Theta(k) = \frac{R_T}{2}\delta(\mathrm{k}), \tag{33}$$

where $m$ indicates the number of terms in the power series, $U(k)$and $\Theta(k)$ are the transformed functions of $\bar{u}(\bar{y})$ and $\bar{\theta}(\bar{y})$, respectively, and $\delta(\mathrm{k})$ is defined as

$$\delta(k) = \begin{cases} 1 & \text{for } k=0 \\ 0 & otherwise \end{cases}, \tag{34}$$

The solutions for $U(k)$ and $\Theta(k)$ are obtained from Eqs. (25) and (30)˜(31), respectively, using the transformed boundary conditions given in Eqs. (26)-(29) and Eqs.(32)-(33).

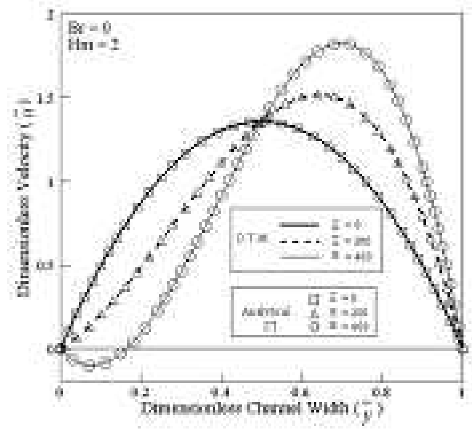


**Fig. 2:** Numerical and analytical results for velocity profiles given various values of $\Xi$ and $B_r = 0, H_m = 2$.

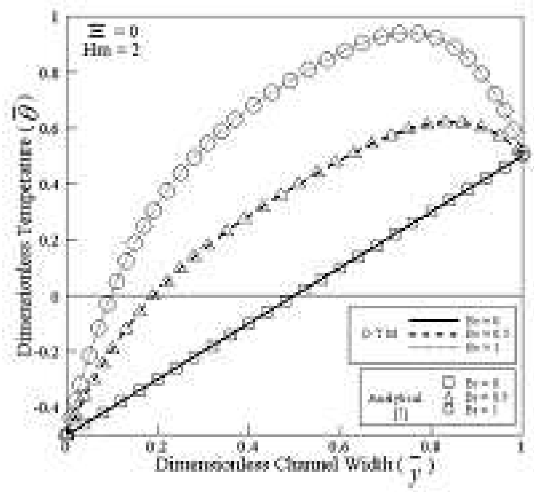


**Fig. 3:** Numerical and analytical results for temperature profiles given various values of $B_r$and$\Xi = 0, H_m = 2$.

## 4 Numerical Results and Discussion

This section analyzes the velocity profile and temperature profile within the parallel-plate vertical channel shown in Fig. 1 subject to both asymmetric and symmetric heating conditions (i.e.,$R_T = 1$ and $R_T$= 0, respectively). Figure 2 presents the velocity profiles obtained using the D.T.M method and the analytical method proposed in [7] for the case in which the viscous dissipation effect is assumed to be negligible, and the channel is heated asymmetrically. It can be seen that for each value of $\Xi$, a good agreement exists between the numerical results and the analytical results.

Similarly, Fig. 3 shows that the temperature profiles computed using the D.T.M method for various values of $B_r$ are also in excellent agreement with the analytical results.

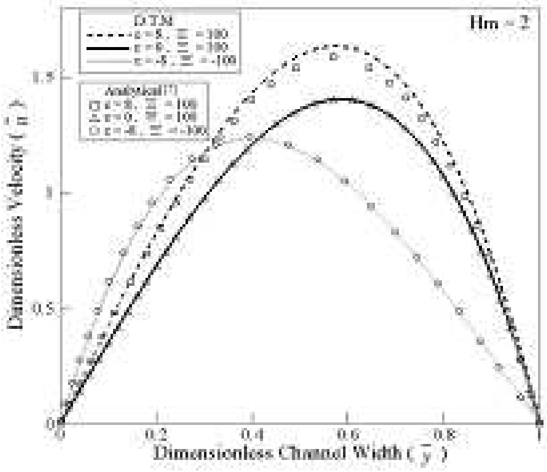


**Fig. 4:** Numerical and analytical results for velocity profiles given various values of$\varepsilon$ ($R_T = 1$).

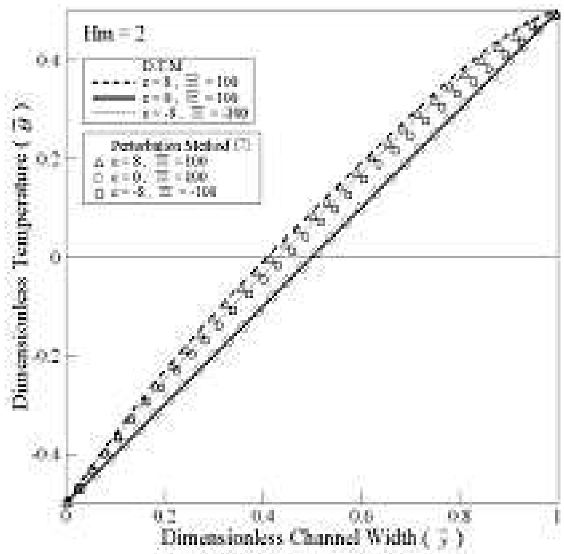


**Fig. 5:** Numerical and analytical results for temperature profiles given various values of $B_r\Xi$ ($R_T = 1$)

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Figure 4 presents the analytical and numerical results for the velocity profile within the channel for values of $\varepsilon$ in the range $-8 \leq \varepsilon \leq 8$ and $\Xi = 100$ or$\Xi = -100$. Note that $H_m = 2$ in every case. It can be seen that for a positive value of $\Xi$, the velocity increases with increasing$\varepsilon$. This result is to be expected since the viscous dissipation effect increases the fluid temperature and therefore gives rise to a greater buoyancy force, which leads in turn to a greater velocity in the upward direction. Conversely, for a negative value of $\Xi$, the velocity reduces as $\varepsilon$ is reduced from $\varepsilon = 0$ to $\varepsilon = -8$.

Figure 5 presents the analytical and numerical results for the temperature profile within the channel for a constant value of $H_m$= 2 and various values of $\varepsilon$ and $\Xi$. The results show that the temperature increases with increasing $\varepsilon$ for both positive and negative values of $\Xi$. Thus, the validity of the D.T.M method as a means of solving the nonlinear differential equations for the velocity and temperature fields is confirmed.

## 5 Conclusion

This study has presented a numerical analysis within a combined forced and free convective magnetohydrodynamic (MHD) flow in a parallel-plate vertical channel subject to Joule heating and viscous dissipation effects. The governing equations for the velocity and temperature fields within the channel have been solved using the differential transformation method (D.T.M). It has been shown that the numerical results are in good agreement with those obtained using an analytical approach. In general, the results have shown that viscous dissipation enhances the flow in the upward direction.

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## References

[1] Tao L.N., On combined free and forced convection in channels. *ASME Journal of Heat Transfer*. **82**, (1960), 233–8.

[2] Hamadah T.T., and Wirtz R.A., Analysis of laminar fully developed mixed convection in a vertical channel with opposing buoyancy. *ASME Journal of Heat Transfer*. **113**, (1991), 507–10.

[3] Barletta, A., Laminar mixed convection with viscous dissipation in a vertical channel. *Int. J. Heat Mass Transfer*. **41**, (1998), 3501–3513.

[4] Pan, B., and Li, B. Q., Effect of magnetic fields on oscillating mixed convection. *Int. J. Heat Mass Transfer*. **41**, (1998), 2705–2710.

[5] Sposito, G., and Ciofalo, M., One-dimensional mixed MHD convection. *Int. J. Heat Mass Transfer*. **49**, (2006), 2939–2949.

[6] Alireza, S., and Sahai, V., Heat transfer in developing magnetohydrodynamic Poiseuille flow and variable transport properties. *Int. J. Heat Mass Transfer*. **33**, (1990), 1711–1720.

[7] Umavathi, J. C., and Malashetty, M. S., Magnetohydrodynamic mixed convection in a vertical channel. *Int. J. Non-Linear Mech.*. **40**, (2005), 91–101.

[8] Kuo, B. L., Application of the differential transformation method to the solutions of the free convection problem. *Applied mathematical and computation*. **165**, (2005), 63–79.
[9] Chen, C. K., and Ho, S. H., Application of differential transformation to eigenvalue problem. *Applied mathematics and computation*. **79**, (1996), 173–188.
[10] Kuo, B. L., Application of the differential transformation method to the solutions of the free convection. *Applied mathematical and computation*. **165**, (2005), 63–79.
[11] Chen, C. K., Lai, H. Y., and Liu, C. C., Application of hybrid differential transformation/finite difference method to nonlinear analysis of micro fixed-fixed beam. *Microsyst Technol*. **15**, (2009), 813–820.
[12] Chen, C. K., Lai, H. Y., and Liu, C. C., Nonlinear Micro Circular Plate Analysis Using Hybrid Differential Transformation / Finite Difference Method. *CMES: Computer Modeling in Engineering & Sciences*, **40**, (2009), 155-174.

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