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# Geometric Calculus-based Postulates for the Derivation and Extension of the Maxwell Equations

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**Abstract:** The geometric algebra of three-dimensional physical space and its associated geometric calculus, enables a compact formulation of Maxwells electromagnetic (EM) equations from a set of familiar and physically relevant postulates. This formulation results in a natural extension of the Maxwell equations yielding wave solutions in addition to the usual EM waves. These additional, non-EM solutions do not conflict with classical EM experiments and have three properties in common with the apparent properties of dark energy. These three properties are that the wave solutions 1) propagate at the speed of light, 2) do not interact with ordinary electric charges or currents, and 3) possess retrograde momentum. By retrograde momentum, we mean that the momentum carried by such a wave is directed oppositely to the direction of energy transport. A gas of such waves generates negative pressure.

**Keywords:** postulates, electrodynamics, multivector field, negative pressure, dark energy, retrograde

## 1 Introduction

By employing geometric algebra (GA) and its associated geometric calculus, this paper extends the Maxwell equations to a multivector field equation with solutions having certain characteristics in common with those attributed to dark energy while maintaining the integrity of classical electrodynamics. We derive the multivector field equation from a set of postulates having a clear correspondence to the principles of classical electrodynamics.

The formulation of the Maxwell equations as a single multivector field equation using geometric algebra has been espoused by many including Hestenes [1], Vold [2], Baylis [3], and Doran and Lasenby [4]. These works are uniformly based on straightforward applications of GA and geometric calculus with the classical Maxwell equations as the starting point and the assumption that the multivector field has components limited to those corresponding to electric and magnetic fields. The four Maxwell equations are shown to be equivalent to the multivector grade components of a single, first order, differential field equation based on the geometric derivative [2,4,5]. A broader range of ideas relating to this work have been explored over the last century in the context of the theory of relativity [6,7], spinors and quaternions [8,9,10], semivectors [11], Clifford numbers [10,12,13], and multivector calculus [14]. Gsponer and Hurni [15] provide a recent review.

There are two main themes in this paper that add to past work. The first is the derivation of the multivector field equation from five postulates and the second is the analysis of energy and momentum conservation for novel solutions of the field equation. To establish notation and needed mathematical relationships, Section 2 discusses the fundamental theorem of calculus. Section 3 presents the five postulates. Section 4 derives the multivector field equation and wave equation from these postulates. Sections 5 and 6 discuss the nature of the fields and external sources and the nature of the field solutions, respectively. Section 7 discusses the continuity equation and Section 8 the conservation equation for energy and momentum. Section 9 compares the energy flux and momentum density for plane wave solutions of the field equation. Finally, Section 10 provides a discussion of findings and Section 11 a conclusion.

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## 2 The Fundamental Theorem of Calculus

Following the notation of Hestenes and Sobczyk [5], the Fundamental Theorem of Calculus using geometric algebra and geometric calculus is

$$\int_M \dot{L}(dV\dot{\partial}_x) = \int_{\partial M} L(dS), \tag{1}$$

where the multivector-valued function $L(A)$ is a general linear function of its argument $A$ and the overdot indicates the action of the geometric derivative. The argument $A$ is also multivector-valued and $L(A)$ may be a function of time $t$ and spatial position $x$. $L(A)$ is assumed to be well behaved over a region $M$ bounded by the surface $\partial M$. $dV$ is the pseudoscalar volume element in $M$ and $dS$ is the pseudovector surface element of the boundary $\partial M$. Finally, the geometric derivative $\partial_x$ is equivalent to and is frequently written as the gradient operator $\nabla$.

In this paper, we use two forms of the fundamental theorem. The first is based on the linear function $L(A) = AF$, where $F = F(x,t)$. Using this $L(A)$ in Equation (1), we obtain

$$\int_M dV\partial_x F = \int_{\partial M} dSF. \tag{2}$$

This basic form of the fundamental theorem coupled with the first two postulates defined below yields a differential equation for the multivector-valued field $F$. The second form of the fundamental theorem is based on the linear function $L(A) = IGI^{-1}AF$ , where $G = G(x,t)$ is another multivector-valued function and $I$ is the unit pseudoscalar of the geometric algebra of $M$. Using this $L(A)$ in Equation (1), we obtain

$$\int_M I\dot{G}I^{-1}dV\dot{\partial}_x\dot{F} = \int_{\partial M} IGI^{-1}dSF. \tag{3}$$

To put Equation (3) in a more convenient form, we use the fact that $dV = I|dV|$ and the convention of Hestenes and Sobczyk [5] that $dS = I|dS|n$, where the unit vector $n$ is the outward-pointing normal to the surface element $dS$. We note that Doran and Lasenby [4] use the convention that $dS = nI|dS|$, which differs from that of Hestenes and Sobczyk in even-dimensional spaces where $n$ and $I$ anticommute. In physical three-dimensional (3-D) space, $n$ and $I$ commute and the distinction is immaterial. Substituting these expressions for $dV$ and $dS$ into Equation (3), rearranging factors, and using the fact that $n$ is its own inverse yields a second form of the fundamental theorem:

$$\int_M dV\dot{G}\dot{\partial}_x\dot{F} = \int_{\partial M} dSn(GnF). \tag{4}$$

This form is used below in the derivation of energy and momentum relationships for the field $F$.

## 3 Five Postulates for a Multivector-Valued Field

We put forth the following postulates as a basis for the theory of a multivector-valued field and associated sources that exist in physical space. As in relativity theory, time $t$ is a parameter measured locally in physical space in terms of the number of cycles of a periodic physical system (a clock) at that location. The postulates are developed in an inertial laboratory frame for clarity to the broadest audience and we express time as a scalar quantity as did Maxwell. This assumption in no sense limits us to a non-relativistic approach. We assume that the region $M$ and its boundary over which we perform integrals are stationary and that the laboratory frame has stationary clocks that are synchronized by the usual procedures of special relativity so that integrals over space are carried out at a fixed time. Important results of the development may be expressed in covariant form by making an algebraic transformation to a spacetime algebra [16] in which time and physical position are both vector quantities in the usual four-dimensional spacetime of indefinite signature.

Postulate 1. *The vacuum of physical space supports a multivector-valued effective field that is a continuous and twice-differentiable function of time and spatial position.*

This postulate expresses the familiar proposition that mathematical field theory may be used to analyze observable phenomena in the vacuum. The qualifier "effective" means that submicroscopic influences such as quantum fluctuations are averaged out to produce smooth functions suitable for the application of the calculus as in classical electrodynamics. For present purposes, the most significant aspect of Postulate 1 is the invocation of geometric algebra through the assumption that the field at any location and time is represented by a multivector of any grade or mixture of grades without exclusion.

Postulate 2. *For any region of physical space enclosed by a boundary, the integral of the effective field over the boundary is equal to the integral of a multivector-valued source density over the enclosed region.*

This postulate is a generalization of the Laws of Gauss and Ampere and, by virtue of the completeness of geometric algebra and geometric calculus, implies both. Because the field may be of any multivector grade, the same may be true for the source of the field. The principle of linear superposition of fields is implicit in Postulate 2, as is the central nature of the field relative to its source.

Postulate 3. *Charges at rest and moving charges (*i.e., *currents), after multiplication by suitable proportionality constants, are external sources of the field.*

This postulate provides a connection between the multivector source density and the electric charges and currents giving rise to electromagnetic fields. The source is external in the sense of classical electrodynamics, where sources are assumed to exist separately from, but give rise to, the field described by the field equations. Using a multivector density with four grade components as stated in Postulate 2 provides for hypothetical magnetic charge and current densities as well as electric [2,3].

Postulate 4. *The time derivative of the multivector-valued field, multiplied by a suitable proportionality constant, is an internal (or intrinsic) source of the field.*

This postulate generalizes the magnetic induction described by Faraday's Law and the electric displacement current whose existence was deduced by Maxwell. This internal source arises from temporal variation of the multivector field rather than from an external apparatus. Like the multivector field, the internal source is associated with the vacuum itself. Just as Maxwell's introduction of a displacement current in his equations of electrodynamics allowed wave solutions for the electromagnetic field, we shall see that Postulate 4 leads to the wave equation in vacuum for all components of the multivector field.

Postulate 5. *The multivector field undergoes wave motion.*

This postulate incorporates the experimental observation of interference phenomena in classical and quantum physics of elementary particle fields at all experimentally accessible wavelengths. It determines the multivector grade of the proportionality constant in Postulate 4.

From these Postulates, we may construct integral, differential, and wave equations satisfied by the multivector-valued field and its sources.

## 4 The Multivector Field Equation

According to Postulate 1, the multivector-valued field $F$ is mathematically well-behaved and, therefore, satisfies the form of the fundamental theorem of calculus given by Equation (2):

$$\int_M dV \partial_x F = \int_{\partial M} dSF, \tag{5}$$

where $M$ is any pseudoscalar region of physical space and $\partial M$ is the pseudovector boundary enclosing $M$. $\partial_x F$ is the geometric derivative [2,5] of $F$. This result makes no supposition regarding the field $F$ other than that it is a continuous and differentiable function of position.

According to Postulate 2, there is a multivector-valued source density $D$ corresponding to the field $F$ such that the integral of the field over a closed surface is equal to the integral of the source density over the enclosed volume:

$$\int_{\partial M} dSF = \int_M dVD. \tag{6}$$

This integral field equation must be satisfied under all circumstances. Combining Equations (5) and (6), we have $\int_M dV \partial_x F = \int_M dVD$. In order that equality of these two integrals be satisfied for an arbitrary volume $M$, it is necessary that the integrands be equal. Therefore, Postulates 1 and 2 imply that the field and the source density satisfy the following differential equation:

$$\partial_x F = D. \tag{7}$$

According to Postulates 3 and 4, the source density $D$ has both external and internal contributions, which we write with subscripts $ex$ and $in$, respectively: $D = D_{ex} + D_{in}$. According to Postulate 4, the internal source density is proportional to the time derivative $\partial_t F$ of the field itself, so $D_{in} = K\partial_t F$. *A priori*, the proportionality constant $K$ might be of any multivector grade and might be written to the right or the left of the time derivative of the field. However, a vector or bivector value for $K$ would provide a reference direction violating the isotropy of space, so $K$ is either scalar or psuedoscalar and therefore commutes with all else. In a vacuum region free of external sources, the multivector differential field equation then reduces to

$$\partial_x F = K\partial_t F. \tag{8}$$

From this equation it is clear that the measurement units of the constant $K$ must be $[L^{-1}T]$, *i.e.*, it is an inverse speed. Following procedures from classical dynamics, we expect that the wave equation may be derived by taking a second geometric derivative of this equation, then on the right side, exchanging the order of differentiation and substituting for the geometric derivative again from Equation (8):

$\partial_x \partial_x F = \partial_x (K\partial_t F) = K\partial_t \partial_x F = K^2 \partial_t^2 F$, so that

$(\partial_x^2 - K^2 \partial_t^2)F = 0.$

This equation would be valid if $K$ were a trivector; however, the sign of the time derivative term would be positive, wrong for a wave equation. Therefore, Postulate 5 implies that $K$ is a scalar rather than a trivector. The wave equation is obtained for $K = \pm c^{-1}$ , where $c$ is a positive constant with units of a speed. We find by reference to the experiments of classical electrodynamics that the appropriate sign for this equation is negative and that $c$ is the speed of light so that $D_{in} = -c^{-1}\partial_t F$.

With this result for in $D_{in}$, we have the usual wave equation in vacuum:

$$(\partial_x^2 - c^{-2}\partial_t^2)F = 0. \tag{9}$$

Because the spatial and temporal second derivatives are both scalar operators, they do not change the grade of any component of $F$. Therefore, Equation (9) shows that each grade of the field $F$ independently satisfies the homogeneous wave equation in a source free region of space.

Using the explicit form of the internal source density moved to left side of Equation (7), the generalized multivector field equation that follows from Postulates 1 through 5 with proportionality constants set for consistency with classical electrodynamics is expressed in the laboratory frame by the single multivector field equation:

$$\partial_x F + c^{-1}\partial_t F = D_{ex}. \tag{10}$$

While each component of $F$ satisfies the wave equation independently, this field equation enforces relationships among the components.

## 5 The Nature of the Fields and External Sources

We need to make an explicit connection to experimental observables by considering the various multivector grade components of $F$ and $D$. We express the field $F$ in terms of its four grade components of as $F = \chi + \bar{E} + \widehat{B} + \widetilde{T}$. Following the notation of Vold [2], the embellishments over the letters, when used, designate vectors with bars, bivectors with arcs, and trivectors with tildes.

For a consistent application of geometric algebra and geometric calculus while maintaining a clear connection to customary notation, it is necessary to distinguish formally between sources on one hand and charges and currents on the other. We define sources as multivector quantities conforming to the Postulates while charges and currents conform to customary experimental definitions. Consistency is achieved by making sources proportional to charges (or currents), with the proportionality constant accounting for both geometric properties (multivector grade) and units of measure.

We adopt the Gaussian (cgs) system [17] to provide guidance on units for the non-standard scalar and pseudoscalar components of $F$. In the Gaussian system, both electric and magnetic fields are measured in statvolts per centimeter, so we assume that the putative scalar and trivector components of $F$ may be measured in statvolts per cm, as well. Similarly, both electric charges and magnetic charges (if they were to exist) are measured in statcoulombs.

The integral field equation in Equation (6) is useful for analyzing symmetric situations and for determining fields from boundary conditions. Examination of static situations with spherical and cylindrical symmetry provides insight into the nature of both $F$ and $D_{ex}$. Equation (6) may be applied in order to 1) establish clearly the relationship between the multivector-valued fields and the electromagnetic fields of the Maxwell equations, 2) illustrate the proper handling of surface and volume integrals in multivector analysis, and 3) derive expressions in Gaussian units for multivector source densities in terms of customary charge and current densities. Because geometric algebra enables simultaneous consideration of four multivector grade components of Equation (6), it is sufficient to consider the single, symmetric case illustrated in Figure 1. An inner cylinder extending to infinity in both directions carries uniform electric and magnetic charges, $\rho_e$ and $\widetilde{\rho}_m$, respectively, moving to the right with uniform axial velocity $\bar{\nu}$. A second, concentric, cylindrical surface $\partial M$ of larger radius and infinite length defines a volume $M$ for application of Equation (6).

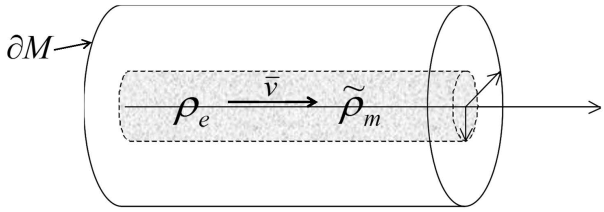


**Fig. 1:** Symmetric geometry for analyzing multivector source terms.

Considering a unit length of the cylinder in Figure 1, applying straightforward symmetry arguments for field components, and comparing each grade of the equality resulting from Equation (6) to standard expressions in Jackson [17] for the electromagnetic fields from such a line source in Gaussian units, we arrive at the following expression for the multivector external source density:

$$D_{ex} = 4\pi c^{-1}(c\rho_e - \bar{J}_e - \widehat{J}_m + c\widetilde{\rho}_m). \tag{11}$$

For the situation in Figure 1, the electric and magnetic current densities are given by $\bar{J}_e = \rho_e\bar{\nu}$ and $\widehat{J}_m = \widetilde{\rho}_m\bar{\nu}$, respectively. Interestingly, $\chi$ and $\widetilde{T}$ both vanish for this configuration.

## 6 The Nature of the Solutions

Substituting Equation (11) into Equation (10) and separating the four grades provides the following set of four field equations:

$$\begin{aligned}
\partial_x \cdot \bar{E} + c^{-1}\partial_t\chi &= 4\pi\rho_e, \\
\partial_x \wedge \chi + \bar{\nabla}\cdot\widehat{B} + c^{-1}\partial_t\bar{E} &= -\frac{4\pi}{c}\bar{J}_e \\
\partial_x \wedge \bar{E} + \partial_x\cdot\widetilde{T} + c^{-1}\partial_t\widehat{B} &= -\frac{4\pi}{c}\widehat{J}_m \\
\partial_x \wedge \widehat{B} + c^{-1}\partial_t\widetilde{T} &= 4\pi\widetilde{\rho}_m \quad .
\end{aligned} \tag{12}$$

The Maxwell equations involve neither the scalar nor the trivector fields $\chi$ and $\widetilde{T}$ allowed by Postulate 1. What we find about $\chi$ and $\widetilde{T}$ from considerations like those in the previous Section is that these fields satisfy the wave equation in vacuum and they do not arise from static sources in either spherically or cylindrically symmetric situations. If we suppose that under some circumstance $\chi$ and $\widetilde{T}$ are not produced at all, then Equations (10) and (12) reduce exactly to the Maxwell equations.

If we account for external sources in the derivation of Equation (9) and separate by grade, we find the following inhomogeneous wave equations:

$$\begin{aligned}(\partial_x^2 - c^{-2}\partial_t^2)\chi &= -\frac{4\pi}{c}(\partial_x \cdot \bar{J}_e + \partial_t \rho_e)\\ (\partial_x^2 - c^{-2}\partial_t^2)\bar{E} &= -4\pi(-\partial_x \wedge \rho_e + c^{-1}\partial_x \cdot \hat{J}_m\\ &\quad - c^{-2}\partial_t \bar{J}_e)\\ (\partial_x^2 - c^{-2}\partial_t^2)\hat{B} &= -4\pi(c^{-1}\partial_x \wedge \bar{J}_e - \partial_x \cdot \tilde{\rho}_m\\ &\quad - c^{-2}\partial_t \hat{J}_m)\\ (\partial_x^2 - c^{-2}\partial_t^2)\widetilde{T} &= -\frac{4\pi}{c}(\partial_x \wedge \hat{J}_m + \partial_t \tilde{\rho}_m) \quad .\end{aligned} \tag{13}$$

Interestingly, the driving terms on the right-hand side of the $\chi$ equation in Equation (13) vanish if we abide by the continuity equation for the external electric charge and current densities, which is well verified by experiment. Therefore, wave solutions exist for $\chi$ but they cannot be driven by ordinary matter. Conversely, the field solutions associated with $\chi$ have no effect on ordinary matter. Similarly, if magnetic charges and currents existed and satisfied a continuity equation, they could not drive the wave solutions for $\widetilde{T}$ as shown by the fourth equation in Equation (13).

In preparation for analyzing the energy and momentum flow of the multivector field, we write a general, circularly polarized, plane wave solution of Equation (10) in free space ($D_{ex} = 0$) as

$$F = (1 + \hat{k})F_0 e^{I\phi} \tag{14}$$

with the usual phase angle $\phi = \omega t - \bar{k} \cdot \bar{x}$ and wave vector $\bar{k} = |\bar{k}|\hat{k} = k\hat{k}$. The field amplitude $F_0$ is an arbitrary multivector constant. It is straightforward to show that Equation (14) is a solution in free space for an arbitrary unit vector $\hat{k}$ as long as the condition $\omega = kc$ is satisfied. There is no restriction on the grade of $F_0$. The factor $(1 + \hat{k})$ couples a wave component of one grade to a component of adjacent grade. The composition of the wave depends on $F_0$. If $F_0$ is a vector $\hat{\epsilon}$ perpendicular to $\hat{k}$, then $F$ is an electromagnetic (EM) wave with oscillating transverse electric (vector) and magnetic (bivector) components. If $F_0$ is a scalar constant, then $F$ oscillates between a scalar-vector pair and a bivector-trivector pair, a decidedly non-electromagnetic (non-EM) wave. The following sections analyze the energy and momentum of such wave solutions.

## 7 Continuity Equation for Energy and Momentum

It is well known in classical electrodynamics (see, *e.g.*, Jackson [17]) that the energy and momentum of electromagnetic fields are quadratic in the field values. Jackson provides a traditional, somewhat involved derivation of this result using the known laws of mechanics and the transfer of energy and momentum from electromagnetic fields to external charge densities. Much more elegant derivations are available in the Lagrangian formulation of field theory via Noether's Theorem, either in traditional field theory or in Lagrangian formulations with geometric algebra [4,18]. It is perhaps less well known that, with the aid of geometric algebra, the continuity and conservation laws for the energy and momentum of the electromagnetic field may be derived directly from the field equations even in the absence of external sources. The derivation is analogous to the way that continuity and conservation laws for probability are derived from the Schrödinger or Dirac equations in quantum mechanics. Here, we follow the method shown by Vold [2] to derive energy and momentum relationships for the multivector field $F$. These relationships are valid whether or not $F$ includes the non-electromagnetic scalar and pseudoscalar components $\chi$ and $\widetilde{T}$.

To begin, we multiply Equation (10) from the left by the reverse $F^\dagger$ of $F$. The reverse [4,5] in geometric algebra plays the role of the Hermitian adjoint in matrix theory and in quantum mechanics. The result is:

$$c^{-1}F^\dagger \partial_t F + F^\dagger \partial_x F = F^\dagger D_{ex}. \tag{15}$$

Next, we take the reverse of Equation (10), noting that $(AB)^\dagger = B^\dagger A^\dagger$ and that scalar and vector quantities, including the operators $\partial_t$ and $\partial_x$, reverse to themselves:

$$c^{-1}\dot{F}^\dagger \dot{\partial}_t + \dot{F}^\dagger \dot{\partial}_x = D_{ex}^\dagger. \tag{16}$$

Multiplying Equation (16) from the right by $F$, we have:

$$c^{-1}\dot{F}^\dagger \dot{\partial}_t F + \dot{F}^\dagger \dot{\partial}_x F = D_{ex}^\dagger F. \tag{17}$$

Adding equations (15) and (17), then applying the rules for the differentiation of products, we obtain:

$$c^{-1}\partial_t(F^\dagger F) + \dot{F}^\dagger \dot{\partial}_x \dot{F} = F^\dagger D_{ex} + D_{ex}^\dagger F. \tag{18}$$

Note that as a scalar operator, the time derivative $\partial_t$ commutes with all geometric quantities and may be brought to the left in the first term of Equation (18). On the other hand, the vector operator $\partial_x$ does not necessarily commute with all components of $F^\dagger$, so it must remain between the factors of $F^\dagger$ and $F$ in the second term. The overdots in this term indicate that the geometric derivative acts both to the left and to the right.

Equation (18) has the apparent structure of a continuity equation for a conserved bilinear quantity $F^\dagger F$. The first term on the left is the rate of change of $F^\dagger F$ at a given point, the second term on the left should be a divergence of the quantity from that point, and the right-hand side of the equation should be the production rate at the point. Such an interpretation is easy to demonstrate when the conserved quantity is a scalar such as energy or probability density. However, in 3-D geometric algebra, the bilinear quantity $F^\dagger F$ may have both scalar and vector components. Given that $F^\dagger F$ reverses to itself, it cannot have either bivector or trivector components [4,5]. Inspection of Equation (18) shows that both the second term on the left and the combined terms on the right hand side have the same property. We shall see how the scalar and vector components represent energy and momentum density, respectively.

First, consider the scalar part of Equation (18) in a source-free region of space:

$$\langle c^{-1}\partial_t(F^\dagger F)\rangle + \langle \dot{F}^\dagger \dot{\partial}_x \dot{F}\rangle = 0, \tag{19}$$

where the angled braces with no subscript indicate the scalar part of the enclosed expression. The first term is $\langle c^{-1}\partial_t(F^\dagger F)\rangle = c^{-1}\partial_t\langle F^\dagger F\rangle$ and the second is $\langle \dot{F}^\dagger \dot{\partial}_x \dot{F}\rangle = \langle \partial_x(FF^\dagger)\rangle = \partial_x \cdot \langle FF^\dagger\rangle_1$, where the subscript 1 on the angled braces indicates the vector (grade 1) component of the enclosed expression. If we use these expressions in Equation (19) and multiply by $c/8\pi$, we obtain the continuity equation

$$\partial_t \rho_\varepsilon + \partial_x \cdot \bar{J}_\varepsilon = 0 \tag{20}$$

for the energy density $\rho_\varepsilon \equiv \langle F^\dagger F/8\pi\rangle$ having flux density $\bar{J}_\varepsilon = c\langle FF^\dagger/8\pi\rangle_1$ in Gaussian units. It is noteworthy that the flow of energy as needed in Equation (20) is in the direction of the vector part of $FF^\dagger$ rather than the vector part of $F^\dagger F$.

By direct calculation with $F = \chi + \bar{E} + \widehat{B} + \widetilde{T}$, we find

$$F^\dagger F = (\chi^2 + \bar{E}^2 - \widehat{B}^2 - \widetilde{T}^2) + 2(\chi\bar{E} + \bar{E}\cdot\widehat{B} - \widehat{B}\widetilde{T}). \tag{21}$$

Given the bivector grade of $\widehat{B}$, we have $\widehat{B}^2 = -B^2$, where $B = |\widehat{B}|$, and from the first set of terms on the right of this equation we obtain the usual expression for the energy density of electric and magnetic fields when the scalar and trivector fields vanish. We expect then that the second set of terms, the vector part of Equation (21), will yield the momentum density of the field. The magnetic field vector $\bar{B}$ in the vector algebra of traditional physics is the dual of the bivector field, *i.e.*, $\widehat{B} = I\bar{B}$. The inner product in the vector part of Equation (21) becomes $\bar{E}\cdot\widehat{B} = -\bar{E}\times\bar{B}$ so we find that the vector part of $F^\dagger F$ is proportional to the negative of the field momentum [2]. In summary, we have the energy density $\rho_\varepsilon$, the momentum density $\bar{J}_{\bar{p}}$, and the energy flux density $\bar{J}_\varepsilon$ given by the following expressions:

$$\begin{aligned} \rho_\varepsilon &= \langle F^\dagger F/8\pi\rangle = \frac{\chi^2 + E^2 + B^2 + T^2}{8\pi} \\ \bar{J}_{\bar{p}} &= -\langle F^\dagger F/8\pi\rangle_1 = \frac{-\chi\bar{E} - \bar{E}\cdot\widehat{B} + \widehat{B}\widetilde{T}}{4\pi} \\ \bar{J}_\varepsilon &= c\langle FF^\dagger/8\pi\rangle_1 = c\frac{\chi\bar{E} - \bar{E}\cdot\widehat{B} - \widehat{B}\widetilde{T}}{4\pi} \end{aligned} \quad . \tag{22}$$

Comparing the second and third equations of (22), we see the well known relationship between energy flux (the Poynting vector) and momentum density for the electromagnetic term $\bar{E}\cdot\widehat{B}$ with momentum pointed in the direction of energy flow as expected [17]. However, a combination of scalar and electric (or of magnetic and pseudoscalar) fields, were they to exist, would have momentum directed oppositely to the flow of energy. In other words such propagating fields would have *retrograde* momentum.

To obtain a continuity equation for energy and momentum density in Gaussian units, we multiply Equation (18) by $c/8\pi$ and rearrange as follows:

$$\partial_t \frac{F^\dagger F}{8\pi} = -\frac{c}{8\pi}\dot{F}^\dagger \dot{\partial}_x \dot{F} + c\frac{F^\dagger D_{ex} + D_{ex}^\dagger F}{8\pi}. \tag{23}$$

We recognize the left-hand side as the time derivative of the energy and momentum density of the electromagnetic field when $F$ consists of only vector and bivector components [2]. The first term on the right relates to the flow of energy and momentum. The second term on the right gives the production of energy and momentum by the external source as may be verified by expanding the term with the aid of Equation (11).

## 8 Conservation Equation for Energy and Momentum

To confirm our understanding of the flow of energy and momentum represented by the geometric derivative term of the continuity equation, we integrate both sides of Equation (23) over an arbitrary volume $M$ enclosed by a boundary $\partial M$:

$$\partial_t \int_M dV \frac{F^\dagger F}{8\pi} = -\frac{c}{8\pi}\int_M dV \dot{F}^\dagger \dot{\partial}_x \dot{F} + c\int_M dV \frac{F^\dagger D_{ex} + D_{ex}^\dagger F}{8\pi}. \tag{24}$$

Next, we apply the fundamental theorem of calculus from Equation (4) to the geometric derivative term of Equation (24), using $G = F^\dagger$ to obtain the following conservation equation:

$$\partial_t \int_M dV \frac{F^\dagger F}{8\pi} = -\int_{\partial M} dS n \frac{cF^\dagger nF}{8\pi} + c\int_M dV \frac{F^\dagger D_{ex} + D_{ex}^\dagger F}{8\pi}. \quad (25)$$

The left-hand side of Equation (25) is the time derivative of the total energy and momentum in $M$. The first integral on the right gives the loss rate of energy and momentum through the boundary $\partial M$, where, as before, $n$ is the outward pointing normal at the surface element $dS$. The second integral on the right is the production rate of energy and momentum within the volume.

We may verify our earlier interpretation of the energy flux density by examining the trivector component of Equation (25). The volume element $dV$ is a pure trivector, so the trivector part of the volume integrals is due to the scalar part of the integrands. Likewise, given that $n$ is normal to $dS$, the combination $dSn$ is a pure trivector, so we need the scalar part of the remaining factors in the surface integral. Then the trivector component of Equation (25) is the energy conservation law for the multivector field:

$$\partial_t \int_M dV \langle \frac{F^\dagger F}{8\pi} \rangle = -\int_{\partial M} dS n \langle \frac{cF^\dagger nF}{8\pi} \rangle + c\int_M dV \frac{\langle F^\dagger D_{ex} + D_{ex}^\dagger F\rangle}{8\pi}. \quad (26)$$

To understand the flow of energy, we analyze the integrand of the surface integral in Equation (26). Apparently, the vector integrand

$$\bar{J}_\varepsilon(n) = n\langle \frac{cF^\dagger nF}{8\pi} \rangle \quad (27)$$

is the flux of energy normal to the surface element $dS$. However, the flow direction at the spatial position of $dS$ is not necessarily normal to $dS$. To find the direction of flow of field energy, we need to orient $dS$ at the given spatial position so that its normal $n$ matches the direction of flow and therefore maximizes the flow through $dS$. It is equivalent to find the unit vector $n = n_0$ that maximizes the scalar $\langle F^\dagger nF\rangle$. Geometric calculus provides a convenient means for this optimization using the geometric derivative with respect to an arbitrary vector $\tau$ in the tangent space at the spatial position of $dS$. We may express the normal to $dS$ as $n = |\tau|^{-1}\tau$. Then we vary $n$ by varying $\tau$. First, we write the scalar to be maximized in terms of $\tau$:

$$\langle F^\dagger nF\rangle = |\tau|^{-1}\langle F^\dagger \tau F\rangle = |\tau|^{-1}\langle \tau FF^\dagger\rangle. \quad (28)$$

Then we take the geometric derivative of Equation (28) with respect to $\tau$ (see Hestenes and Sobczyk [5]):

$$\begin{aligned}\partial_\tau \langle F^\dagger nF\rangle &= (\partial_\tau |\tau|^{-1})\langle \tau FF^\dagger\rangle + |\tau|^{-1}\partial_\tau\langle \tau FF^\dagger\rangle \\ &= -|\tau|^{-3}\tau\langle \tau FF^\dagger\rangle + |\tau|^{-1}\langle FF^\dagger\rangle_1.\end{aligned}$$

The extremum occurs for $\tau = \tau_0$ when this derivative vanishes, which yields the condition:

$$|\tau_0|^{-3}\tau_0\langle \tau_0 FF^\dagger\rangle = |\tau_0|^{-1}\langle FF^\dagger\rangle_1.$$

This condition, given $n_0 = |\tau_0|^{-1}\tau_0$, reduces to

$$n_0\langle F^\dagger n_0 F\rangle = \langle FF^\dagger\rangle_1.$$

Using this result in Equation (27) gives the maximum energy flow

$$\bar{J}_\varepsilon(n_0) = n_0\langle \frac{cF^\dagger n_0 F}{8\pi} \rangle = c\langle \frac{FF^\dagger}{8\pi} \rangle_1$$

in agreement with the energy flux density in Equation (22) derived from the continuity equation.

## 9 Comparing Energy Flux and Momentum Density of Wave Solutions

Having confirmed the expression for the energy flux given in Equation (22) through analysis of the conservation equation in the previous section, we may confidently compare the energy flux and momentum density given by Equation (22) for the plane wave solution of Equation (14). For simplicity, we compare the direction of the energy flux given by $\langle FF^\dagger\rangle_1$ with the direction of the momentum density given by $-\langle F^\dagger F\rangle_1$ for the EM and non-EM solutions described at the end of Section 6. The reverse of the plane wave solution of Equation (14) is

$$F^\dagger = e^{-I\phi}F_0^\dagger(1+\hat{k})$$

from which we obtain:

$$\begin{aligned}F^\dagger F &= 2F_0^\dagger(1+\hat{k})F_0 \\ FF^\dagger &= (1+\hat{k})F_0F_0^\dagger(1+\hat{k}).\end{aligned}$$

**Table 1:** Non-EM plane waves have retrograde momentum.

| *Plane wave* | $F_0$ | *Energy flux direction* $\langle FF^\dagger\rangle_1$ | *Momentum density direction* $-\langle F^\dagger F\rangle_1$ |
|---|---|---|---|
| EM | $\hat{\epsilon}(\perp \hat{k})$ | $2\hat{k}$ | $2\hat{k}$ |
| Non-EM | 1 | $2\hat{k}$ | $-2\hat{k}$ |

Table 1 compares the directions of the energy flux and the momentum density given by the vector parts of these expressions for the EM and non-EM solutions. For both solutions, the energy flux is in the direction of $\hat{k}$. This result matches intuition given that the contribution to the energy density as seen in Equation (22) is positive for all four grade components of the field and that the wave peaks for any solution move in the $\hat{k}$ direction by virtue of the phase angle in Equation (14). Furthermore, the momentum density of the EM wave is also in the $\hat{k}$ direction, as it must be for consistency with classical electrodynamics.

However, the direction of the momentum density for the non-EM wave is distinctly non-intuitive, being directed oppositely, or retrograde, to the direction of energy flux.

## 10 Discussion

Expansion and inspection of the terms in Equation (25) show that they correspond to the usual results of classical electrodynamics [17] for the flow of energy and momentum when $F$ is an EM wave having only vector and bivector components. In particular, the flow of energy is in the direction $\hat{k}$ of the wave vector and the vector momentum density points in the same direction. However, Equation (25) is valid for any multivector field $F$ satisfying the wave equations of Equation (13). It is quite remarkable that non-EM wave solutions consisting of scalar and (longitudinal) vector components have energy flow also in the direction of $\hat{k}$ but that their momentum density points in the opposite direction. We say that these waves carry retrograde momentum, *i.e.*, momentum directed oppositely to the energy flow. Such waves would generate negative pressure. In rebounding from a surface, they would pull rather than push on the surface. Wave solutions consisting of pseudoscalar and bivector components have the same property.

It is important to examine further the question of the coexistence of EM and non-EM solutions of Equations (10) and (12) and the issue of consistency with experiment. It can be argued that the first of the equations in (12), relating the divergence of the electric field to the electric charge density, has been verified by experiment in the absence of the time derivative of the scalar field $\chi$ and, therefore, the time derivative of $\chi$ cannot exist above an upper limit set by these experiments.

The counter to this argument involves two points. First, the equation involving the divergence of the electric field is part of the larger set of relations of Equation (12). The other relations in Equation (12) require that any time-varying scalar field propagates at the speed of light with an accompanying longitudinal electric field, just as a time-varying transverse electric field propagates with an accompanying transverse magnetic field. As an illustration, consider a linearly polarized plane wave solution $F = (1 + \hat{k})F_0 \cos\phi$ analogous to the circularly polarized wave of Equation (14). With a scalar amplitude $F_0$, this solution is such a propagating wave (which may be called a *scalarelectric* wave). The divergence of its longitudinal electric field (non-EM) exactly balances the time derivative of its scalar field, so the first equation of Equation (12) is satisfied by this wave without the external charge density term on the right side. On the other hand, classical EM experiments show that either the divergence of the electric field arising from an external charge density or the divergence of the electric field of a freely propagating EM wave balance the first equation of (12) without need for the time-varying scalar field. Because the equation is linear, the solutions for these EM and non-EM situations may be superimposed without affecting each other.

The second point is related to the discussion following Equation (13). Electromagnetic experiments are performed with electric charge densities and currents that satisfy the continuity equation and, hence, do not interact electromagnetically with the scalar field. These experiments cannot detect either the scalar field or the longitudinal electric field associated with it. In this way, the non-EM waves may exist without contradicting experiment. The initial argument stated above for the vanishing of the time derivative of $\chi$ stems from the implicit assumption that all components of the electric field are measurable in classical EM experiments. This assumption is understandable given the traditional definition of the electric field in terms of the force on an electrically charged particle. However, the definition of the electric field via the five postulates in this paper admits a more general vector component $\langle F\rangle_1$ of the multivector field having a longitudinal, non-EM component that is not generated or measurable in classical EM experiments.

It is tempting to dismiss the non-EM, retrograde solutions described above as non-physical. However, the currently accepted cosmological theory [19] that the preponderance of the energy density in the universe is dark energy with negative pressure brings some interest to the topic of retrograde momentum. Cosmological theory frequently makes use of a broad-brush characterization of a uniform region of matter and energy in terms of the equation of state $p = we$ relating pressure $p$ to energy

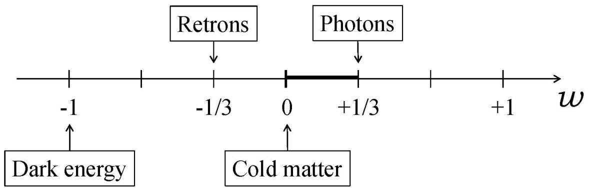


**Fig. 2:** Matter and energy may be characterized by an equation of state parameter $w$.

density $e$ via a proportionality constant $w$ known as the equation of state parameter. Figure 2 illustrates values of $w$ for various states of matter and energy. Ordinary matter (dark bar in Figure 2) has values ranging from 0 for cold matter to +1/3 for highly relativistic particles and photons. Dark energy has a value of about -1, giving a negative pressure for a positive energy density. The non-EM solutions described in this paper, if quantized, would be analogous to photons but with the opposite sign for the equation of state parameter, *i.e.*, $w = -1/3$. Such quantized states with retrograde momentum are indicated as *retrons* in Figure 2.

We cannot suggest that the non-EM multivector field solutions described here are candidates for dark energy given the apparent difference in values of the equation of state parameter. However, the fact that the non-EM solutions propagate at the speed of light, seem not to interact with ordinary matter, and exhibit retrograde momentum give them enough similarity to dark energy to warrant further study of their properties. Transforming the results of this paper to covariant form in spacetime algebra will facilitate such study.

## 11 Conclusion

Geometric algebra and geometric calculus enable for the first time an efficient formulation of classical electrodynamics from a set of familiar and physically meaningful postulates. It is certainly intriguing that the standard assumptions of classical electrodynamics represented by the five postulates lead naturally without further elaboration to a field equation admitting both standard EM solutions with the usual energy-momentum relationship and non-EM solutions with retrograde momentum. A further point of interest is that the non-EM solutions coexist with the classical EM solutions with no apparent conflict with experiment. These results are due to the elegance and power of geometric algebra and geometric calculus.

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