Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.12785/sjm/020102

$\mathscr{I}P$-Separation Axioms in Ideal Bitopological Ordered Spaces I

A. Kandil[1], *O. Tantawy*[2], *S. A. El-Sheikh*[3] *and M. Hosny*[3,*]

[1] Department of Mathematics, Faculty of Science, Helwan University, Egypt
[2] Department of Mathematics, Faculty of Science, Zagazig University, Cairo, Egypt
[3] Department of Mathematics, Faculty of Education, Ain Shams University, 11757 beside Tabary school, Roxy, Cairo, Egypt.

Received: 7 Jun. 2014, Revised: 21 Sep. 2014, Accepted: 23 Sep. 2014
Published online: 1 Jan. 2015

Abstract: The aim of this paper is to use the concept of ideal $\mathscr{I}$ to study ideal bitopological ordered spaces $(X,\tau_1,\tau_2,R,\mathscr{I})$. Clearly, if $\mathscr{I}=\{\phi\}$, then every ideal bitopological ordered spaces are bitopological ordered spaces. In addition, if $\mathscr{I}=\{\phi\}$ and R is the equality relation "Δ", then every ideal bitopological ordered spaces are bitopological spaces. Therefore, these spaces are generalization of the bitopological ordered spaces and bitopological spaces. In this and a subsequent paper, we use the notion of $\mathscr{I}$-increasing (decreasing) sets which based on the ideal $\mathscr{I}$, to introduce separation axioms in ideal bitopological ordered spaces. Whereas this paper is devoted to the axioms $\mathscr{I}PT_i,(i=0,1,2)$ in part II the axioms $\mathscr{I}PT_i,(i=3,4,5)$ and $\mathscr{I}PR_j$-ordered spaces, $j=0,1,2,3,4$ are introduced and studied. Some important results related these separations have obtained and the relationship between these axioms and the previous one has been given.

Keywords: Bitopological ordered spaces, separation axioms, ideals.
Mathematics Subject Classification (2010): 54F05, 54E55, 54D10.

1 Introduction

Topological spaces have been generalized by many ways. A topological ordered space (X,τ,R) is a set X endowed with both a topology τ and a partially order relation R on X. The study of order relations in topological spaces was initiated by Nachbin [10] in 1965.

In 1963 Kelly [7] was introduced a bitopological space (X,τ_1,τ_2) as a richer structure than topological space. In 1971 Singal and Singal [12] were studied the bitopological ordered space (X,τ_1,τ_2,R) which is a generalization of the study of general topological space, bitopological space and topological ordered space. Every bitopological ordered space (X,τ_1,τ_2,R) can be regarded as a bitopological space (X,τ_1,τ_2) if R is the equality relation "Δ" and every bitopological space (X,τ_1,τ_2) can be regarded as a topological space (X,τ) if $\tau_1=\tau_2=\tau$. Also, every bitopological ordered space (X,τ_1,τ_2,R) can be regarded as a topological ordered space (X,τ,R) if $\tau_1=\tau_2=\tau$.

The object of the present paper is to use the ideal properties to study ideal bitopological ordered space $(X,\tau_1,\tau_2,R,,\mathscr{I})$ which is a generalization of the study of bitopological ordered spaces (X,τ_1,τ_2,R) and bitopological space (X,τ_1,τ_2). Every ideal bitopological ordered space $(X,\tau_1,\tau_2,R,\mathscr{I})$ can be regarded as a bitopological ordered space (X,τ_1,τ_2,R) if $\mathscr{I}=\{\phi\}$ and can be regarded as bitopological space (X,τ_1,τ_2) if $\mathscr{I}=\{\phi\},R$ is the equality relation "Δ". Separation axioms $\mathscr{I}PT_i, i=0,1,2$ are studied on the space $(X,\tau_1,\tau_2,R,\mathscr{I})$ which based on the notion of $\mathscr{I}$-increasing (decreasing) sets [2]. In addition, the relationship between these axioms and the axioms in [6, 12] have been obtained. Moreover, we show that the properties of being $\mathscr{I}PT_i$-ordered spaces, $i=0,1,2$ are preserved under a bijective, P-open and order (reverse) embedding mappings (see Theorems 3.2, 3.6). Furthermore, it is proved that the property of being $\mathscr{I}PT_i$-ordered spaces, $i=0,1,2$ is hereditary property (see Theorems 3.4, 3.7).

* Corresponding author e-mail: moona_hosny@yahoo.com

2 Preliminaries

In this section, we collect the relevant definitions and results from bitopological ordered spaces, lower separation axioms and mappings.

Definition 2.1.[10] Let (X,R) be a poset. A set $A \subseteq X$ is said to be

1. Increasing if for every $a \in A$ and $x \in X$ such that aRx, then $x \in A$.
2. Decreasing if for every $a \in A$ and $x \in X$ such that xRa, then $x \in A$.

Definition 2.2. A mapping $f:(X,R) \to (Y,R^*)$ is called

1. Increasing (decreasing) if $\forall x_1,x_2 \in X$ such that $x_1Rx_2 \Rightarrow f(x_1)R^*f(x_2)(f(x_2)R^*f(x_1))$ [10].
2. Order embedding if $\forall x_1,x_2 \in X, x_1Rx_2 \Leftrightarrow f(x_1)R^*f(x_2)$ [13].
3. Order reverse embedding if $\forall x_1,x_2 \in X, x_1Rx_2 \Leftrightarrow f(x_2)R^*f(x_1)$ [1].

Definition 2.3. [4] Let X be a non-empty set. A class τ of subsets of X is called a topology on X iff τ satisfies the following axioms.

1. $X, \phi \in \tau$.
2. An arbitrary union of the members of τ is in τ.
3. The intersection of any two sets in τ is in τ.

The members of τ are then called τ-open sets, or simply open sets. The pair (X,τ) is called a topological space. A subset A of a topological space (X,τ) is called a closed set if its complement A' is an open set.

Definition 2.4.[7] A bitopological space (bts, for short) is a triple (X,τ_1,τ_2), where τ_1 and τ_2 are arbitrary topologies for a set X.

Definition 2.5.[8,11] A function $f:(X_1,\tau_1,\tau_2) \to (X_2,\eta_1,\eta_2)$ is called

1. p.continuous (respectively p.open, p.closed) if $f:(X_1,\tau_i) \to (X_2,\eta_i), i=1,2$ are continuous (respectively open, closed).
2. p.homeomorphism if $f:(X_1,\tau_i) \to (X_2,\eta_i), i=1,2$ are homeomorphism.

Definition 2.6.[12] A bitopological ordered space (bto-space, for short) has the form (X,τ_1,τ_2,R), where (X,R) is a poset and (X,τ_1,τ_2) is a bts.

Definition 2.7.[12]A bto-space (X,τ_1,τ_2,R) is said to be

1. Lower pairwise $T_1(LPT_1$, for short)-ordered space if for every $a,b \in X$ such that $a\overline{R}b$, there exists an increasing τ_i-nbd U of a such that $b \notin U, i=1$ or 2.
2. Upper pairwise $T_1(UPT_1$, for short)-ordered space if for every $a,b \in X$ such that $a\overline{R}b$, there exists a decreasing τ_i-nbd V of b such that $a \notin V, i=1$ or 2.

Definition 2.8.[12] A bto-space (X,τ_1,τ_2,R) is said to be PT_0-ordered space if it is LPT_1 or UPT_1 ordered space.

Definition 2.9.[12] A bto-space (X,τ_1,τ_2,R) is said to be pairwise $T_1(PT_1$, for short), if it is LPT_1 and UPT_1-ordered space.

Definition 2.10.[12] A bto-space (X,τ_1,τ_2,R) is said to be pairwise $T_2(PT_2$, for short), if for every $a,b \in X$ such that $a\overline{R}b$, there exist an increasing τ_i-nbd U of a and a decreasing τ_j-nbd V of b such that $U \cap V = \phi$.

Definition 2.11.[5] A non-empty collection $\mathscr{I}$ of subsets of a set X is called an ideal on X, if it satisfies the following conditions

1. $A \in \mathscr{I}$ and $B \in \mathscr{I} \Rightarrow A \cup B \in \mathscr{I}$.
2. $A \in \mathscr{I}$ and $B \subseteq A \Rightarrow B \in \mathscr{I}$.

Definition 2.12.[2] Let (X,R) be a poset and $\mathscr{I} \subseteq P(X)$ be an ideal on X. Then, a set $A \subseteq X$ is called:

1. $\mathscr{I}$-increasing set iff $aR \cap A' \in \mathscr{I}\ \forall a \in A$, where $aR = \{b:(a,b) \in R\}$.
2. $\mathscr{I}$-decreasing set iff $Ra \cap A' \in \mathscr{I}\ \forall a \in A$, where $Ra = \{b:(b,a) \in R\}$.

Theorem 2.1.[2] Let $f:(X,R,\mathscr{I}) \to (Y,R^*,f(\mathscr{I}))$ be a bijective function and order embedding. Then for every $\mathscr{I}$-increasing (decreasing) subset A of X, $f(A)$ is $f(\mathscr{I})$-increasing (decreasing) subset of Y.

Corollary 2.1.[2] Let $f:(X,R,\mathscr{I}) \to (Y,R^*,f(\mathscr{I}))$ be a bijective function and order embedding. If $B \subseteq Y$ is $f(\mathscr{I})$-increasing (decreasing), then $f^{-1}(B)$ is $\mathscr{I}$-increasing (decreasing) subset of X.

Theorem 2.2.[2] Let $f:(X,R,\mathscr{I}) \to (Y,R^*,f(\mathscr{I}))$ be a bijective function and reverse embedding. Then for every $\mathscr{I}$-increasing (decreasing) subset A of X, $f(A)$ is $f(\mathscr{I})$-decreasing (increasing) subset of Y.

Corollary 2.2.[2] Let $f:(X,R,\mathscr{I}) \to (Y,R^*,f(\mathscr{I}))$ be a bijective function and order reverse embedding. If $B \subseteq Y$ is $f(\mathscr{I})$-increasing (decreasing), then $f^{-1}(B)$ is $\mathscr{I}$-decreasing (increasing) subset of X.

3 $\mathscr{I}P$-Separation axioms

The aim of this section is to introduce new separation axioms $\mathscr{I}PT_i$-ordered spaces, $i=0,1,2$ on the space $(X,\tau_1,\tau_2,R,\mathscr{I})$ which based on the notion of $\mathscr{I}$-increasing (decreasing) sets [2]. In addition, the relationship between these axioms and the axioms in [12] are obtained. Moreover, it is proved that the property of being $\mathscr{I}PT_i$-ordered spaces, $i=0,1,2$ is invariant under a bijective, P-open and order embedding mapping (order reverse embedding mapping). Furthermore, it is proved that the property of being $\mathscr{I}PT_i$-ordered spaces, $i=0,1,2$ is hereditary property.

Definition 3.1. A space $(X,\tau_1,\tau_2,R,\mathscr{I})$ is called an ideal bitopological ordered space if (X,τ_1,τ_2,R) is a bitopological ordered space and $\mathscr{I} \subseteq P(X)$ is an ideal on X.

Remark 3.1. Every ideal bitopological ordered space

$(X,\tau_1,\tau_2,R,\mathscr{I})$ can be regarded as a bitopological ordered space (X,τ_1,τ_2,R) if $\mathscr{I}=\{\phi\}$ and can be regarded as bitopological space (X,τ_1,τ_2) if $\mathscr{I}=\{\phi\},R$ is the equality relation ”Δ”.

Definition 3.2. An ideal bitopological ordered space $(X,\tau_1,\tau_2,R,\mathscr{I})$ is said to be

1. $\mathscr{I}$ lower PT_1($\mathscr{I}LPT_1$, for short)-ordered space if for every $a,b\in X$ such that $a\overline{R}b$, there exists an $\mathscr{I}$-increasing τ_i-open set U such that $a\in U$ and $b\notin U, i=1$ or 2.
2. $\mathscr{I}$ upper PT_1($\mathscr{I}UPT_1$, for short)-ordered space if for every $a,b\in X$ such that $a\overline{R}b$, there exists an $\mathscr{I}$-decreasing τ_i-open set V such that $b\in V$ and $a\notin V, i=1$ or 2.

Definition 3.3. $(X,\tau_1,\tau_2,R,\mathscr{I})$ is said to be $\mathscr{I}PT_0$-ordered space if it is $\mathscr{I}LPT_1$ or $\mathscr{I}UPT_1$ ordered space.

Example 3.1. Let $X=\{1,2,3,4\}, R=\Delta\cup\{(1,4),(1,3),(2,3),(4,3)\}, \mathscr{I}=\{\phi,\{1\},\{3\},\{1,3\}\}, \tau_1=\{X,\phi,\{1\},\{1,2\},\{1,4\},\{1,2,4\}\}, \tau_2=\{X,\phi,\{2,3\},\{1,2,3\},\{2,3,4\}\}$. Then, $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}UPT_0$-ordered space and consequently it is $\mathscr{I}PT_0$-ordered space.

Example 3.2. In Example 3, let $\mathscr{I}=\{\phi,\{3\},\{4\},\{3,4\}\}, \tau_1=\{X,\phi,\{3\},\{1,2\},\{1,2,3\},\{1,2,4\}\}, \tau_2=\{X,\phi,\{3,4\},\{1,3,4\},\{2,3,4\}\}$. Then, $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}LPT_0$-ordered space and consequently it is $\mathscr{I}PT_0$-ordered space.

The following proposition gives the relationship between Definition 3.3 and Definition 2.8 [12].

Proposition 3.1. Let $(X,\tau_1,\tau_2,R,\mathscr{I})$ be an ideal bitopological ordered space. Then,
PT_0-ordered spaces $\Rightarrow \mathscr{I}PT_0$-ordered spaces.

Proof. The proof follows directly from the definitions of PT_0-ordered spaces and $\mathscr{I}PT_0$-ordered spaces.

Example 3.1 shows that $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_0$-ordered space, but not PT_0-ordered space since, it is not UT_1-ordered space and not LT_1-ordered space (as, $1\overline{R}2$, all decreasing τ_i-open sets which contain 2 also containing 1. Also, $3\overline{R}2$, all increasing τ_i-open sets which contain 3 also containing 2).

Definition 3.4. An ideal bitopological ordered space $(X,\tau_1,\tau_2,R,\mathscr{I})$ is said to be $\mathscr{I}PT_1$-ordered space iff it is $\mathscr{I}LPT_1$ and $\mathscr{I}UPT_1$ ordered space.

Example 3.3. Let $\tau_1=\{X,\phi,\{3\},\{2,3\},\{3,4\},\{2,3,4\}\}, \tau_2=\{X,\phi,\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$ in Example 3.1. Then, $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_1$-ordered space.

The following theorem studies the relationship between the current Definitions 3.3, 3.4 and the previous Definition 2.9.

Theorem 3.1. Let $(X,\tau_1,\tau_2,R,\mathscr{I})$ be an ideal bitopological ordered space. Then,
PT_1-ordered spaces $\Rightarrow \mathscr{I}PT_1$-ordered spaces $\Rightarrow \mathscr{I}PT_0$-ordered space.

Proof. The proof follows directly from the definitions of PT_1-ordered spaces, $\mathscr{I}PT_0$-ordered space and $\mathscr{I}PT_1$-ordered spaces.

Example 3.3 shows that $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_1$-ordered space, but not PT_1-ordered space, since it is not UT_1-ordered space (as $1\overline{R}2$ and all decreasing τ_i-open sets which contain 2 also containing 1).

Example 3.1 (Example 3.2) shows that $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_0$-ordered space, but not $\mathscr{I}PT_1$-ordered space.

The following theorem shows that the property of being $\mathscr{I}PT_i$-ordered space, $i=0,1$ is preserved by a bijective, P-open and order (reverse) embedding mapping.

Theorem 3.2. If $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_i$-ordered space, $f:(X,\tau_1,\tau_2,R,\mathscr{I})\to(Y,\eta_1,\eta_2,R^*,f(\mathscr{I}))$ is a bijective, P-open and order embedding mapping (order reverse embedding mapping). Then, $(Y,\eta_1,\eta_2,R^*,f(\mathscr{I}))$ is $f(\mathscr{I})PT_i$-ordered space, $i=0,1$.

Proof.
We prove the theorem in case $i=0$, order embedding mapping and the other case is similar. Let $(X,\tau_1,\tau_2,R,\mathscr{I})$ and $(Y,\eta_1,\eta_2,R^*,f(\mathscr{I}))$ be two ideal bitopological ordered spaces such that $(X,\tau_1,\tau_2,R,\mathscr{I})$ is a $\mathscr{I}PT_0$-ordered space and $f:(X,\tau_1,\tau_2,R,\mathscr{I})\to(Y,\eta_1,\eta_2,R^*,f(\mathscr{I}))$ be a bijective, P-open and order embedding mapping. Let $y_1,y_2\in Y$ such that $y_1\overline{R^*}y_2$. Then, there exist $x_1,x_2\in X$ such that $f(x_1)=y_1, f(x_2)=y_2$ and $x_1\overline{R}x_2$. As $(X,\tau_1,\tau_2,R,\mathscr{I})$ is a $\mathscr{I}PT_0$-ordered space, then there exists an $\mathscr{I}$-increasing τ_i-open set O_{x_1} such that $x_2\notin O_{x_1}$ or an $\mathscr{I}$-decreasing τ_i-open set O_{x_2} such that $x_1\notin O_{x_2}$. Since f is P-open and by Theorem 2.1, $f(O_{x_1})$ is $f(\mathscr{I})$-increasing η_i-open set contains $y_1,y_2\notin f(O_{x_1})$ or $f(O_{x_2})$ is $f(\mathscr{I})$-decreasing η_i-open set contains $y_2,y_1\notin f(O_{x_2})$. Hence, $(Y,\eta_1,\eta_2,R^*,f(\mathscr{I}))$ is a $f(\mathscr{I})PT_0$-ordered space.

Let $Y\subseteq X$ and R be a relation on X. Then, $R_Y:=R\cap(Y\times Y)$ is a relation on Y and is called the relation induced by R on Y. If a relation has any properties of reflexivity, transitivity, symmetry and anti-symmetry, then the properties are inherited by induced relations [9].

If $(X,\tau,\mathscr{I})$ is an ideal topological space and A is a subset of X, then $(A,\tau_A,\mathscr{I}_A)$, where τ_A is the relative topology on A and $\mathscr{I}_A = \{A \cap J : J \in \mathscr{I}\}$, is an ideal topological subspace [3].

Theorem 3.3. Let $(X,R,\mathscr{I})$ be an ideal ordered space. If $A \subseteq X, (A,R_A,\mathscr{I}_A)$ is an ideal ordered subspace of $(X,R,\mathscr{I})$ and B is an $\mathscr{I}$-increasing (decreasing) set, then $B \cap A$ is an $\mathscr{I}_A$-increasing (decreasing) set.

Proof.
The proof for both parts are similar. So, we only present the proof for the part not in the parentheses. We want to prove $B \cap A$ is an $\mathscr{I}_A$-increasing set (i.e if the complement of $B \cap A$ with respect to A is $A \setminus (B \cap A)$, then $xR_A \cap [A \setminus (B \cap A)] \in \mathscr{I}_A \ \forall x \in B \cap A$). So,

$$\begin{aligned} xR_A \cap [A \setminus (B \cap A)] &= xR_A \cap [(X \setminus B) \cap A] \\ &= (xR \cap A) \cap [B^{'} \cap A] \\ &= xR \cap B^{'} \cap A. \end{aligned}$$

Since B is an $\mathscr{I}$-increasing set, so $xR \cap B^{'} \in \mathscr{I} \ \forall x \in B$. Consequently $xR \cap B^{'} \cap A \in \mathscr{I}_A \ \forall x \in B \cap A$, which follows that $B \cap A$ is an $\mathscr{I}_A$-increasing set.

The following theorem shows that the property of being $\mathscr{I}PT_i$,-ordered space $, i = 0,1$ is a hereditary property.

Theorem 3.4. Let $(X,\tau_1,\tau_2,R,\mathscr{I})$ be $\mathscr{I}PT_i$-ordered space. Then every subspace of $\mathscr{I}PT_i$-ordered space is also $\mathscr{I}_APT_i$-ordered space, $(i = 0,1)$.

Proof.
We give a proof in case $(i = 0$ and the case $i = 1$ is similar). Let $(X,\tau_1,\tau_2,R,\mathscr{I})$ be $\mathscr{I}PT_0$-ordered space, $(A,\tau_{1A},\tau_{2A},R_A,\mathscr{I}_A)$ be any subspace of $(X,\tau_1,\tau_2,R,\mathscr{I})$ and $a,b \in A$ such that $a\overline{R_A}b$, it follows that $a\overline{R}b$. Since $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_0$-ordered, then there exists an $\mathscr{I}$-increasing τ_i-open set U such that $a \in U$ and $b \notin U$ or there exists an $\mathscr{I}$-decreasing τ_i-open set V such that $b \in V$ and $a \notin V, i = 1$ or 2. By Theorem 3.3 there exists ($\mathscr{I}_A$-increasing τ_{iA}-open set G such that $a \in G = A \cap U$ and $b \notin G = A \cap U$) or there exists ($\mathscr{I}_A$-decreasing τ_{iA}-open set H such that $b \in H = A \cap V$ and $a \notin H = A \cap V$). Hence, $(A,\tau_{1A},\tau_{2A},R_A,\mathscr{I}_A)$ is $\mathscr{I}_APT_0$-ordered.

Definition 3.5. An ideal bitopological ordered space $(X,\tau_1,\tau_2,R,\mathscr{I})$ is said to be $\mathscr{I}PT_2$-ordered space iff for all $a,b \in X$ such that $a\overline{R}b$, there exists an $\mathscr{I}$-increasing τ_i-open set O_a and an $\mathscr{I}$-decreasing τ_j-open set O_b such that $O_a \cap O_b \in \mathscr{I}$.

Example 3.4. In Example 3.1 take $\mathscr{I} = \{\phi,\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{1,2,4\}\} (\mathscr{I} = P(X)), \tau_1 = \{X,\phi,\{3\},\{4\},\{2,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\}, \tau_2 = \{X,\phi,\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$. It is clear that $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_2$-ordered space.

The following theorem studies the relationship between Definitions 3.4, 3.5 and Definition 2.10 [12].

Theorem 3.5. Let $(X,\tau_1,\tau_2,R,\mathscr{I})$ be an ideal bitopological ordered space. Then,
PT_2-ordered spaces $\Rightarrow \mathscr{I}PT_2$-ordered spaces $\Rightarrow \mathscr{I}PT_1$-ordered space.

Proof. The proof follows directly from the definitions of PT_2-ordered spaces, $\mathscr{I}PT_1$-ordered space and $\mathscr{I}PT_2$-ordered spaces.

Example 3.4 shows that $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_2$-ordered space, but not PT_2-ordered space as, $1\overline{R}2$, and all increasing τ_2-open sets which contain 1 are the sets $X,\{1,3,4\}$, intersect the only decreasing τ_1-open set X which contains 2.

Example 3.3 shows that $(X,\tau_1,\tau_2,R,\mathscr{I})$ is $\mathscr{I}PT_1$-ordered space, but not $\mathscr{I}PT_2$-ordered space as, $1\overline{R}2$ and all $\mathscr{I}$-increasing τ_2-open sets which contains 1 and not contain 2 are the sets $\{1,4\},\{1,3,4\}$ and all $\mathscr{I}$-decreasing τ_1-open set which contains 2 are the sets $X,\{2,3,4\}$, while $\{1,4\} \cap \{2,3,4\} = \{4\} \notin \mathscr{I}, \{1,3,4\} \cap \{2,3,4\} = \{3,4\} \notin \mathscr{I}, \{1,4\} \cap X = \{1,4\} \notin \mathscr{I}, \{1,3,4\} \cap X = \{1,3,4\} \notin \mathscr{I}$.

On account of Proposition 3.1, Theorems 3.1, 3.5 and [6,12], we have the following corollary.

Corollary 3.1. For an ideal bitopological ordered space $(X,\tau_1,\tau_2,R,\mathscr{I})$, we have the following implications:

$$\begin{array}{ccccc} \mathscr{I}PT_2-\textit{ordered space} & \Rightarrow & \mathscr{I}PT_1-\textit{ordered space} & \Rightarrow & \mathscr{I}PT_0-\textit{ordered space} \\ \Uparrow & & \Uparrow & & \Uparrow \\ PT_2-\textit{ordered space} & \Rightarrow & PT_1-\textit{ordered space} & \Rightarrow & PT_0-\textit{ordered space} \\ \Downarrow & & \Downarrow & & \Downarrow \\ P^*T_2-\textit{ordered space} & \Rightarrow & P^*T_1-\textit{ordered space} & \Rightarrow & P^*T_0-\textit{ordered space} \end{array}$$

The following theorem shows that the property of being $\mathscr{I}PT_2$-ordered space is preserved by a bijective, P-open and order (reverse) embedding mapping.

Theorem 3.6. If $(X,\tau_1,\tau_2,R,\mathscr{I})$ is a $\mathscr{I}PT_2$-ordered space, $f : (X,\tau_1,\tau_2,R,\mathscr{I}) \to (Y,\eta_1,\eta_2,R^*,f(\mathscr{I}))$ is a bijective, P-open and order (reverse) embedding mapping. Then, $(Y,\eta_1,\eta_2,R^*,f(\mathscr{I}))$ is $f(\mathscr{I})PT_2$-ordered space.

Proof. We give a proof in the case of order embedding mapping and the case of order reverse embedding mapping is similar.

Let $y_1,y_2 \in Y$ be such that $y_1\overline{R^*}y_2$. Then, there exist $x_1,x_2 \in X, f(x_1) = y_1, f(x_2) = y_2$ and $x_1\overline{R}x_2$. As $(X,\tau_1,\tau_2,R,\mathscr{I})$ is a $\mathscr{I}PT_2$-ordered space, then there exist an $\mathscr{I}$-increasing τ_i-open set U contains x_1 and an $\mathscr{I}$-decreasing τ_j-open set V contains x_2 such that $U \cap V \in \mathscr{I}$. Since f is P-open and by Theorem 2.1, $f(U)$ is $f(\mathscr{I})$-increasing η_i-open set contains $y_1 = f(x_1)$ and $f(V)$ is $f(\mathscr{I})$-decreasing η_j-open set contains $y_2 = f(x_2)$ such that $f(U) \cap f(V) = f(U \cap V) \in f(\mathscr{I})$. Hence, $(Y,\eta_1,\eta_2,R^*,f(\mathscr{I}))$ is $f(\mathscr{I})PT_2$-ordered space.

The following theorem shows that the property of being $\mathscr{I}PT_2$-ordered space is a hereditary property.

Theorem 3.7. Let $(X, \tau_1, \tau_2, R, \mathscr{I})$ be $\mathscr{I}PT_2$-ordered space. Then, every subspace of $\mathscr{I}PT_2$-ordered space is also $\mathscr{I}_APT_2$-ordered space.

Proof.
Let $(X, \tau_1, \tau_2, R, \mathscr{I})$ be a $\mathscr{I}PT_2$-ordered space, $(A, \tau_{1A}, \tau_{2A}, R_A, \mathscr{I}_A)$ be any subspace of $(X, \tau_1, \tau_2, R, \mathscr{I})$ and $a, b \in A$ such that $a\overline{R_A}b$. Since $(X, \tau_1, \tau_2, R, \mathscr{I})$ is $\mathscr{I}PT_2$-ordered, then there exists an $\mathscr{I}$-increasing τ_i-open set O_a and an $\mathscr{I}$-decreasing τ_j-open set O_b such that $O_a \cap O_b \in \mathscr{I}$. By Theorem 3.3 there exists an $\mathscr{I}_A$-increasing τ_{iA}-open set G such that $a \in G = A \cap O_a$ and an $\mathscr{I}_A$-decreasing τ_{jA}-open set H such that $b \in H = A \cap O_b$, $G \cap H = A \cap (O_a \cap O_b) \in \mathscr{I}_A$. Hence, $(A, \tau_{1A}, \tau_{2A}, R_A, \mathscr{I}_A)$ is $\mathscr{I}_APT_2$-ordered.

4 Conclusion

In this paper, we use the notion of $\mathscr{I}$-increasing (dec) sets [2], which based on the notion of ideal $\mathscr{I}$, to generate new separation axioms $\mathscr{I}P\tau_i, i = 0, 1, 2$ on ideal bitopological ordered space $(X, \tau_1, \tau_2, R, \mathscr{I})$. These types of separation axioms are a generalization of the previous one [6, 12]. Some properties of these separation have been obtained. In the future, we study the separation axioms $\mathscr{I}P\tau_i, i = 3, 4, 5$ and $\mathscr{I}PR_j$-ordered spaces, $j = 0, 1, 2, 3, 4$.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] M. A. Abo Elhamayel Abo Elwafa, On bitopological Ordered Spaces, Master Thesis, Mansoura University, Faculty of Science, Mansoura, Egypt 2009.
[2] S. A. El-Sheikh and M. Hosny, $\mathscr{I}$-increasing (decreasing) sets and $\mathscr{I}P^*$-separation axioms in bitopological ordered spaces, Pensee Journal 76 (3) 2014 429−443.
[3] E. Ekici, T. Noiri, connectedness in ideal topological spaces, Novi Sad J. Math. 38(2) 2008 65−70.
[4] R. Engelking, General Topology, Warszwawa 1977.
[5] D. Jankovic, T.R. Hamlet, New topologies from old via ideals, The American Mathematical Monthly 97 1990 295−310.
[6] A. Kandil, O. Tantawy, S. A. El-Sheikh and M. Hosny, New approach to bitopological ordered spaces, Jökull Journal 63 (6) 2013 10−25.
[7] J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. 13 1963 71−89.
[8] D. N. Misra and K. K. Dube, Pairwise R_0-space, Ann. Soc. Sci. Bruxelles 87 1976 3−15.
[9] G. M. Murdeshwar, General Topology, New Age International (P) Ltd., Publishers 1990.
[10] L. Nachbin, Topology and order, Van NostrandMathematical studies, Princeton, New Jersey 1965.
[11] M.K. Singal and A.R. Singal, Some more separation axioms in bitopological spaces. Ann. Soc. Sci. Bruelles 84 1970 207−230.
[12] M. K. Singal and A.R. Singal, Bitopological ordered spaces, Math. student XXXIX 1971 440−447.
[13] X. Q. Xu and Y. M. Liu, Regular relations and strictly completly regular ordered spaces, Topology and its applications 135 2004 1−12.