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Some Subclasses of p-Valent Functions Defined by Generalised Fractional Differintegral Operator - I

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Abstract: A generalized extended fractional differintegral operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}(z\in\triangle; p\in\mathbb{N}; \mu,\eta\in\mathbb{R}; \mu<p+1; -\infty<\lambda<\eta+p+1)$ was introduced and studied by Goyal and Prajapat [3]. In this paper, by applying this operator we define a class $\mathcal{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ for a different subordination function and obtain some interesting results.

Keywords: Analytic function; Multivalent function; Differential subordination; Generalised fractional differintegral operator; Generalised hypergeometric function; Hadamard product (or convolution).

1 Introduction and definitions

Let $\mathcal{A}_p$ denote the class of functions normalized by

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in\mathbb{N}=\{1,2,3,\ldots\}), \quad (1)$$

which are analytic and p-valent in the open disk $\triangle=\{z: z\in\mathbb{C} \text{ and } |z|<1\}$.

A function $f(z)\in\mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\alpha)$ of p-valently starlike functions of order α in $\triangle$, if $\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha(0\leqq\alpha<p;\ z\in\triangle)$. Furthermore, a function $f(z)\in\mathcal{A}_p$ is said to be in the class $\mathcal{K}_p(\alpha)$ of p-valently convex functions of order α in $\triangle$, if $\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\ (0\leqq\alpha<p;\ z\in\triangle)$. Indeed, it follows that $f(z)\in\mathcal{K}_p(\alpha)\Leftrightarrow\frac{zf'(z)}{p}\in\mathcal{S}_p^*(\alpha)$ $(0\leqq\alpha<p;\ z\in\triangle)$. We note that $\mathcal{S}_p^*(\alpha)\subseteq\mathcal{S}_p^*(0)\equiv\mathcal{S}_p^*$ and $\mathcal{K}_p(\alpha)\subseteq\mathcal{K}_p(0)\equiv\mathcal{K}_p(0\leqq\alpha<p)$, where $\mathcal{S}_p^*$ and $\mathcal{K}_p$ denote the subclass of $\mathcal{A}_p$ consisting of functions which are p-valently starlike in $\triangle$ and p-valently convex in $\triangle$, respectively (see, for details, [2]; see also [17] and [1]).

If $f(z)$ and $g(z)$ are analytic in $\triangle$, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as

$$f\prec g \text{ in } \triangle \text{ or } f(z)\prec g(z) \qquad (z\in\triangle),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\triangle$ with $w(0)=0$ and $|w(z)|<1$ in $\triangle$ such that $f(z)=g(w(z)), z\in\triangle$. It is known that

$$f(z)\prec g(z) \qquad (z\in\triangle) \quad \Rightarrow \quad f(0)=g(0)$$

and

$$f(\triangle)\subset g(\triangle).$$

In particular, if the function $g(z)$ is univalent in $\triangle$, then we have the following equivalence (cf., e.g., [10]):

$$f(z)\prec g(z) \qquad (z\in\triangle) \quad \Leftrightarrow \quad f(0)=g(0)$$

and

$$f(\triangle)\subset g(\triangle).$$

Furthermore, $f(z)$ is said to be subordinate to $g(z)$ in the disk $\triangle_r=\{z\in\mathbb{C}: |z|<r\}$ if the function $f_r(z)=f(rz)$ is subordinate to the function $g_r(z)=g(rz)$ in $\triangle$. It follows from the Schwarz lemma that if $f\prec g$ in $\triangle$, then $f\prec g$ in $\triangle_r$ for every $r(0<r<1)$.

The general theory of differential subordination introduced by Miller and Mocanu is given in [9]. Namely,

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if $\Psi:\Omega\to\mathbb{C}$ (where $\Omega\subseteq\mathbb{C}^2$) is an analytic function, h is analytic and univalent in $\triangle$, and if ϕ is analytic in $\triangle$ with $(\phi(z),z\phi'(z))\in\Omega$ when $z\in\triangle$, then we say that ϕ satisfies a first-order differential subordination provided that

$$\Psi(\phi(z),z\phi'(z))\prec h(z)\quad (z\in\triangle)\quad\text{and}\quad \Psi(\phi(0),0)=h(0). \tag{2}$$

We say that a univalent function $q(z)$ is a dominant of the differential subordination (2) if $\phi(0)=q(0)$ and $\phi(z)\prec q(z)$ for all analytic functions $\phi(z)$ that satisfy the differential subordination (2). A dominant $\bar{q}(z)$ is called as the best dominant of (2), if $\bar{q}(z)\prec q(z)$ for all dominants $q(z)$ of (2)[9, 10].

For functions $f_j(z)\in\mathscr{A}_p$, given by

$$f_j(z)=z^p+\sum_{n=1}^{\infty}a_{p+n,j}z^{p+n}\qquad (j\in 1,2;\ p\in\mathbb{N}),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1\star f_2)(z)=z^p+\sum_{n=1}^{\infty}a_{p+n,1}a_{p+n,2}z^{p+n}=(f_2\star f_1)(z)\qquad (p\in\mathbb{N}\ z\in\triangle).$$

In our present investigation, we shall also make use of the generalised hypergeometric functions ${}_2F_1$ and ${}_3F_2$ defined by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\qquad (a,b,c\in\mathbb{C},\ c\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}), \tag{3}$$

and

$${}_3F_2(a,b,c;d,e;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{z^n}{n!}\qquad (a,b,c,d,e\in\mathbb{C},\ d,e\notin\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}), \tag{4}$$

where $(\kappa)_n$ denote the Pochhammer symbol (or the shifted factorial) given in terms of Gamma function Γ, by

$$(\kappa)_n=\frac{\Gamma(\kappa+n)}{\Gamma(\kappa)}=\begin{cases}1 & (n=0),\\ \kappa(\kappa+1)\cdots(\kappa+n-1) & (n\in\mathbb{N}).\end{cases}$$

We note that the series defined by (3) and (4) converges absolutely for $z\in\triangle$ and hence ${}_2F_1$ and ${}_3F_2$ represent analytic functions in the open unit disk $\triangle$ (see, for details, [[22], Chapter 14]).

We recall here the following generalised fractional integral and generalised fractional derivative operator due to Srivastava et al. [19] (see also [15]).

Definition 1. *For real numbers $\lambda>0,\mu$ and η, Saigo hypergeometric fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ is defined by*

$$I_{0,z}^{\lambda,\mu,\eta}f(z)=\frac{z^{-\lambda-\mu}}{\Gamma(\lambda)}\int_0^z(z-t)^{\lambda-1}\times\ {}_2F_1\left(\lambda+\mu,-\eta;\lambda;1-\frac{t}{z}\right)f(t)dt,$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, with the order

$$f(z)=O(|z|^{\varepsilon})\qquad (z\longrightarrow 0;\varepsilon>\max\{0,\mu-\eta\}-1),$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed y requiring $\log(z-t)$ to be real when $(z-t)>0$.

Definition 2. *Under the hypotheses of Definition 1, Saigo hypergeometric fractional derivative operator $\mathfrak{S}_{0,z}^{\lambda,\mu,\eta}$ is defined by*

$$\mathfrak{S}_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\left\{z^{\lambda-\mu}\int_0^z(z-t)^{-\lambda}\times{}_2F_1\left(\mu-\lambda,1-\eta;1-\lambda;1-\frac{t}{z}\right)f(t)dt\right\} & (0\le\lambda<1);\\ \frac{d^n}{dz^n}\mathfrak{S}_{0,z}^{\lambda-n,\mu,\eta}f(z) & (n\le\lambda<n+1;n\in\mathbb{N}),\end{cases}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.

It may be remarked that

$$I_{0,z}^{\lambda,-\lambda,\eta}f(z)=D_z^{-\lambda}f(z)\qquad (\lambda>0)$$

and

$$\mathfrak{S}_{0,z}^{\lambda,\lambda,\eta}f(z)=D_z^{\lambda}f(z)\qquad (0\le\lambda<1),$$

where $D_z^{-\lambda}$ denotes fractional integral operator and D_z^{λ} denotes fractional derivative operator considered by Owa [12].

Recently Goyal and Prajapat [3] introduced generalized fractional differintegral operator $S_{0,z}^{\lambda,\mu,\eta}:\mathscr{A}_p\longrightarrow\mathscr{A}_p$, by

$$S_{0,z}^{\lambda,\mu,\eta}f(z)=\begin{cases}\frac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}z^{\mu}\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z) & (0\le\lambda<\eta+p+1,z\in\triangle);\\ \frac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)}z^{\mu}I_{0,z}^{-\lambda,\mu,\eta}f(z) & (-\infty<\lambda<0,z\in\triangle).\end{cases} \tag{5}$$

It is easily seen from (5) that for a function $f \in \mathscr{A}_p$, we have

$$\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n} a_{p+n} z^{p+n}$$
$$= z^p \, {}_3F_2(1,1+p,1+p+\eta-\mu;1+p-\mu,1+p+\eta-\lambda;z) * f(z)$$
$$(z \in \triangle; p \in \mathbb{N}; \mu,\eta \in \mathbb{R}; \mu < p+1; -\infty < \lambda < \eta+p+1) \quad (6)$$

Note that

$$\mathcal{S}_{0,z}^{0,0,0} f(z) = f(z),$$
$$\mathcal{S}_{0,z}^{1,1,1} f(z) = \mathcal{S}_{0,z}^{1,0,0} f(z) = \frac{zf'(z)}{p}$$

and

$$\mathcal{S}_{0,z}^{2,1,1} f(z) = \frac{zf'(z)+z^2f''(z)}{p^2}.$$

We also note that

$$\mathcal{S}_{0,z}^{\lambda,\lambda,\eta} f(z) = \mathcal{S}_{0,z}^{\lambda,\mu,0} f(z) = \Omega_z^{\lambda,p} f(z),$$

where $\Omega_z^{\lambda,p}$ is an extended fractional differintegral operator studied very recently by [14]. On the other hand, if we set

$$\lambda = -\alpha,\ \mu = 0,\ \eta = \beta - 1,$$

in (6) and using

$$I_{0,z}^{\alpha,0,\beta-1} f(z) = \frac{1}{z^\beta \Gamma(\alpha)} \int_0^z t^{\beta-1}\left(1-\frac{t}{z}\right)^{\alpha-1} f(t)dt,$$

we obtain following $p-$valent generalization of multiplier transformation operator [5,7]:

$$\mathscr{Q}_{\beta,p}^{\alpha} f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^\beta} \times \int_0^z t^{\beta-1}\left(1-\frac{t}{z}\right)^{\alpha-1} f(t)dt$$
$$= z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p+\beta+n)\Gamma(p+\alpha+\beta)}{\Gamma(p+\alpha+\beta+n)\Gamma(p+\beta)} a_{p+n} z^{p+n}$$

On the other hand, if we set

$$\lambda = -1,\ \mu = 0,\ \text{and}\ \eta = \beta - 1.$$

in (6), we obtain the generalized Bernardi-Libera Livingston integral operator $\mathscr{F}_{\beta,p} : \mathscr{A}_p \longrightarrow \mathscr{A}_p$ $(\beta > -p)$ defined by

$$\mathcal{S}_{0,z}^{-1,0,\beta-1} f(z) = \mathscr{F}_{\beta,b} f(z) = \frac{p+\beta}{z^\beta} \int_0^z t^{\beta-1} f(t)dt$$
$$= z^p + \sum_{n=1}^{\infty} \frac{p+\beta}{p+\beta+n} a_{p+n} z^{p+n}$$
$$= z^p \, {}_2F_1(1,p+\beta;p+\beta+1;z)$$
$$(\beta > -p; z \in \triangle). \quad (8)$$

It is easily seen from (6) that

$$z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z))' = (p+\eta-\lambda)(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)) - (\eta-\lambda)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)) \quad (9)$$

and it follows from (6) and (8) that

$$z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} \mathscr{F}_{\beta,p}(f)(z))' = (p+\eta+\beta)(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)) - (\eta+\beta)(\mathcal{S}_{0,z}^{\lambda,\lambda,\eta} \mathscr{F}_{\beta,p}(f)(z)). \quad (10)$$

Using the generalized fractional differintegral operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$, we now introduce the following subclass of $\mathscr{A}_p$:

Definition 3. *For fixed parameters A,B $(-1 \leq B < A \leq 1)$ and $0 \leq \alpha < p$, we say that a function $f(z) \in \mathscr{A}_p$ is in the class $\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ if it satisfies the following subordination condition:*

$$\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)} - \alpha\right) \prec \frac{1+Az}{1+Bz}$$
$$(z \in \triangle; p \in \mathbb{N}; \mu,\eta \in \mathbb{R}; \mu < p+1; -\infty < \lambda < \eta+p+1). \quad (11)$$

For $A = 1, B = -1$, we have

$$\frac{1}{p-\alpha}\left(\frac{z\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)} - \alpha\right) \prec \frac{1+z}{1-z}.$$

For convenience, we write

$$\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;1,-1) = \mathscr{V}_p^{\lambda,\mu,\eta}(\alpha) = \left\{ f(z) \in \mathscr{A}_p : \Re\left(\frac{z\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)}\right) > \alpha, \ 0 \leqq \alpha < p, z \in \triangle \right\}.$$

We further observe that

$$\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B) = \mathscr{V}_p^{\lambda,\mu,\eta}(0;A+\frac{\alpha}{p}(B-A),B);$$
$$(\beta > -p; \alpha+\beta > -p7)$$
$$\mathscr{V}_p^{0,0,0}(\alpha) = \mathscr{S}_p^*(\alpha)$$

and

$$\mathscr{V}_p^{1,0,0}(\alpha) = \mathscr{K}_p(\alpha).$$

Srivastava et. al [16] have studied some interesting properties of class $\mathscr{V}_p^{\lambda,\mu,0}(\alpha) = \mathscr{S}_\lambda(\alpha)$ $(0 \leq \lambda < 1;\ 0 \leq \alpha < 1)$ by using the techniques of Hadamard product.

In the present paper several sharp inclusion relationships and other interesting properties of the class $\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ are found for $\eta \in \mathbb{R}, \mu < p+1$ and for all admissible non-negative values of λ and also for negative values of λ by using the techniques of differential subordination. Mapping properties of a variety of operators involving the operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$ are also investigated.

2 A set of preliminary lemmas

We denote by $\mathscr{P}(\gamma)$ the class of functions $\varphi(z)$ given by

$$\varphi(z)=1+b_1z+b_2z^2+\cdots \tag{12}$$

which are analytic in $\triangle$ and satisfy the following inequality:

$$\Re(\varphi(z))>\gamma \qquad (0\leqq\gamma<1;\ z\in\triangle).$$

In order to prove our main results, we recall the following lemmas.

Lemma 1.[4,11] *Let the function $h(z)$ be analytic and convex (univalent) in $\triangle$ with $h(0)=1$. Suppose also that the function $\phi(z)$ given by*

$$\phi(z)=1+c_1z+c_2z^2+\cdots$$

is analytic in $\triangle$. If

$$\phi(z)+\frac{z\phi'(z)}{\gamma}\prec h(z) \qquad (z\in\triangle;\Re(\gamma)\geqq 0;\gamma\neq 0), \tag{13}$$

then

$$\phi(z)\prec\psi(z)=\frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1}h(t)dt\prec h(z) \qquad (z\in\triangle)$$

and $\psi(z)$ is the best dominant of (13).

Lemma 2.[10] *If $-1\leqq B<A\leqq 1,\beta>0$, and the complex number γ is constrained by*

$$\Re(\gamma)\geqq-\beta(1-A)/(1-B),$$

then the following differential equation:

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz} \qquad (z\in\triangle)$$

has a univalent solution in $\triangle$ given by

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}dt}-\dfrac{\gamma}{\beta}, & (B\neq 0)\\[2ex] \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)dt}-\dfrac{\gamma}{\beta}, & (B=0)\end{cases} \tag{14}$$

If the function $\phi(z)$ given by

$$\phi(z)=1+c_1z+c_2z^2+\cdots$$

is analytic in $\triangle$ and satisfies the following subordination:

$$\phi(z)+\frac{z\phi'(z)}{\beta\phi(z)+\gamma}\prec\frac{1+Az}{1+Bz} \qquad (z\in\triangle), \tag{15}$$

then

$$\phi(z)\prec q(z)\prec\frac{1+Az}{1+Bz} \qquad (z\in\triangle)$$

and $q(z)$ is the best dominant of (15).

Lemma 3.[21] *For real or complex numbers a,b and c $(c\neq 0,-1,-2,\ldots)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z)$$

$$(Re(c)>Re(b)>0); \tag{16}$$

$${}_2F_1(a,b;c;z)={}_2F_1(b,a;c;z); \tag{17}$$

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1(a,c-b;c;\frac{z}{z-1}); \tag{18}$$

$$(a+1)\,{}_2F_1(1,a;a+1;z)=(a+1)+az\,{}_2F_1(1,a+1;a+2;z) \tag{19}$$

and

$${}_2F_1(a,b;\frac{a+b+1}{2};\frac{1}{2})=\frac{\sqrt{\pi}\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}. \tag{20}$$

Lemma 4.[20] *If $\phi_j(z)\in P(\gamma_j)$ $(0\leqq\gamma_j<1;j=1,2)$, then $(\phi_1*\phi_2)(z)\in P(\gamma_3)$, $\gamma_3=1-2(1-\gamma_1)(1-\gamma_2)$ and the bound γ_3 is the best possible.*

Lemma 5.[11] *Suppose that the function $\Psi:\mathbb{C}^2\times\Delta\to\mathbb{C}$ satisfy the condition*

$$\Re(\Psi(ix,y;z))\leqq\varepsilon \tag{21}$$

for $\varepsilon>0$, real $x,y\leqq-\frac{(1+x^2)}{2}$ and for all $z\in\Delta$. If $\phi(z)$, given by (12) *is analytic in Δ and $\Re(\Psi(\phi(z)),z\phi'(z);z)>\varepsilon$, then $\Re(\phi(z))>0$ in Δ.*

3 Inclusion relationships for function class $\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$

Unless otherwise mentioned, we assume throughout the sequel that

$$-1\leq B<A\leq 1,\ 0\leq\alpha<p,$$

$$(z\in\triangle;p\in\mathbb{N};\mu,\eta\in\mathbb{R};\mu<p+1;-\infty<\lambda<\eta+p+1).$$

Theorem 1.*Let $f(z)\in\mathscr{V}_p^{\lambda+1,\mu,\eta}(\alpha;A,B)$,*

$$(p-\alpha)(1-A)+(\alpha+\eta-\lambda)(1-B)\geq 0 \tag{22}$$

and the function $Q(z)$ be defined on $\triangle$ by

$$Q(z)=\begin{cases}\int_0^1 t^{p+\eta-\lambda-1}(\frac{1+Btz}{1+\beta z})^{(p-\alpha)(A-B)/B}dt & (B\neq 0),\\ \int_0^1 t^{p+\eta-\lambda-1}\exp\left(A(p-\alpha)(t-1)z\right)dt & (B=0).\end{cases} \tag{23}$$

Then

$$\frac{1}{p-\alpha}\left(\frac{z\left(S_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{\left(S_{0,z}^{\lambda,\mu,\eta}f(z)\right)}-\alpha\right) \prec \frac{1}{p-\alpha}\left(\frac{1}{Q(z)}-\alpha-\eta+\lambda\right) = q(z)\prec\frac{1+Az}{1+Bz}\quad (z\in\triangle), \tag{24}$$

$$\mathscr{V}_p^{\lambda+1,\mu,\eta}(\alpha;A,B)\subset\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B),$$

and $q(z)$ is the best dominant of (24).

If, in addition to (22) one has $A\leq-\frac{(\alpha+\eta-\lambda+1)B}{p-\alpha}$ with $-1\leq B<0$, then

$$\mathscr{V}_p^{\lambda+1,\mu,\eta}(\alpha;A,B)\subset\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;1-2\rho,-1), \tag{25}$$

where

$$\rho=\frac{1}{p-\alpha}\left[(p+\eta-\lambda)\times\left\{{}_2F_1(1,\frac{(p-\alpha)(B-A)}{B};p+\eta-\lambda+1;\frac{B}{B-1})\right\}^{-1}-\alpha-\eta+\lambda\right].$$

The bound in (25) is the best possible.

Proof. Let $f(z)\in\mathscr{V}_p^{\lambda+1,\mu,\eta}(\alpha;A,B)$, and $g(z)$ be defined by

$$g(z)=z\left(\frac{\left(S_{0,z}^{\lambda,\mu,\eta}f(z)\right)}{z^p}\right)^{\frac{1}{p-\alpha}}\qquad (z\in\triangle). \tag{26}$$

Write $r_1=\sup\{r:g(z)\neq0,0<|z|<r<1\}$. Then $g(z)$ is single-valued and analytic in $|z|<r_1$. Taking logarithmic differentiation in (26), it follows that the function

$$\phi(z)=\frac{zg'(z)}{g(z)}=\frac{1}{p-\alpha}\left(\frac{z\left(S_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{\left(S_{0,z}^{\lambda,\mu,\eta}f(z)\right)}-\alpha\right) \tag{27}$$

is of the form (12) and is analytic in $|z|<r_1$. Using the identity (9) in (27) and again carrying out logarithmic differentiation in the resulting equation, we get

$$\phi(z)+\frac{z\phi'(z)}{(p-\alpha)\phi(z)+\alpha+\eta-\lambda} = \frac{1}{p-\alpha}\left(\frac{z\left(S_{0,z}^{\lambda+1,\mu,\eta}f(z)\right)'}{\left(S_{0,z}^{\lambda+1,\mu,\eta}f(z)\right)}-\alpha\right) \prec\frac{1+Az}{1+Bz}\quad (|z|<r_1). \tag{28}$$

Hence, by using Lemma ?? we find that

$$\phi(z)\prec\frac{1}{p-\alpha}\left(\frac{1}{Q(z)}-\alpha-\eta+\lambda\right)=q(z)\prec\frac{1+Az}{1+Bz}\quad (|z|<r_1), \tag{29}$$

where $q(z)$ is the best dominant of (24) and $Q(z)$ is given by (23). The remaining part of the proof can now be deduced on the same lines as in [[13], Theorem 1]. This evidently completes the proof.

Taking $A=1,B=-1,\eta=0$ and $p=1$ in Theorem 1 we get the following result which both extends and sharpens the work of Srivastava et al. [16].

Corollary 1. *If $-\infty<\max\{\lambda,\frac{\lambda}{2}\}\leq\alpha<1$, then*

$$\mathscr{S}_{\lambda+1}(\alpha)\subseteq\mathscr{S}_{\lambda}(\gamma)\subseteq\mathscr{S}_{\lambda}(\alpha),$$

where $\gamma=(1-\lambda)\left[{}_2F_1(1,2(1-\alpha);2-\lambda;\frac{1}{2})\right]^{-1}+\lambda$. The value of γ is the best possible.

Theorem 2. *Let β be a real number satisfying*

$$(p-\alpha)(1-A)+(\eta+\beta+\alpha)(1-B)\geq0.$$

(i) If $f(z)\in C\mathscr{V}_p^{\lambda}(\alpha;A,B)$, then

$$\frac{1}{p-\alpha}\left(\frac{z\left(S_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'}{\left(S_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)}-\alpha\right) \prec\frac{1}{p-\alpha}\left(\frac{1}{Q(z)}-\eta-\beta-\alpha\right) =\widetilde{q}(z)\prec\frac{1+Az}{1+Bz}\qquad (z\in\triangle), \tag{30}$$

where

$$Q(z)=\begin{cases}\int_0^1 t^{p+\eta+\beta-1}(\frac{1+Btz}{1+\beta z})^{(p-\alpha)(A-B)/B}dt & (B\neq0)\\ \int_0^1 t^{p+\eta+\beta-1}\exp\left(A(p-\alpha)(t-1)z\right)dt & (B=0)\end{cases}$$

and $\widetilde{q}(z)$ is the best dominant of (30). Consequently, the operator $\mathscr{F}_{\beta,p}$ maps the class $\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ into itself.

(ii) If $-1\leq B<0$ and

$$\beta\geq\max\left\{\frac{(p-\alpha)(B-A)}{B}-p-\eta-1,\ -\frac{(p-\alpha)(1-A)}{1-B}-\alpha-\eta\right\}, \tag{31}$$

then the operator $\mathscr{F}_{\beta,p}$ *maps the class* $\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;A,B)$ *into the class* $\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha;1-2\rho,-1)$, *where*

$$\rho = \frac{1}{p-\alpha}\Bigg[(\eta+\beta+p) \times \left\{{}_2F_1(1,\frac{(p-\alpha)(B-A)}{B};\eta+\beta+p+1;\frac{B}{B-1})\right\}^{-1} -\eta-\beta-\alpha\Bigg].$$

The bound ρ *is the best possible.*

*Proof.*Upon replacing

$$g(z) \text{ by } z\left(\frac{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)}{z^p}\right)^{\frac{1}{p-\alpha}}, (z\in\triangle)$$

in (26) and carrying out logarithmic differentiation it follows that the function $\phi(z)$ given by

$$\phi(z) = \frac{zg'(z)}{g(z)} = \frac{1}{p-\alpha}\left(\frac{z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)'}{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)} - \alpha\right) \tag{32}$$

is of the form (12) and is analytic in $|z|<r_1$. Using the identity (10) in (32) and the fact that $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\neq 0$ in $0<|z|<1$, we get

$$\frac{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)}{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)} = \frac{\eta+\beta+p}{(p-\alpha)\phi(z)+\eta+\beta+\alpha} \quad (|z|<r_1). \tag{33}$$

Again, by taking logarithmic differentiation in (33) and using (32) in the resulting equation, we deduce that

$$\frac{1}{p-\alpha}\left(\frac{z\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)}-\alpha\right) = \phi(z)+\frac{z\phi'(z)}{(p-\alpha)\phi(z)+\eta+\beta+\alpha} \prec \frac{1+Az}{1+Bz} \quad (|z|<r_1).$$

The remaining part of the proof is similar to that of [[13], Theorem 1] and we choose to omit the details.

Putting $A=1$ and $B=-1$ in Theorem 2, we get

Corollary 2.*If* β *is a real number satisfying* $\beta\geq\max\{p-2\alpha-\eta-1,-\alpha-\eta\}$, *then*

$$\mathscr{F}_{\beta,p}(\mathscr{V}_p^{\lambda,\mu,\eta}(\alpha))\subset\mathscr{V}_p^{\lambda,\mu,\eta}(\sigma),$$

where $\sigma=(\eta+\beta+p)\left[{}_2F_1(1,2(p-\alpha);\eta+\beta+p+1;\frac{1}{2})\right]^{-1}-\eta-\beta$. *The result is the best possible.*

In particular, when $\eta=0$, Corollary 3.4 gives [[14], Corollary 1.7]. Further, for $\eta=0$ and $\lambda=0$, Corollary 2 gives the following result which, in turn, the first half of Remark 2 [[13],p.330].

Corollary 3.*If* β *is a real number satisfy* $\beta\geq\max\{p-2\alpha-1,-\alpha\}$, *then*

$$\mathscr{F}_{\beta,p}(\mathscr{S}_p^*(\alpha))\subset\mathscr{S}_p^*(\sigma),$$

where $\sigma=(\beta+p)[\,{}_2F_1(1,2(p-\alpha);\beta+p+1;\frac{1}{2})]^{-1}-\beta$. *The value of* σ *is the best possible.*

It is interest to note that, by setting $\beta=0$ in corollary 3, we have the further consequence [[18], Corollary 7].

4 Some properties of the operator $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}$

Now we discuss some properties of the operator $\mathscr{S}_{0,z}^{\lambda,\mu,\eta}$.

Theorem 3.*Let* $\delta>0,\eta\in\mathbb{R},\mu<p+1,-\infty<\lambda<p,p\neq 1$ *and the function* $f(z)\in\mathscr{A}_p$ *satisfies the following subordination:*

$$(1-\delta)\frac{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)}{z^p}+\delta\frac{\left(\mathscr{S}_{0,z}^{\lambda+1,\mu,\eta}f(z)\right)}{z^p}\prec\frac{1+Az}{1+Bz} \quad (z\in\triangle). \tag{34}$$

Then

$$\Re\left[\left(\frac{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)}{z^p}\right)^{\frac{1}{m}}\right]>\chi_1^{\frac{1}{m}} \quad (m\in\mathbb{N};\ z\in\triangle), \tag{35}$$

where

$$\chi_1=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1(1,1;\frac{p+\eta-\lambda}{\delta}+1;\frac{B}{B-1}) & (B\neq 0),\\ 1-\frac{(p+\eta-\lambda)A}{p+\eta-\lambda+\delta}, & (B=0).\end{cases}$$

The result is the best possible.

*Proof.*For $f(z)\in\mathscr{A}_p$, consider the function given by

$$\phi(z)=\frac{\left(\mathscr{S}_{0,z}^{\lambda,\mu,\eta}f(z)\right)}{z^p} \quad (z\in\triangle). \tag{36}$$

Then $\phi(z)$ is of the form (12) and analytic in $\triangle$. By differentiating (36) and making use of the identity (9), we obtain

$$\phi(z)+\frac{\delta}{p+\eta-\lambda}z\phi'(z)\prec\frac{1+Az}{1+Bz} \quad (z\in\triangle).$$

Now, by applying Lemma **??** we get

$$\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)}{z^p} \prec Q(z)$$
$$= \frac{p+\eta-\lambda}{\delta} z^{-\frac{p+\eta-\lambda}{\delta}} \int_0^z t^{\frac{p+\eta-\lambda}{\delta}-1}\left(\frac{1+At}{1+Bt}\right)dt$$
$$= \begin{cases} \frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1(1,1;\frac{p+\eta-\lambda}{\delta}+1;\frac{Bz}{1+Bz}) \\ \qquad\qquad (B\neq 0), \\ 1+\frac{(p+\eta-\lambda)}{p+\eta-\lambda+\delta}Az \qquad (B=0), \end{cases}$$

where we have also made a change of variable followed by the use of identities (16) and (18). The remaining part of the proof can be deduced on the same lines as in [[13], Theorem 4]. The proof of Theorem 3 is thus completed.

Upon setting $A=1-2\alpha, (0\le\alpha<1)$, $B=-1, m=1, \eta=0$ and $\lambda=0$ in Theorem 3, we state the following

Corollary 4. *For $\delta>0$, if*

$$\Re\left((1-\delta)\frac{f(z)}{z^p}+\delta\frac{(zf'(z))}{pz^p}\right)>\alpha,$$

then

$$\Re\left(\frac{f(z)}{z^p}\right)>\alpha+(1-\alpha)\left[{}_2F_1\left(1,1;\frac{p}{\delta}+1;\frac{1}{2}\right)-1\right].$$

Upon setting $A=1-2\alpha, (0\le\alpha<1)$, $B=-1$, $m=1, \eta=0$ and $\lambda=-1$ in Theorem 3we state the following

Corollary 5. *For $\delta>0$, if* $\Re\left(\frac{(1-\delta)}{z^p}\left[\frac{p+1}{z}\int_0^z f(\xi)d\xi\right]+\delta\frac{f(z)}{z^p}\right)>\alpha$, *then*

$$\Re\left(\frac{1}{z^p}\left[\frac{p+1}{z}\int_0^z f(\xi)d\xi\right]\right)$$
$$>\alpha+(1-\alpha)\left[{}_2F_1\left(1,1;\frac{p+1}{\delta}+1;\frac{1}{2}\right)-1\right].$$

Theorem 4. *Let $\delta>0, \eta\in\mathbb{R}, \mu<p+1, -\infty<\lambda<p+1, p\neq 1$ and $f(z)\in\mathscr{A}_p$. If the function $\mathscr{F}_{\beta,p}(f)(z)$ be defined by (8) satisfies*

$$(1-\delta)\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)}{z^p}+\delta\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)\right)}{z^p}\prec\frac{1+Az}{1+Bz} \qquad (z\in\triangle), \quad (37)$$

then

$$\Re\left[\left(\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)}{z^p}\right)^{\frac{1}{m}}\right]>\varsigma_1^{\frac{1}{m}} \quad (m\in\mathbb{N};\ z\in\triangle), \quad (38)$$

where

$$\varsigma_1=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1(1,1;\frac{p+\eta+\beta}{\delta}+1;\frac{B}{B-1}) \\ \qquad\qquad (B\neq 0), \\ 1-\frac{(p+\eta+\beta)A}{p+\eta+\beta+\delta}, \qquad (B=0).\end{cases}$$

Proof. For $f(z)\in\mathscr{A}_p$, consider the function given by

$$\psi(z)=\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta}\mathscr{F}_{\beta,p}(f)(z)\right)}{z^p} \qquad (z\in\triangle). \qquad (39)$$

Then $\psi(z)$ is of the form (12) and analytic in $\triangle$. By differentiating (39) and making use of the identity (10), we obtain

$$\psi(z)=\frac{\delta}{p+\eta+\beta}z\,\psi'(z)\prec\frac{1+Az}{1+Bz} \qquad (z\in\triangle).$$

The remaining part of the proof of Theorem 4 is similar to that of Theorem 3 and we omit the details.

Upon setting $A=1-2\alpha, (0\le\alpha<1)$, $B=-1, m=\delta=1, \eta=0$ and $\lambda=0$ in Theorem 4 we state the following

Corollary 6. *If $\Re\left(\frac{f(z)}{z^p}\right)>\alpha$, then*

$$\Re\left(\frac{1}{z^p}\left[\frac{p+\beta}{z^\beta}\int_0^z \xi^{\beta-1}f(\xi)d\xi\right]\right)$$
$$>\alpha+(1-\alpha)\left[{}_2F_1\left(1,1;p+\beta+1;\frac{1}{2}\right)-1\right].$$

Upon setting $A=1-2\alpha, (0\le\alpha<1)$, $B=-1, \eta=0$ and $m=\delta=\lambda=1$ in Theorem 4 we state the following

Corollary 7. *If $\Re\left(\frac{(zf'(z))}{pz^p}\right)>\alpha$ then*

$$\Re\left(\frac{1}{pz^p}\left[z\{\frac{p+\beta}{z^\beta}\int_0^z \xi^{\beta-1}f(\xi)d\xi\}'\right]\right)$$
$$>\alpha+(1-\alpha)\left[{}_2F_1\left(1,1;p+\beta+1;\frac{1}{2}\right)-1\right].$$

In particular, for $\beta=0$, Corollary 7 gives

Corollary 8. *If $\Re\left(\frac{f(z)}{z^p}\right)>\alpha$, then*

$$\Re\left(\frac{f(z)}{z^p}\right)>\alpha+(1-\alpha)\left[{}_2F_1\left(1,1;p+1;\frac{1}{2}\right)-1\right].$$

5 Some properties of the operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$ involving convolution

The proof of the following theorem is similar to that of [[13], Theorem 3] and we omit its proof.

Theorem 5. *Let $\delta > 0$ and $-1 \leqq B_j < A_j \leqq 1\,(j=1,2)$. If each of the function $f_j(z) \in \mathcal{A}_p$ $(j=1,2)$ satisfies*

$$(1-\delta)\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f_j(z)\right)}{z^p} + \delta\frac{\left(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f_j(z)\right)}{z^p} \prec \frac{1+A_j z}{1+B_j z} \quad (j=1,2;\ z\in\triangle),$$

then

$$(1-\delta)\frac{\left(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} H(z)\right)}{z^p} + \delta\frac{\left(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} H(z)\right)}{z^p} \prec \frac{1+(1-2\eta_0)z}{1-z} \quad (z\in\triangle),$$

where

$$H(z) = \mathcal{S}_{0,z}^{\lambda,\mu,\eta}(f_1 \star f_2)(z) \quad (z\in\triangle) \tag{40}$$

and

$$\eta_0 = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \times \left[1 - \frac{1}{2}\,{}_2F_1\left(1,1;\frac{p+\eta-\lambda}{\delta}+1;\frac{1}{2}\right)\right].$$

The results is the best possible when $B_1 = B_2 = -1$.

We now state

Theorem 6. *Let $f_j(z) \in \mathcal{A}_p$ $(j=1,2)$. If the functions $\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f_j(z)/z^p \in \mathcal{P}(\eta_j)$ $(0 \leqq \eta_j < 1;\ \ j=1,2)$, then the function $H(z)$, given by (40) satisfies*

$$\Re\left(\frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} H(z)}{\mathcal{S}_{0,z}^{\lambda,\mu,\eta} H(z)}\right) > 0 \ (z\in\triangle),$$

provided

$$(1-\eta_1)(1-\eta_2) < \frac{2(p-\lambda)+1}{2\left[\begin{array}{c}\{{}_2F_1(1,1;p+\eta-\lambda+1;\frac{1}{2})-2\}^2\\ +2(p+\eta-\lambda)\end{array}\right]}. \tag{41}$$

Proof. By the hypothesis on $f_j(z)$ it follows from Lemma 2.4

$$\begin{aligned}&\Re\left(\frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} H(z)}{z^p} + \frac{z}{p-\lambda}\left(\frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} H(z)}{z^p}\right)'\right)\\ &= \Re\left(\frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f_1(z)}{z^p} \star \frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f_2(z)}{z^p}\right)\\ &> 1-2(1-\eta_1)(1-\eta_2) \ (z\in\triangle),\end{aligned} \tag{42}$$

which in view of Lemma 1 for

$$\gamma = p+\eta-\lambda,\ A = -1+2(1-\eta_1)(1-\eta_2), \quad \text{and} \quad B=-1$$

yields

$$\Re\left(\frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} H(z)}{z^p}\right) > 1+2(1-\eta_1)(1-\eta_2) \times \left[{}_2F_1\left(1,1;p+\eta-\lambda;\frac{1}{2}\right)-2\right] \ (z\in\triangle). \tag{43}$$

Again, from (43) and Theorem 4.1 for

$$A = -1-4(1-\eta_1)(1-\eta_2)\left[{}_2F_1\left(1,1;p+\eta-\lambda;\frac{1}{2}\right)-2\right],$$
$$B=-1, \delta=1 \quad \text{and} \quad m=1,$$

we deduce that

$$\Re(\vartheta(z)) > 1-2(1-\eta_1)(1-\eta_2) \times \left[{}_2F_1\left(1,1;p+\eta-\lambda;\frac{1}{2}\right)-2\right]^2 \ (z\in\triangle), \tag{44}$$

where $\vartheta(z) = \mathcal{S}_{0,z}^{\lambda,\mu,\eta} H(z)/z^p$. Now, if we let

$$\phi(z) = \frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} H(z)}{\mathcal{S}_{0,z}^{\lambda,\mu,\eta} H(z)} \ (z\in\triangle),$$

then $\phi(z)$ is of the form (12) is analytic in $\triangle$ and a simple computation shows that

$$\begin{aligned}&\frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} H(z)}{z^p} + \frac{z}{p+\eta-\lambda}\left(\frac{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} H(z)}{z^p}\right)'\\ &= \vartheta(z)\left[\phi^2(z) + \frac{z\phi'(z)}{p+\eta-\lambda}\right]\\ &= \Psi(\phi(z), z\phi'(z); z),\end{aligned} \tag{45}$$

where $\Psi(u,v;z) = \vartheta(z)(u^2 + (v/(p+\eta-\lambda)))$. Thus by using (42) in (45), we get

$$\Re\left(\Psi(\phi(z), z\phi'(z); z)\right) > 1-2(1-\eta_1)(1-\eta_2) \ (z\in\triangle).$$

Now for all real $x, y \leq -\frac{1}{2}(1+x^2,)$ we have

$$\begin{aligned}\Re(\Psi(ix,y;z)) &= \left(\frac{y}{p+\eta-\lambda} - x^2\right)\Re(\vartheta(z))\\ &\leqq -\frac{1}{2(p+\eta-\lambda)}\\ &\quad \times(1+(2(p+\eta-\lambda)+1)x^2)\Re(\vartheta(z))\\ &\leqq -\frac{1}{2(p+\eta-\lambda)}\Re(\vartheta(z))\\ &\leqq 1-2(1-\eta_1)(1-\eta_2) \ (z\in\triangle),\end{aligned}$$

by (41) and (44). Hence by making use of Lemma 2.5 we get $\Re(\phi(z)) > 0$ in $\triangle$; that is

$$\Re\left(\frac{S_{0,z}^{\lambda+1,\mu,\eta}H(z)}{S_{0,z}^{\lambda,\mu,\eta}H(z)}\right) > 0 \ (z\in\triangle).$$

This completes the proof.

Setting $\lambda = 0$ and $\eta = 0$ in Theorem 5.2, we get the following corollary which, in turn, yields the corresponding work of Lashin [[6], Theorem 1] for $p = 1$.

Corollary 9. *Let $f_j(z)\in\mathscr{A}_p$ $(j = 1,2)$. If the functions $f_j'(z)/pz^{p-1}\in\mathscr{P}(\eta_j)$ $(0\leqq\eta_j<1;\ j=1,2)$, then the function $(f_1\star f_2)(z)\in\mathscr{S}_p^*$, provided*

$$(1-\eta_1)(1-\eta_2) < \frac{2p+1}{2[\{{}_2F_1(1,1;p+1;\frac{1}{2})-2\}^2+2p]}. \tag{46}$$

Theorem 7. *Let $f_j(z)\in\mathscr{A}_p$ $(j = 1,2)$. If the functions $H(z)$, given by (40) satisfies*

$$\Re\left(\frac{S_{0,z}^{\lambda+1,\mu,\eta}H(z)}{z^p}\right) > 1-\frac{2(p+\eta-\lambda)+1}{[\{{}_2F_1(1,1;p+\eta-\lambda+1;\frac{1}{2})-2\}^2+2(p+\eta-\lambda)]} \quad (z\in\triangle),$$

then

$$\Re\left(\frac{S_{0,z}^{\lambda+1,\mu,\eta}G_{\lambda,\eta}(z)}{S_{0,z}^{\lambda,\mu,\eta}G_{\lambda,\eta}(z)}\right) > 0 \ (z\in\triangle),$$

where

$$G_{\lambda,\eta}(z) = (p+\eta-\lambda)z^{\lambda-\eta}\int_0^z \frac{H(t)}{t^{\lambda-\eta+1}}\,dt \ (z\in\triangle).$$

Proof. From the definition of the function $G_{\lambda,\eta}(z)$, we see that

$$\Re\left(\frac{S_{0,z}^{\lambda+1,\mu,\eta}H(z)}{z^p}\right) = \Re\left(\frac{S_{0,z}^{\lambda+1,\mu,\eta}G_{\lambda,\eta}(z)}{z^p}+\frac{z}{p+\eta-\lambda}\left(\frac{S_{0,z}^{\lambda+1,\mu,\eta}G_{\lambda,\eta}(z)}{z^p}\right)'\right) > 1-\frac{2(p+\eta-\lambda)+1}{[\{{}_2F_1(1,1;p+\eta-\lambda+1;\frac{1}{2})-2\}^2+2(p+\eta-\lambda)]} \quad (z\in\triangle),$$

and the proof of Theorem 5.4 is completed similar to Theorem 5.2.

For $\lambda = 0$ and $\eta = 0$ in Theorem 5.4, we obtain the following result which yields the corresponding work of Lashin [[6], Theorem 3] for $p = 1$.

Corollary 10. *Let $f_j(z)\in\mathscr{A}_p$ $(j = 1,2)$. If*

$$\Re\left(\frac{(f_1\star f_2)'(z)}{pz^{p-1}}\right) > 1-\frac{2p+1}{[\{{}_2F_1(1,1;p+1;\frac{1}{2})-2\}^2+2p]} \quad (z\in\triangle),$$

then

$$G_{0,0}(z) = p\int_0^z \frac{(f_1\star f_2)(t)}{t}\,dt\in\mathscr{S}_p^*.$$

6 Concluding Remarks

Putting $\eta = 0$ in Theorem 3.1, 3.3, 4.1, 4.4, 5.1, 5.2 and 5.4, we get the corresponding theorems and consequences of Patel and Mishra [14].

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