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A Refinement of Grüss Inequality via Cauchy-Schwarz's Inequality for Discrete Random Variables

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Abstract: If X and Y are discrete random variables in finite case, then using the inequality of Cauchy-Schwarz, we will obtain another inequality expressed by the variance and covariance. The aim of this paper is to obtain a new refinement of discrete version of Grüss inequality. In the final we show that we can structure the set of random variables with equal probabilities as a Hilbert space and as a seminormed vector space.

Keywords: Covariance, Hilbert space, refinement of Grüss inequality, seminormed vector space, variance.

1 Introduction

The integral variant of Grüss inequality (see, [11]), besides applications in mathematical analysis, has some statistical applications. The discrete version of Grüss inequality (see, [2], [13], [14], [15], [19], [23]) has the following form:

$$\left|\frac{1}{n}\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\sum_{i=1}^{n}x_i\frac{1}{n}\sum_{i=1}^{n}y_i\right|\leq\frac{1}{4}\left(\Gamma_1-\gamma_1\right)\left(\Gamma_2-\gamma_2\right),\quad(1)$$

where x_i, y_i are real numbers, $\gamma_1\leq x_i\leq\Gamma_1$ and $\gamma_2\leq y_i\leq\Gamma_2$, for all $i=\overline{1,n}$.

In [24], Pečarić showed some remarks on the Ostrowski generalization of Chebyshev's inequality by the Chebyshev functional. There are many articles which treated this inequality in integral variant (see, [5], [6], [7], [8], [19], [23]). We will focus attention on the discrete version of Grüss inequality and motivated by its usefulness, we will study this inequality in the context of elements of statistics, using the concepts of variance and covariance for the random variables.

The variance of a random variable $X=\begin{pmatrix}x_i\\p_i\end{pmatrix}_{1\leq i\leq n}$ with probabilities $P(X=x_i)=p_i=\frac{1}{n}$, for any $i=\overline{1,n}$, is second central moment, the expected value of the squared deviation from mean $\mu_X=E[X]=\frac{1}{n}\sum_{i=1}^{n}x_i$:

$$Var(X)=E\left[(X-\mu_X)^2\right]=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_X)^2.$$

The expression for the variance can be expanded thus:

$$Var(X)=E\left[X^2\right]-E^2\left[X\right].$$

We denote by $\mathbb{RV}$ the set of random variables $X=\begin{pmatrix}x_i\\p_i\end{pmatrix}_{1\leq i\leq n}$ with probabilities $P(X=x_i)=p_i=\frac{1}{n}$, for any $i=\overline{1,n}$.

The covariance is a measure of how much two random variables changes together and is defined as:

$$Cov(X,Y)=E[(X-E[X])(Y-E[Y])],$$

and is equivalent to the form

$$Cov(X,Y)=E[XY]-E[X]E[Y].$$

Using the inequality of Cauchy-Schwarz for discrete random variables, we find the inequality given by

$$|Cov(X,Y)|^2\leq Var(X)Var(Y),$$

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or in the form

$$|Cov(X,Y)| \leq \sqrt{Var(X)Var(Y)}.$$

Two variables have a strong statistical relationship each other if they appear to move together. According to [9], correlation is a measure of a linear relationship between two variables, X and Y, and is measured by the correlation coefficient, given by:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}.$$

It is easy to see that $-1 \leq \rho(X,Y) \leq 1$.

2 Main results

For beginning, we will present some properties of the discrete random variables in finite case. If X and Y are discrete random variables in finite case, and a, b are real numbers, then it is easy to see, using the definitions for the variance and covariance, that there is the following relation:

$$Var(aX+bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y), \tag{2}$$

If we take $a = b = 1$ and $a = 1$, $b = -1$, in relation (2), then we obtain the equalities:

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y), \tag{3}$$

and

$$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y). \tag{4}$$

Remark 2.1. From relations (3) and (4), we find the parallelogram law in terms of variance, namely

$$Var(X+Y) + Var(X-Y) = 2Var(X) + 2Var(Y). \tag{5}$$

Also, if X, Y, Z and T are discrete random variables in finite case, and a, b, c and d are real numbers, then there is the following equality:

$$\begin{aligned} &Cov(aX+bY, cZ+dT) = \\ &= acCov(X,Z) + adCov(X,T) + \\ &+ bcCov(Y,Z) + bdCov(Y,T) \end{aligned} \tag{6}$$

Theorem 2.1. If X, Y and Z are discrete random variables in finite case, with $X \neq kZ$, then we have the inequality:

$$\begin{aligned} 0 &\leq \frac{[Cov(X,Y)Cov(X,Z) - Cov(Y,Z)Var(X)]^2}{Var(X)Var(Z) - [Cov(X,Z)]^2} \leq \\ &\leq Var(X)Var(Y) - [Cov(X,Y)]^2 \end{aligned} \tag{7}$$

Proof. For the discrete random variables X, Y and Z given in finite case, with $Var(X) \neq 0$, we take the following random variable:

$$W = \frac{Cov(X,Y) + \lambda Cov(X,Z)}{Var(X)}X - Y - \lambda Z.$$

We calculate the variance of random variable W, thus:

$$Var(W) = Var\left(\left(\frac{Cov(X,Y)}{Var(X)}X - Y\right) - \lambda\left(\frac{Cov(X,Z)}{Var(X)}X - Z\right)\right),$$

and applying relation (4), we have:

$$\begin{aligned} &Var(W) = Var\left(\frac{Cov(X,Y)}{Var(X)}X - Y\right) + \lambda^2 Var\left(\frac{Cov(X,Z)}{Var(X)}X - Z\right) - \\ &-2\lambda Cov\left(\frac{Cov(X,Y)}{Var(X)}X - Y, \frac{Cov(X,Z)}{Var(X)}X - Z\right) = \\ &= Var(Y) - \frac{[Cov(X,Y)]^2}{Var(X)} + \lambda^2\left(Var(Z) - \frac{[Cov(X,Z)]^2}{Var(X)}\right) - \\ &-2\lambda Cov\left(\frac{Cov(X,Y)}{Var(X)}X - Y, \frac{Cov(X,Z)}{Var(X)}X - Z\right). \end{aligned}$$

We deduce the following inequality

$$\begin{aligned} &Cov\left(\frac{Cov(X,Y)}{Var(X)}X - Y, \frac{Cov(X,Z)}{Var(X)}X - Z\right) = \\ &= \frac{Cov(X,Y)Cov(X,Z)}{Var(X)Var(X)}Cov(X,X) - \\ &- \frac{Cov(X,Y)Cov(X,Z)}{Var(X)} - \frac{Cov(X,Z)Cov(X,Y)}{Var(X)} + Cov(Y,Z) = \\ &= Cov(Y,Z) - \frac{Cov(X,Y)Cov(X,Z)}{Var(X)}. \end{aligned}$$

Returning to calculate the variance for random variable W, we have:

$$\begin{aligned} &Var(W) = Var(Y) - \frac{[Cov(X,Y)]^2}{Var(X)} + \lambda^2\left(Var(Z) - \frac{[Cov(X,Z)]^2}{Var(X)}\right) - \\ &-2\lambda\left(Cov(Y,Z) - \frac{Cov(X,Y)Cov(X,Z)}{Var(X)}\right). \end{aligned}$$

Therefore, we deduce the equality

$$\begin{aligned} &Var(X)Var(W) = Var(X)Var(Y) - [Cov(X,Y)]^2 + \\ &+\lambda^2(Var(X)Var(Z) - [Cov(X,Z)]^2) - \\ &-2\lambda(Var(X)Cov(Y,Z) - Cov(X,Y)Cov(X,Z)). \end{aligned}$$

Since $Var(X)Var(W) \geq 0$, it follows that

$$\begin{aligned} &\lambda^2(Var(X)Var(Z) - [Cov(X,Z)]^2) - 2\lambda(Var(X)Cov(Y,Z) - \\ &\quad -Cov(X,Y)Cov(X,Z)) + Var(X)Var(Y) - [Cov(X,Y)]^2 \geq 0, \end{aligned}$$

for every $\lambda \in \mathbb{R}$.

Taking into account that $Var(X)Var(Z) - [Cov(X,Z)]^2 \neq 0$, because $X \neq kZ$, this implies that

$$\begin{aligned} &(Var(X)Var(Z) - [Cov(X,Z)]^2)\left(Var(X)Var(Y) - [Cov(X,Y)]^2\right) \geq \\ &\geq (Var(X)Cov(Y,Z) - Cov(X,Y)Cov(X,Z))^2. \end{aligned} \tag{8}$$

Consequently, we obtain the inequality of the statement. □

Remark 2.2. (a) By replacement with the correlation coefficient in inequality (8), we deduce the inequality:

$$\left[1-\rho^2(X,Y)\right]\left[1-\rho^2(X,Z)\right] \geq (\rho(Y,Z)-\rho(X,Y)\rho(X,Z))^2. \quad (9)$$

(b) Let X, Y and Z be discrete random variables in finite case, with $Var(Y) \neq 0$ and $Var(Z) \neq 0$. If we take the following random variable: $W = X - \frac{Cov(X,Y)}{Var(Y)}Y - \lambda Z$, then we have the inequality:

$$\begin{aligned} &0 \leq \tfrac{[Cov(X,Y)Cov(Y,Z)-Cov(X,Z)Var(Y)]^2}{Var(Y)Var(Z)} \leq \\ &\leq Var(X)Var(Y) - |Cov(X,Y)|]^2 \end{aligned} \quad (10)$$

3 Applications

Let $x_1, x_2, ..., x_n$ be real numbers, assume $\gamma_1 \leq x_i \leq \Gamma_1$, for all $i = \overline{1,n}$ and the average $\mu_X = \frac{1}{n}\sum_{i=1}^{n} x_i$.

In 1935, Popoviciu proved the folllowing inequality

$$Var(X) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_X)^2 \leq \frac{1}{4}(\Gamma_1-\gamma_1)^2. \quad (11)$$

From the relation $Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \frac{1}{n}\sum_{i=1}^{n} y_i$ and using the inequality of Cauchy-Schwarz for discrete random variables given by $|Cov(X,Y)| \leq \sqrt{Var(X)Var(Y)}$ and inequality (11), we deduce the inequality of Grüss.

Bhatia and Davis shows in [1], the following inequality

$$Var(X) = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_X)^2 \leq (\Gamma_1-\mu_X)(\mu_X-\gamma_1), \quad (12)$$

which represents an improvement of Popoviciu's inequality, because $(\Gamma_1-\gamma_1)^2 \geq 4(\Gamma_1-\mu_X)(\mu_X-\gamma_1)$. Therefore, we will have the first improvement of Grüss inequality given by the following relation:

$$\begin{aligned} &\left|\tfrac{1}{n}\sum_{i=1}^{n} x_i y_i - \tfrac{1}{n}\sum_{i=1}^{n} x_i \tfrac{1}{n}\sum_{i=1}^{n} y_i\right| \leq \\ &\leq \sqrt{(\Gamma_1-\mu_X)(\mu_X-\gamma_1)(\Gamma_2-\mu_Y)(\mu_Y-\gamma_2)} \leq \\ &\leq \tfrac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2). \end{aligned} \quad (13)$$

If X, Y and Z are discrete random variables in finite case, with $X \neq kZ$, then we have from inequality (7), the following relation:

$$\begin{aligned} &[Cov(X,Y)]^2 + \tfrac{[Cov(X,Y)Cov(X,Z)-Cov(Y,Z)Var(X)]^2}{Var(X)Var(Z)-[Cov(X,Z)]^2} \\ &\leq Var(X)Var(Y). \end{aligned} \quad (14)$$

Let $x_1, x_2, ..., x_n$, $y_1, y_2, ..., y_n$, $z_1, z_2, ..., z_n$ be real numbers, assume $x_i \neq kz_i$, for all $i = \overline{1,n}$ and for any real number k. Then applying inequality (14), we deduce second refining of Grüss inequality, given by

$$\left[\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \frac{1}{n}\sum_{i=1}^{n} y_i\right]^2 + S \leq Var(X)Var(Y), \quad (15)$$

where $S = \frac{[A-B]^2}{C}$, with:

$$A = \left(\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \frac{1}{n}\sum_{i=1}^{n} y_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i z_i - \frac{1}{n}\sum_{i=1}^{n} x_i \frac{1}{n}\sum_{i=1}^{n} z_i\right),$$

$$B = \left(\frac{1}{n}\sum_{i=1}^{n} y_i z_i - \frac{1}{n}\sum_{i=1}^{n} y_i \frac{1}{n}\sum_{i=1}^{n} z_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right),$$

and

$$\begin{aligned} C = &\left(\tfrac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\tfrac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right)\left(\tfrac{1}{n}\sum_{i=1}^{n} z_i^2 - \left(\tfrac{1}{n}\sum_{i=1}^{n} z_i\right)^2\right) - \\ &-\left(\tfrac{1}{n}\sum_{i=1}^{n} x_i z_i - \tfrac{1}{n}\sum_{i=1}^{n} x_i \tfrac{1}{n}\sum_{i=1}^{n} z_i\right)^2. \end{aligned}$$

Remark 3.1. In [15], Kechriniotis and Delibasis demonstrated other refinements of the discrete version of Grüss inequality. Zitikis presented in [25] a probabilistic interpretation and another bound for Grüss inequality.

1. If X and Y are discrete random variables in finite case, then there is the following inequality

$$\sqrt{Var(X+Y)} \leq \sqrt{Var(X)} + \sqrt{Var(Y)} \quad (16)$$

Proof. From relation (1), we have:

$$\begin{aligned} &Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = \\ &= \left(\sqrt{Var(X)} + \sqrt{Var(Y)}\right)^2 - \\ &-2\left(\sqrt{Var(X)Var(Y)} - Cov(X,Y)\right). \end{aligned}$$

Applying the inequality of Cauchy-Schwarz for discrete random variables given by

$$|Cov(X,Y)| \leq \sqrt{Var(X)Var(Y)},$$

it follows that

$$Var(X+Y) \leq \left(\sqrt{Var(X)} + \sqrt{Var(Y)}\right)^2,$$

which implies the inequality of the statement. □

Remark 3.2. Inequality (16) in terms of sums becomes

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i-\mu_X-\mu_Y)^2} \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_X)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i-\mu_Y)^2}.$$

Dividing by $\sqrt{\frac{1}{n}}$ and making the following substitutions: $x_i - \mu_X = a_i$ and $y_i - \mu_Y = b_i$, we obtain the inequality

$$\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i{}^2} + \sqrt{\sum_{i=1}^{n} b_i{}^2},$$

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which is the fact the Minkowski inequality, in the case $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} b_i = 0$.

2. If X and Y are discrete random variables in finite case, then there is the following inequality

$$\sqrt{Var(X-Y)} \geq \left|\sqrt{Var(X)} - \sqrt{Var(Y)}\right| \qquad (17)$$

Proof. From relation (3), we have

$$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y) = \\ = (\sqrt{Var(X)} - \sqrt{Var(Y)})^2 + \\ +2(\sqrt{Var(X)Var(Y)} - Cov(X,Y)).$$

Applying the inequality of Cauchy-Schwarz for discrete random variables, we obtain

$$Var(X-Y) \geq (\sqrt{Var(X)} - \sqrt{Var(Y)})^2$$

which implies the inequality of the statement.□

By Ostrowski's inequality (see, [22]), we can estimate the deviation of the values of a smooth function from its mean value. In the same way, Florea and Niculescu established in [10] a variant of Ostrowski's inequality in a normed vector space. But, the set of real numbers is a normed vector space. Therefore, we can write, in terms of random variables, thus:

$$|x_i - E[X]| \leq \tfrac{1}{n}\left[\left(i - \tfrac{n+1}{2}\right)^2 + \tfrac{n^2-1}{4}\right] \max_{1\leq k\leq n-1} |x_{k+1} - x_k|. \qquad (18)$$

This inequality suggests an estimation of the variance, which is given below.

3. If X is a discrete random variable in finite case, then there is the following inequality

$$Var(X) \leq \frac{(n^2-1)(7n^2-8)}{60n^2} \max_{1\leq k\leq n-1} |x_{k+1} - x_k|^2, \qquad (19)$$

where X is a random variable given by $X = \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{1\leq i\leq n}$ with probabilities $P(X = x_i) = p_i = \frac{1}{n}$.

Proof. From relation (18), we have

$$Var(X) = E[(X - E[X])^2] = E[|X - E[X]|^2] = \\ = \tfrac{1}{n}\sum_{i=1}^{n} |x_i - E[X]|^2 \leq \\ \leq \tfrac{1}{n^3} \max_{1\leq k\leq n-1} |x_{k+1} - x_k|^2 \left[\sum_{i=1}^{n}\left[\left(i - \tfrac{n+1}{2}\right)^2 + \tfrac{n^2-1}{4}\right]^2\right] = \\ = \tfrac{(n^2-1)(7n^2-8)}{60n^2} \max_{1\leq k\leq n-1} |x_{k+1} - x_k|^2.$$

Here, we use the equalities from [4]:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2,$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

□

4. If X and Y are discrete random variables in finite case, then there is the following inequality

$$|Cov(X,Y)| \leq \\ \leq \tfrac{(n^2-1)(7n^2-8)}{60n^2} \max_{1\leq k\leq n-1} |x_{k+1} - x_k| \max_{1\leq k\leq n-1} |y_{k+1} - y_k|, \qquad (20)$$

where X and Y are two random variables given by $X = \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{1\leq i\leq n}$ and $Y = \begin{pmatrix} y_i \\ q_i \end{pmatrix}_{1\leq i\leq n}$ with probabilities $P(X = x_i) = p_i = \frac{1}{n}$ and $P(Y = y_i) = q_i = \frac{1}{n}$.

Proof. Applying the inequality of Cauchy-Schwarz for discrete random variables, $|Cov(X,Y)| \leq \sqrt{Var(X)Var(Y)}$ and using the inequality (19), for $Var(X)$ and $Var(Y)$, we deduce inequality (20). □

Remark 3.3. In fact inequality (20) is another Grüss type inequality.

In ([24], Corollary 5), Pečarić gave another result, which characterizes the variance. More precisely: if $x_1 \leq x_2 \leq ... \leq x_n$ or $x_1 \geq x_2 \geq ... \geq x_n$, then

$$n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2 \leq \left\lfloor\frac{n}{2}\right\rfloor\left(n - \left\lfloor\frac{n}{2}\right\rfloor\right)(x_n - x_1)^2. \qquad (21)$$

This inequality helps us to find an estimation of the variance, which is given below.

5. If X is a discrete random variable in finite case, in the above conditions, then there is the following inequality

$$Var(X) \leq \frac{1}{n^2}\left\lfloor\frac{n}{2}\right\rfloor\left(n - \left\lfloor\frac{n}{2}\right\rfloor\right)(\Gamma_1 - \gamma_1)^2, \qquad (22)$$

where $\gamma_1 \leq x_i \leq \Gamma_1$, for all $i = \overline{1,n}$.

Proof. From relation (21) it is easy to see that inequality (22) is demonstrated. □

6. If X and Y are discrete random variables in finite case, in the above conditions, then there is the following inequality

$$|Cov(X,Y)| \leq \frac{1}{n^2}\left\lfloor\frac{n}{2}\right\rfloor\left(n - \left\lfloor\frac{n}{2}\right\rfloor\right)(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2), \qquad (23)$$

where $\gamma_1 \leq x_i \leq \Gamma_1$, $\gamma_2 \leq y_i \leq \Gamma_2$, for all $i = \overline{1,n}$.

Proof. From inequality $|Cov(X,Y)| \leq \sqrt{Var(X)Var(Y)}$ and using the inequality (22) for $Var(X)$ and $Var(Y)$, we deduce inequality (23). □

This inequality is a refinement of Grüss inequality due to Biernacki, Pidek and Ryll-Nardzewski (see, [2], [13]). In [16] and [17], the Lukaszyk-Karmowski metric is a

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function defining a distance between two random variables or two random vectors.

In case when the random variables X and Y are characterized by discrete probability distribution, the Lukaszyk-Karmowski metric D is defined as:

$$D(X,Y) = \sum_i \sum_j |x_i - y_j| P(X = x_i) P(Y = y_i).$$

Next we will use another metric for the set $\mathbb{RV}$. We can look the set $\mathbb{RV}$ as a vector space. The natural way is by introducing and using the standard inner product on $\mathbb{RV}$. The inner product of any two random variables X and Y is defined by

$$\langle X,Y \rangle = Cov(X,Y). \tag{24}$$

The inner product of X with itself is always non-negative. This product allows us to define the "length" of a random variable X through square root:

$$\|X\| = \sqrt{\langle X,X \rangle} = \sqrt{Cov(X,X)} = \sqrt{Var(X)}. \tag{25}$$

This length function satisfies the required properties of a seminorm and is called the Euclidean seminorm on $\mathbb{RV}$. A seminorm allowed to assign zero length to some non-zero vectors. The set $\mathbb{RV}$ with this seminorm is called seminormed vector space. Finally, one can use the norm to define a metric on $\mathbb{RV}$ by

$$d(X,Y) = \|X - Y\| = \sqrt{Var(X - Y)}.$$

This distance function is the Euclidean metric on $\mathbb{RV}$. From relation (16), we have

$$\sqrt{Var(X-Z)} = \sqrt{Var\left((X-Y)+(Y-Z)\right)} \le$$
$$\le \sqrt{Var(X-Y)} + \sqrt{Var(Y-Z)},$$

so, we obtain the inequality of triangle

$$d(X,Z) \le d(X,Y) + d(Y,Z).$$

Properties related to a Hilbert space can be found in [12], and several inequalities in pseudo-Hilbert spaces can be found in [3]. Consequently, the set of random variables $\mathbb{RV}$ forming a structure of Hilbert space, and a seminormed vector space.

4 Some final remarks on Grüss inequality for variance, covariance and coefficient of variation

Izumio and Pečarić (see, [13]) shows the following inequality:

$$\left|\frac{1}{n}\sum_{i=1}^n x_i y_i - \frac{1}{n}\sum_{i=1}^n x_i \frac{1}{n}\sum_{i=1}^n y_i\right| \le \le (\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2) \max_{1\le k\le n-1} \frac{k(n-k)}{n^2},$$

which proved another improvement of Grüss inequality, because $\max\limits_{1\le k\le n-1} \frac{k(n-k)}{n^2} \le \frac{1}{4}$.

In [14], Izumio, Pečarić and Tepeš found others extensions of Grüss inequality. We selected two of them:

$$\left|\frac{1}{n}\sum_{i=1}^n x_i y_i - \frac{1}{n}\sum_{i=1}^n x_i \frac{1}{n}\sum_{i=1}^n y_i\right| \le$$
$$\le \frac{1}{2}\sqrt{(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2)\frac{1}{n}\sum_{i=1}^n \left|x_i - \frac{1}{n}\sum_{i=1}^n x_i\right| \frac{1}{n}\sum_{i=1}^n \left|y_i - \frac{1}{n}\sum_{i=1}^n y_i\right|},$$

$$\left|\frac{1}{n}\sum_{i=1}^n x_i y_i - \frac{1}{n}\sum_{i=1}^n x_i \frac{1}{n}\sum_{i=1}^n y_i\right| \le (\Gamma_2 - \gamma_2)\frac{1}{n}\sum_{i=1}^n \left|x_i - \frac{1}{n}\sum_{i=1}^n x_i\right|,$$

In terms of covariance, we obtain the following inequalities:

$$|Cov(X,Y)| \le (\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2) \max_{1\le k\le n-1} \frac{k(n-k)}{n^2},$$

$$|Cov(X,Y)| \le \frac{1}{2}\sqrt{(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2) E[|X - E[X]|] E[|Y - E[Y]|]} \tag{26}$$

and

$$|Cov(X,Y)| \le \frac{(\Gamma_2 - \gamma_2)}{2} E[|X - E[X]|]. \tag{27}$$

But, we have the relation $Cov(X,X) = Var(X)$, from inequality (26) or (27), it follows that

$$Var(X) \le \frac{1}{2}(\Gamma_1 - \gamma_1) E[|X - E[X]|] \tag{28}$$

Using inequality (18), we deduce that

$$E[|X - E[X]|] \le \frac{n^2 - 1}{3n} \max_{1\le k\le n-1} |x_{k+1} - x_k|.$$

Therefore, we rewrite inequalities (26), (27) and (28), in the following way:

$$|Cov(X,Y)| \le$$
$$\le \frac{n^2-1}{6n}\sqrt{(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2) \max_{1\le k\le n-1}|x_{k+1} - x_k| \max_{1\le k\le n-1}|y_{k+1} - y_k|},$$

$$|Cov(X,Y)| \le \frac{n^2 - 1}{6n}(\Gamma_2 - \gamma_2) \max_{1\le k\le n-1} |x_{k+1} - x_k|,$$

and

$$Var(X) \le \frac{n^2 - 1}{6n}(\Gamma_1 - \gamma_1) \max_{1\le k\le n-1} |x_{k+1} - x_k|.$$

In [20], Mitrinović and Vasić mentioned the following inequality:

$$\min_{1\le i<k\le n} (x_k - x_i)^2 \le \frac{12}{n(n^2-1)}\left[\sum_{i=1}^n x_i^2 - \frac{1}{n}\left(\sum_{i=1}^n x_i\right)^2\right], \tag{29}$$

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and Pečarić in [23] showed the following result:

$$\left|\frac{1}{n}\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\sum_{i=1}^{n}x_i\frac{1}{n}\sum_{i=1}^{n}y_i\right|\le$$

$$\le\frac{1}{12}(n^2-1)\max_{1\le k\le n-1}|x_{k+1}-x_k|\max_{1\le k\le n-1}|y_{k+1}-y_k|, \tag{30}$$

Relation (29) provides a lower bound for the variance, namely:

$$\frac{n^2-1}{12}\min_{1\le i<k\le n}(x_k-x_i)^2\le Var(X),$$

and inequality (30) give an upper bound for the covariance, thus:

$$|Cov(X,Y)|\le$$
$$\le\tfrac{1}{12}\left(n^2-1\right)\max_{1\le k\le n-1}|x_{k+1}-x_k|\max_{1\le k\le n-1}|y_{k+1}-y_k|. \tag{31}$$

Let $CV(X)\equiv\frac{\sqrt{Var(X)}}{E[X]}$ be the coefficient of variation of random variable $X=\begin{pmatrix}x_i\\p_i\end{pmatrix}_{1\le i\le n}$ with probabilities $P(X=x_i)=p_i=\frac{1}{n}$, for any $i=\overline{1,n}$, with $\gamma_1\le x_i\le\Gamma_1$, for all $i=\overline{1,n}$.

Masuyama proved that the inequality

$$CV(X)\le\frac{1}{2}\left(\sqrt{\frac{\Gamma_1}{\gamma_1}}-\sqrt{\frac{\gamma_1}{\Gamma_1}}\right), \tag{32}$$

is equivalent to the well-known Pólya-Szegő inequality

$$\frac{\sum_{i=1}^{n}x_i^2\sum_{i=1}^{n}y_i^2}{\left(\sum_{i=1}^{n}x_iy_i\right)^2}\le\frac{(\Gamma_1\Gamma_2+\gamma_1\gamma_2)^2}{4\Gamma_1\Gamma_2\gamma_1\gamma_2},$$

which is in fact a simple consequence of Grüss inequality, according to [19].

In [18] was given a refinement of inequality (32), thus:

$$CV(X)\le\frac{(\Gamma_1-E[X])(E[X]-\gamma_1)}{E[X]}\le\frac{1}{2}\left(\sqrt{\frac{\Gamma_1}{\gamma_1}}-\sqrt{\frac{\gamma_1}{\Gamma_1}}\right).$$

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References

[1] R. Bhatia, C. Davis, A better bound on the variance, The American Mathematical Monthly **107**, 353-357 (2000).

[2] P. Cerone, S.S. Dragomir, Mathematical inequalities: a perspective, CRC Press, Taylor and Francis Group, New York, (2011).

[3] L. Ciurdariu, Some inequalities in pseudo-Hilbert spaces, Tamkang Journal of Mathematics **42**, 438-492 (2011).

[4] J.H. Conway, R. Guy, The Book of Numbers, Springer, (1996).

[5] S.S. Dragomir, A generalization of Grüss's inequality in inner product spaces and applications, Journal of Mathematical Analysis and Applications **237**, 74-82 (1999).

[6] S.S. Dragomir, Some integral inequalities of Grüss type, Indian Journal of Pure and Applied Mathematics **31**, 397-415 (2000).

[7] S.S. Dragomir, Some Grüss type inequalities in inner product spaces, Journal of Inequalities in Pure and Applied Mathematics **4**, (2003).

[8] N.Elezović, L. Marangunić, J. Pečarić, Some improvements of Grüss type inequality, Journal of Mathematical Inequalities **1**, 425-436 (2007).

[9] J.R. Evans, Statistics, Data Analysis and Decision Modeling, Pearson Prentice Hall, New Jersey, 2007.

[10] A. Florea, C.P. Niculescu, A note on Ostrowski's inequality, Journal of Inequalities and Applications **5**, 459-468 (2005).

[11] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx$, Math. Z., **39**, 215-226 (1935).

[12] P. Halmos, Finite dimensional vector spaces, Springer, (1974).

[13] S. Izumio, J. Pečarić, Some extensions of Grüss's inequality and its applications, Nikohai Math. J. **13**, 159-166 (2002).

[14] S. Izumio, S., J. Pečarić, B. Tepeš, Some extensions of Grüss's inequality, Math. J. Toyama Univ. **26**, 61-73 (2003).

[15] A. Kechriniotis, K. Delibasis, On generalizations of Grüss inequality in inner product spaces and applications, Journal of Inequalities and Applications, 2010, Article ID 167091.

[16] S. Lukaszyk, Measurement metric, examples of approximation applications in experimental mechanics, PhD Thesis, Cracow University of Technology, submitted december, **31**, (2001).

[17] S. Lukaszyk, A new concept of probability metric and its applications in approximation of scattered data sets, Computational Mechanics **33**, 299-304 (2003).

[18] N. Minculete, N. Bârsan-Pipu, A. Raţiu, Characterization of some bounds for several statistical indicators, Jökull Journal **63**, 271-276 (2013).

[19] D.S. Mitrinović, J. Pečarić, Comments on an inequality of M. Masuyama, SUT J. Math. (Formerly TRU Math.) **27**, 89-91 (1991).

[20] D.S. Mitrinović, P.M. Vasić, Analytic inequalities, Springer-Verlag, Berlin/Heidelberg/New York, (1970).

[21] D.S. Mitrinovic, J. Pečarić, A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, (1992).

[22] A. Ostrowski, Über die Absolutabweichung einer differenzierbaren Funktion von ihrem Integalmittelwert, Comment. Math. Helv. **10**, 226-227 (1938).

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[23] J. Pečarić, On the Ostrowski Generalization of Čebyšev's Inequality, Journal of Mathematical Analysis and Application **102**, 479-487 (1984).
[24] J. Pečarić, Some further remarks on the Ostrowski Generalization of Čebyšev's Inequality, Journal of Mathematical Analysis and Application **123**, 18-33 (1987).
[25] R. Zitikis, Grüss inequality, its probabilistic interpretation, and sharper bound, Journal of Mathematical Inequalities, **3**, 15-20 (2009).

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