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Nonlinear Lattice Solitons in Saturable Media

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Abstract: Using computational methods, the numerical existence and nonlinear stability of fundamental solitons in saturable media on crystal and certain type of quasicrystal lattices are investigated. In a certain parameter regime of the lattice depth and the propagation constant, the first nonlinear band-gap structures are obtained and the effect of the DC bias field (external electric field) and the lattice depth to the gap width are analyzed.

Keywords: Saturable NLS, Penrose quasicrystal.

1 Introduction

In the last few years, optical solitons have become the main area for studying solitons' interactions and they are responsible for most of the progress on soliton phenomena because of the ease with which sophisticated experiments can be conducted in a laboratory environment that offers precise control over almost every parameter. Furthermore, the ability to sample the waves directly as they propagate and the availability of numerous material systems that are fully characterized by a set of simple equations result in a field in which theory and experiments make rapid progress.

Recently, there has been considerable interest in studying solitons in systems with periodic potentials or lattices, in particular, those which can be generated in nonlinear optical materials [1,2]. In periodic lattices, solitons can typically form when their propagation constants (or eigenvalues) are within a certain region, so called gap; a concept that is borrowed from the Floquet-Bloch theory for linear propagation. However, the external potential of complex systems can be much more general and physically richer than a periodic lattice. For example, atomic crystals can have various irregularities such as defects and edge dislocations or quasicrystal structures which have long-range orientational order but no translational symmetry [3,4]. In general, when the lattice periodicity is slightly perturbed, the band-gap structure and soliton properties also become slightly perturbed and solitons are expected to exist in much the same way as in the perfectly periodic case [5,6]. On one hand, little is known about the spectrum with quasi-periodic potentials [7]. On the other hand, recently in [8], Nonlinear Schrödinger (NLS) equation with external potentials (lattices) possessing crystal and quasicrystal structures are studied and the numerical existence of fundamental solitons is shown. In the same study, the first nonlinear band-gap structure of the fundamental lattice solitons are computed and their stability properties are investigated by numerical methods. (1+1)D nonlinear Schrödinger equation with saturable nonlinearity is studied both analytically and numerically in the current literature [9]-[11].

In this paper, the numerical existence and nonlinear stability of solitons in crystal and certain quasicrystal potentials (lattices) are investigated. The model is following saturable (2+1)D nonlinear Schrödinger equation with an external potential

$$iu_z(x,y,z)+\Delta u-\frac{E_0 u(x,y,z)}{1+V(x,y)+|u(x,y,z)|^2}=0 \qquad (1)$$

In optics, $u(x,y,z)$ corresponds to the complex-valued, slowly varying amplitude of the electric field in the xy-plane propagating in the z direction, $\Delta u = u_{xx}+u_{yy}$ corresponds to diffraction, E_0 is the applied DC bias field which induces the saturable nonlinearity and $V(x,y)$ is an external optical potential that can be written as the intensity of a sum of N phase-modulated plane waves, i.e. (see [12]),

$$V_N(x,y)=\frac{V_0}{N^2}\left|\sum_{n=0}^{N-1} e^{i\left(x\cos\frac{2\pi n}{N}+y\cos\frac{2\pi n}{N}\right)}\right|^2 \qquad (2)$$

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The potentials for $N = 2,3,4,6$ yield periodic lattices which correspond to standard 2D crystal structures whereas $N = 5,7$ correspond to quasicrystals. In particular, the quasicrystal with $N = 5$ is often called the Penrose tiling.

Recently, Freedman et al. observed solitons in Penrose and other quasicrystal lattices generated by the optical induction method [15].

For the solution of this equation, a fixed-point spectral computational method (spectral renormalization method [8]) is employed which uses the ansatz $u(x,y,z) = f(x,y)e^{-i\mu z}$ where $f(x,y)$ is a real valued function and μ is the propagation constant (eigenvalue) and solves it iteratively in the Fourier space.

In this paper, the numerical existence of fundamental solitons on the periodic and quasicrystal lattices is shown and the band-gap structures are found for periodic and Penrose type potentials. The effects of the potential depth and applied bias field on the gap width are investigated.

Next, the nonlinear stability of the fundamental solitons is investigated. For this reason, the change of the soliton power with respect to the eigenvalue (soliton power P versus μ) is analyzed in order to predict collapse according to the slope condition. The long time (distance) behaviors of the fundamental solitons are also observed by direct simulations of saturable NLS equation, i.e. whether they conserve their shapes, locations and maximum amplitudes or not. To study the nonlinear stability, Eq. (1) is directly computed over a long distance (Finite difference method is used on derivatives u_{xx} and u_{yy}; fourth-order Runge-Kutta method is used to advance in z.) for both periodic and Penrose type potentials. The initial conditions were fundamental solitons with %1 random noise in the amplitude and phase.

2 Spectral Renormalization Method

Spectral renormalization method is essentially a Fourier iteration method. The idea of this method was proposed by Petviashvili in [26]. The idea behind the Petviashvili's method is to transform the underlying governing equation to Fourier space and determine a convergence factor based upon the degree (homogeneity) of a single nonlinear term. This method has been extensively used to find localized solutions in a wide range of nonlinear systems. This method can be successfully applied to nonlinear systems only if the degree of the nonlinearity is fixed in the associated evolution equation. In fact, in nonlinear optics, many equations involve nonlinearities with different homogeneities, such as cubic-quintic, or even lack of homogeneity, as in saturable nonlinearity.

Ablowitz and Musslimani [27] proposed a generalized numerical scheme for computing solitons in nonlinear waveguides called spectral renormalization. The essence of the method is to transform the governing equation into Fourier space and find a nonlinear nonlocal integral equation coupled to an algebraic equation. The coupling prevents the numerical scheme from diverging. The optical mode is then obtained from an iteration scheme, which converges rapidly. This method can efficiently be applied to a large class of problems including higher order nonlinear terms with different homogeneities.

In this section, we consider the saturable (2+1)D nonlinear Schrödinger equation with a potential:

$$iu_z(x,y,z) + \Delta u - \frac{E_0 u(x,y,z)}{1+V(x,y)+|u(x,y,z)|^2} = 0 \qquad (3)$$

Using the ansatz $u(x,y,z) = f(x,y)e^{-i\mu z}$ gives

$$\mu f e^{-i\mu z} + (f_{xx} + f_{yy})e^{-i\mu z} - \frac{E_0 f e^{-i\mu z}}{1+V+|f|^2} = 0\,. \qquad (4)$$

Multiplying both sides of this equation by $e^{i\mu z}$ results in

$$\mu f + f_{xx} + f_{yy} - \frac{E_0 f}{1+V+|f|^2} = 0\,. \qquad (5)$$

By applying Fourier transformation, one obtains

$$\mu \hat{f} - (k_x^2 + k_y^2)\hat{f} - \mathscr{F}\left(\frac{E_0 f}{1+V+|f|^2}\right) = 0\,. \qquad (6)$$

To prevent singularities in the denominator in future calculations, the term $r\hat{f}$ is added to and subtracted from (6):

$$(\mu + r)\hat{f} - (k_x^2 + k_y^2 + r)\hat{f} - \mathscr{F}\left(\frac{E_0 f}{1+V+|f|^2}\right) = 0\,. \qquad (7)$$

Solving for the $\hat{f}$ in the second term above yields

$$\hat{f} = \frac{(\mu + r)\hat{f} - \mathscr{F}\left(\frac{E_0 f}{1+V+|f|^2}\right)}{k_x^2 + k_y^2 + r}\,. \qquad (8)$$

In order to get a solution that its amplitude does not grow nor tends to zero with each iteration we introduce a new variable $w(x,y)$ such that $f(x,y) = \lambda w(x,y)$. Making the substitution $\hat{f}(x,y) = \lambda \hat{w}(x,y)$ gives

$$\hat{w} = \frac{(\mu + r)\hat{w} - \mathscr{F}\left(\frac{E_0 w}{1+V+|\lambda|^2|w|^2}\right)}{k_x^2 + k_y^2 + r}\,. \qquad (9)$$

where λ is a non-zero constant to be determined. We use the fixed point iteration method in order to find w. For this purpose, $\hat{w}$ can be calculated using the following iteration scheme:

$$\hat{w}_n = \frac{(\mu + r)\hat{w}_{n-1} - \mathscr{F}\left(\frac{E_0 w_{n-1}}{1+V+|\lambda|^2|w_{n-1}|^2}\right)}{k_x^2 + k_y^2 + r}\ , \quad n \in Z^+ \qquad (10)$$

with the initial condition taken as a Gaussian

$$w_0 = e^{-((x-x_0)^2+(y-y_0)^2)} \qquad (11)$$

Here, x_0 and y_0 define the location of the initial condition. In order to center the initial condition on the lattice maximum (the one that appears at the center of the lattice), one should take $x_0 = y_0 = 0$.

The iteration continues until $|\hat{w}_n - \hat{w}_{n-1}| < 10^{-12}$ and that the obtained $\hat{w}$ satisfies (9) with an absolute error less than 10^{-6}. Convergence is obtained quickly when the mode is strongly localized in the band gap. This occurs when the linearized spectrum is in the semi-infinite band gap.

However, λ is unknown and hence must be calculated for each iteration.

Multiplying (9) by $k_x^2 + k_y^2 + r$ leads to

$$(k_x^2 + k_y^2 + r)\hat{w} = (\mu + r)\hat{w} - \mathscr{F}\left(\frac{E_0 w}{1 + V + |\lambda|^2|w|^2}\right) . \tag{12}$$

After moving all terms to the left side, one has

$$(k_x^2 + k_y^2 - \mu)\hat{w} + \mathscr{F}\left(\frac{E_0 w}{1 + V + |\lambda|^2|w|^2}\right) = 0 . \tag{13}$$

Multiplying by the conjugate of $\hat{w}$, i.e. by $\hat{w}^*$ results in

$$(k_x^2 + k_y^2 - \mu)|\hat{w}|^2 + \mathscr{F}\left(\frac{E_0 w}{1 + V + |\lambda|^2|w|^2}\right)\hat{w}^* = 0 . \tag{14}$$

Finally, by integrating this equation, one gets

$$\iint_{-\infty}^{\infty}(k_x^2 + k_y^2 - \mu)|\hat{w}|^2 dk + \iint_{-\infty}^{\infty}\mathscr{F}\left(\frac{E_0 w}{1 + V + |\lambda|^2|w|^2}\right)\hat{w}^* dk = 0 . \tag{15}$$

The equation (15) can not be solved analytically for λ. We need a numerical method such as the Newton-Raphson method for finding roots of the equation (15). Employing the Newton-Raphson Method to the equation (15) gives following iteration scheme:

$$\lambda_n = \lambda_{n-1} - \frac{F(\lambda_{n-1})}{F'(\lambda_{n-1})} \ , \ n \in Z^+ \tag{16}$$

Here $F(\lambda)$ and $F' = \frac{dF}{d\lambda}$ are given as follows:

$$F(\lambda) = \iint_{-\infty}^{\infty}(k_x^2 + k_y^2 - \mu)|\hat{w}|^2 dk + \iint_{-\infty}^{\infty}\mathscr{F}\left(\frac{E_0 w}{1 + V + \lambda^2|w|^2}\right)\hat{w}^* dk , \tag{17}$$

$$F'(\lambda) = \iint_{-\infty}^{\infty}(-2\lambda)\mathscr{F}\left(\frac{E_0 w|w|^2}{(1 + V + \lambda^2|w|^2)^2}\right)\hat{w}^* dk . \tag{18}$$

When $n = 1$ an initial quess for λ_0 is required. We chose λ_0 to be 1 and the stopping criterions are $|\lambda_n - \lambda_{n-1}| < 10^{-12}$ and that the obtained λ satisfies (15) with an absolute error less than 10^{-6}.

Once $\hat{w}$ is obtained from (10) by means of (16) to calculate λ for each iteration, the desired soliton is $f(x,y) = \lambda w(x,y) = \lambda \mathscr{F}^{-1}(\hat{w})$.

The first nonlinear gap is the edge of the parameter regime of the potential depth V_0 and eigenvalue μ for which the numerical method converges to a localized mode. For Kerr nonlinearity, one can observe that, in the band region, beyond a certain threshold value of the eigenvalue, the numerical method yields an extended state which is called the *Bloch wave* region. This phenomenon doesn't occur for saturable nonlinearity. Presence of saturation helps the modes to stay localized but the convergence is not achieved beyond a threshold value of μ.

3 Numerical Existence of Fundamental Solitons

In this section, fundamental lattice solitons obtained by means of the spectral renormalization method will be demonstrated. A comparison of the amplitudes and the profiles of the solitons obtained on each lattice maxima and an observation of the effect of the increment of potential depth V_0 on the periodic lattice are also analyzed in this section.

In Fig. 1, 3D view of the soliton obtained on the lattice maximum with the parameters $E_0 = 8$, potential depth $V_0 = 1$ and the eigenvalue $\mu = 4$ is depicted. In the same figure, contour plot of the soliton superimposed on the initial condition and underlying periodic lattice ($N = 4$) is shown. Fig. 2 and Fig. 3 display the same outcomes for Penrose-5 and Penrose-7 potentials respectively. Note that for each case, the initial condition is taken on the lattice maximum and after the iteration, a localized mode is obtained at the same location.

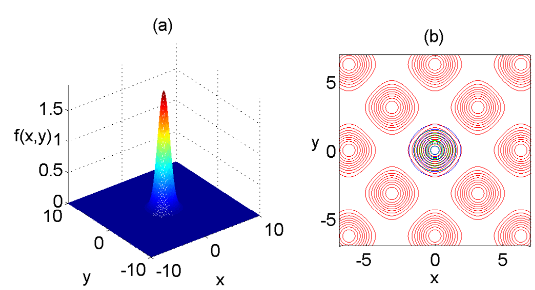

Fig. 1: (Color online) (a) 3D view of the soliton on the periodic lattice ($N = 4$) maximum with $E_0 = 8$, $V_0 = 1$ and $\mu = 4$; and (b) its contour plot (blue) superimposed on the initial condition (green) and underlying lattice (red).

In order to compare the maximum amplitudes and the profiles of the fundamental solitons obtained on the

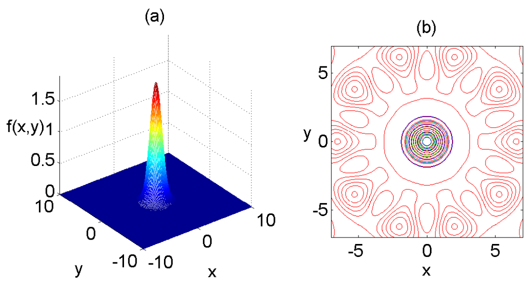

Fig. 2: (Color online) (a) 3D view of the soliton on the Penrose-5 lattice ($N = 5$) maximum with $E_0 = 8$, $V_0 = 1$ and $\mu = 4$; and (b) its contour plot (blue) superimposed on the initial condition (green) and underlying lattice (red).

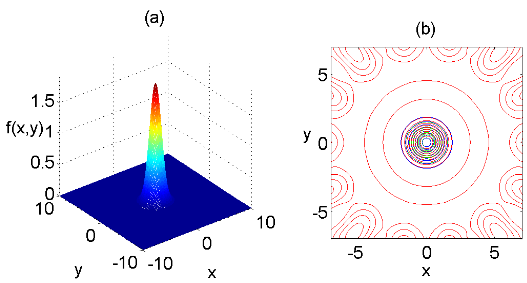

Fig. 3: (Color online) (a) 3D view of the soliton on the Penrose-7 lattice ($N = 7$) maximum with $E_0 = 8$, $V_0 = 1$ and $\mu = 4$; and (b) its contour plot (blue) superimposed on the initial condition (green) and underlying lattice (red).

periodic, Penrose-5 and Penrose-7 lattices, on axis mode profiles are depicted in Fig. 4.

It can be observed from Fig. 4 that the amplitudes of the fundamental solitons slightly increase as the numerical value of N increases. On the other hand, the shapes of the fundamental solitons do not exhibit any difference as N increases despite the fact that the fundamental lattice solitons in Kerr medium show changes in shape. For example, Penrose-5 solitons are shown to have dimples when their eigenvalues are close to gap edge [8].

In order to investigate the effect of the potential depth (V_0) on the fundamental solitons, the periodic lattice case is considered and it is shown that the amplitude of the fundamental soliton located on the lattice maximum decreases as the potential depth increases. An example for this fact is illustrated in Fig. 5 where $N = 4$ (periodic lattice) and V_0 is increased from 1 to 4. This also holds true for Penrose-5 and Penrose-7 solitons.

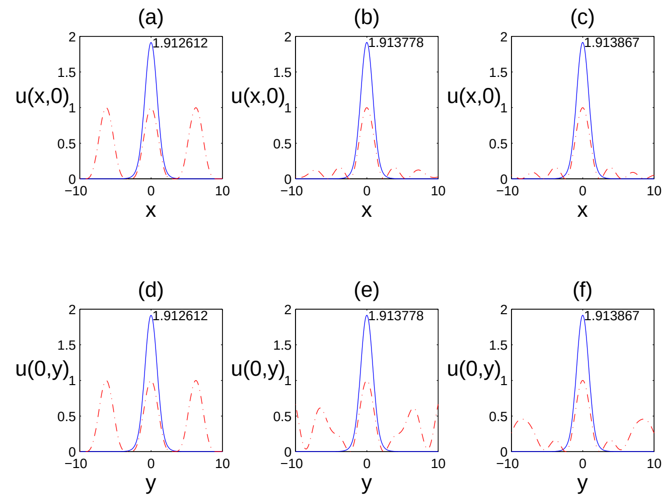

Fig. 4: (Color online) On axis profiles of the solitons obtained with $E_0 = 8$, $V_0 = 1$ and $\mu = 4$ at the maximum of the potentials: (a) $N = 4$, (b) $N = 5$, (c) $N = 7$ along x-axis; (d) $N = 4$, (e) $N = 5$, (f) $N = 7$ along y-axis.

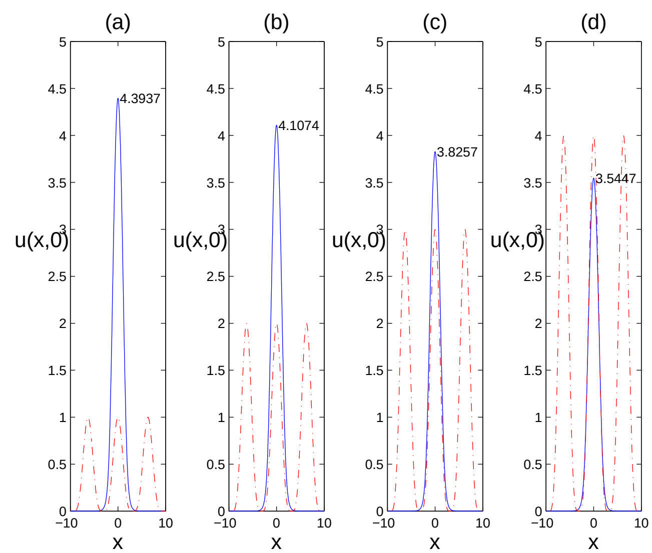

Fig. 5: (Color online) On axis profiles of the solitons at the maximum of the periodic potential ($N = 4$) along x-axis obtained with $E_0 = 8$, $\mu = 2.2$ and (a) $V_0 = 1$; (b) $V_0 = 2$; (c) $V_0 = 3$; (d) $V_0 = 4$.

4 Band-gap Structures of Fundamental Lattice Solitons

Band-gap structure is a linear concept that first appeared in Floquet theory as well as condensed matter theory (i.e. diffraction of X-rays through atomic crystals).

In nonlinear optics, the propagation of the soliton on a lattice depends on the depth of the lattice and the propagation constant (or the eigenvalue). In the spectrum of the propagation, there are strips (or regions), where a localized structure is obtained (called *gap*s) and strips where the solution is not a localized structure but an extended state (called *band*s). The whole spectrum is called *band-gap structure* of the lattice or the potential.

The first nonlinear gap of a lattice is the region where localized solutions (solitons) exist. In a certain parameter regime of the potential depth V_0 and the propagation constant μ, spectral renormalization method converges to a localized bound state, i.e. to the solution of (1) that is bounded and decays to zero. By fixing the potential depth and increasing the eigenvalue, both the convergence and the localization of the solution are checked. In this way, the first nonlinear gap edge of the lattices considered in this work is obtained. In this case, although the mode stays localized during the iteration (due to the saturation term) but the convergence is not achieved beyond a certain threshold value of the eigenvalue μ which is considered as the gap edge.

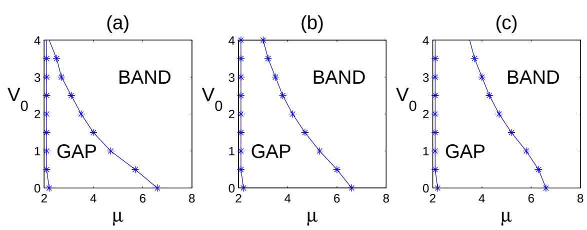

Fig. 6: (Color online) Band-gap structure of the solitons obtained at the lattice maximum with $E_0 = 8$ and the potentials: (a) $N = 4$; (b) $N = 5$; (c) $N = 7$.

In Fig. 6, the band-gap structures are depicted for periodic, Penrose-5 and Penrose-7 solitons in separate graphs, located on the (absolute) lattice maxima for the fixed value of $E_0 = 8$. For a more clear comparison of the band-gap structures, those obtained band-gap structures are depicted on top of each other in Fig. 7.

It is observed, especially from the Fig. 7, that increasing N expands the gap region. This shows the fact that the gap regions are wider for quasicrystal type lattices than that of the periodic type lattice.

In order to observe the effect of the bias field E_0 to the band-gap structure, the potential depth $V_0 = 1$ is fixed and by the use of the SR method, the existence of fundamental solitons for increasing values of E_0 for the periodic, Penrose-5 and Penrose-7 lattices are investigated.

For small E_0 values, no soliton could be obtained. As E_0 gets greater, the number of eigenvalues for which a soliton exists increases. In other words, the gap width increases as E_0 increases.

Above mentioned facts are demonstrated in Fig. 8.

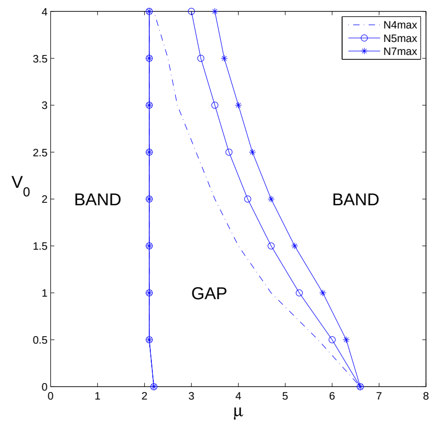

Fig. 7: (Color online) Band-gap structure of the solitons obtained with $E_0 = 8$ and the potentials $N = 4$, $N = 5$ and $N = 7$ at the lattice maximum (superimposed)

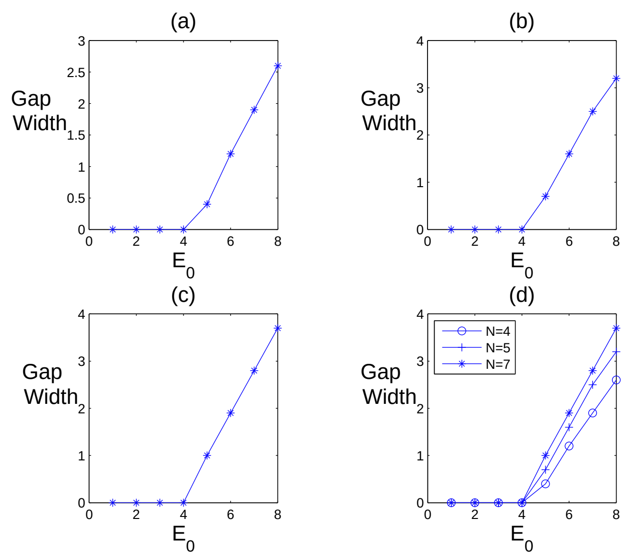

Fig. 8: (Color online) Gap width versus E_0 graph of the solitons obtained at the lattice maximum with $V_0 = 1$ and the potentials: (a) $N = 4$; (b) $N = 5$; (c) $N = 7$; (d) $N = 4$, $N = 5$, $N = 7$ (superimposed).

In order to explore the effect of the potential depth on the gap width, the potential depth V_0 versus gap width graphs for the periodic potential $N = 4$ is plotted. Those graphs for fundamental solitons on lattice maximum are shown in Fig. 9. It can be seen that the gap width decreases as V_0 increases. Same phenomenon occurs for quasicrystals as well.

When Fig. 6 and Fig. 7 are considered, it is observed that the Penrose-7 type potential brings out a bigger gap region than the Penrose-5 type potential and the Penrose-5

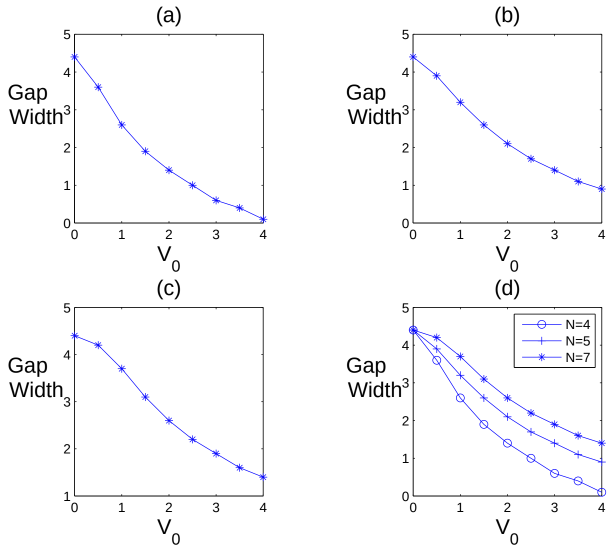

Fig. 9: (Color online) Gap width versus V_0 graph of the solitons obtained at the lattice maximum with $E_0 = 8$ and the potentials: (a) $N = 4$; (b) $N = 5$; (c) $N = 7$; (d) $N = 4$, $N = 5$, $N = 7$ (superimposed).

type potential brings out a bigger gap region than periodic $N = 4$ type potential. This holds also true for other cases, i.e. for different E_0 values. So, one can claim that the gap widths increase as N increases.

5 Nonlinear Stability Analysis of Fundamental Solitons

In this section, the nonlinear stability properties of the fundamental lattice solitons are explored. One way to do this is the power analysis. The *power* is defined as

$$P = \iint_{-\infty}^{\infty} |f(x,y)|^2 dxdy \quad (19)$$

and plays an important role in determining the stability of the soliton $f(x,y)$. In [28], Vakhitov and Kolokolov proved that a necessary condition for the linear stability of the soliton $f(x;\mu)$ is

$$\frac{dP}{d\mu} < 0\,. \quad (20)$$

In other words, the soliton is stable only if its power decreases with increasing propagation constant μ. This condition is called the *slope condition*.

Key analytic results on nonlinear stability were obtained in [29,**?**]. They proved that the necessary conditions for nonlinear stability are the slope condition (20) and the spectral condition. Furthermore, it is well known that a necessary condition for collapse in the 2D cubic NLS equation is that the power of the beam exceeds the critical power $P_c \approx 11.7$ [31].

The fundamental solitons of the NLS equation can become unstable in two ways:

1. Is the slope condition not satisfied, this leads to a *focusing instability*.
2. Is the spectral condition associated with the eigenvalue problem (see [32]) not satisfied, this leads to a *drift instability*.

In order to investigate the nonlinear stability, first approach is trying to predict collapse by using the VK criterion by plotting the soliton power P versus the propagation constant μ graph.

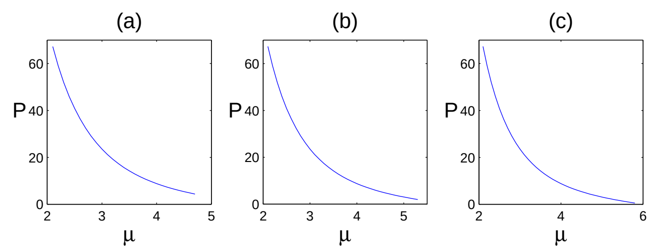

Fig. 10: (Color online) Power versus μ graph of the solitons obtained at the lattice maximum with $E_0 = 8$, $V_0 = 1$ and the potentials: (a) $N = 4$; (b) $N = 5$; (c) $N = 7$.

As it can be seen from Fig. 10, the power decreases as μ increases for all three different potentials. Consequently, all the solitons obtained from the numerical solution of the Eq. (1) by the spectral renormalization method explained in the previous section are expected to be stable as they all satisfy the Vakhitov-Kolokolov stability criterion. However, the VK-condition is not sufficient. The soliton may undergo a drift instability, that is, the soliton might move from the lattice maximum towards a nearby lattice minimum during the direct simulation. One way to check this is to let the soliton move along the z-axis and see whether it conserves its maximum amplitude and its location.

In this context,the center of mass is monitored as

$$(<x>,<y>) = \frac{1}{P}\iint_{-\infty}^{\infty}(x,y)|u|^2 dxdy\,. \quad (21)$$

Here, $P \equiv P[u] := \iint_{-\infty}^{\infty}|u|^2 dxdy$ is the soliton power.

To study whether there exists a drift instability or not, Eq. (1) is computed over a long distance. For this purpose, a random noise of 1% is added to the soliton; finite difference method is used on derivatives u_{xx} and u_{yy}, and fourth order Runge-Kutta method is employed to advance in z.

A fundamental soliton is considered nonlinearly stable if it conserves its location and maximum amplitude during the direct simulation. Therefore, the maximum amplitude and centers of mass versus the propagation distance graphs are plotted for the fundamental solitons.

6 Stability Analysis of Solitons on Lattice Maximum

Following are sample outcomes of the nonlinear stability test for some parameters and different potentials. More precisely, Fig. 11 depicts the evolution of the soliton obtained on the lattice maximum of the periodic potential ($N = 4$) with $E_0 = 8$, $V_0 = 1$ and $\mu = 4$, the change of x and y coordinates of the center of mass over z, cross section along the diagonal axis of the soliton superimposed on the potential at $z = 10$ and contour plot of the soliton superimposed on the potential after the propagation; Fig. 12 and Fig. 13 depict the same but for the Penrose-5 and Penrose-7 potentials respectively. Note that for each case, the initial condition is taken on the lattice maximum, so are the obtained solitons.

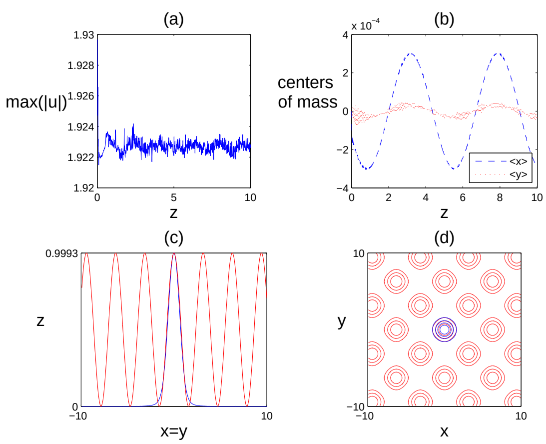

Fig. 11: (Color online) Evolution of the soliton obtained with $E_0 = 8, V_0 = 1, \mu = 4$ and the potential $N = 4$. (a) Peak amplitude $max_{x,y}|u(x,y,z)|$ as a function of the propagation distance z; (b) Change of x and y coordinates of the center of mass over z; (c) Cross section along the diagonal axis of the soliton at the maximum superimposed on the potential $N = 4$ at $z = 10$; (d) Contour plot of the soliton at the maximum superimposed on the potential $N = 4$ at $z = 10$.

As it can be seen from Fig. 11, Fig. 12 and Fig. 13, peak amplitudes of the fundamental solitons slightly oscillate with the propagation distance z and the centers of mass almost stay at the same place. This suggests that the fundamental solitons for all three lattices are nonlinearly stable.

In contrary, the existence of nonlinearly unstable solitons of the NLS equation with Kerr nonlinearity for the periodic potential $N = 4$ and Penrose-5 potential were shown in [8].

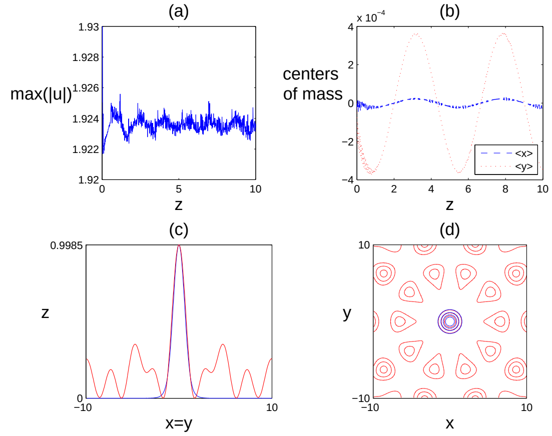

Fig. 12: (Color online) Evolution of the soliton obtained with $E_0 = 8, V_0 = 1, \mu = 4$ and the potential $N = 5$. (a) Peak amplitude $max_{x,y}|u(x,y,z)|$ as a function of the propagation distance z; (b) Change of x and y coordinates of the center of mass over z; (c) Cross section along the diagonal axis of the soliton at the maximum superimposed on the potential $N = 5$ at $z = 10$; (d) Contour plot of the soliton at the maximum superimposed on the potential $N = 5$ at $z = 10$.

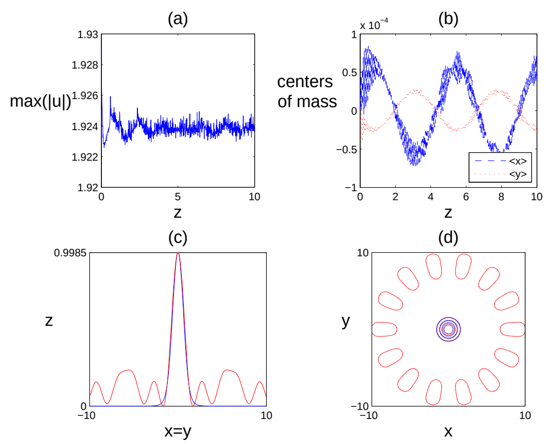

Fig. 13: (Color online) Evolution of the soliton obtained with $E_0 = 8, V_0 = 1, \mu = 4$ and the potential $N = 7$. (a) Peak amplitude $max_{x,y}|u(x,y,z)|$ as a function of the propagation distance z; (b) Change of x and y coordinates of the center of mass over z; (c) Cross section along the diagonal axis of the soliton at the maximum superimposed on the potential $N = 7$ at $z = 10$; (d) Contour plot of the soliton at the maximum superimposed on the potential $N = 7$ at $z = 10$.

7 Conclusion

The purpose of this study was to investigate the existence and stability properties of solitons in periodic ($N = 4$) and certain quasicrystal ($N = 5$ and $N = 7$) lattices.

Firstly, the solutions of saturable NLS equation with an external potential are obtained by means of the spectral renormalization method. The band-gap structure of the lattice solitons are obtained for periodic and Penrose type potentials. The effects of the bias field E_0 and the potential depth V_0 on the gap width are investigated and depicted in some figures and it is shown that on one hand, increasing the bias field E_0 expands the gap width, on the other hand, increasing the potential depth V_0 decreases the gap width. Another result is that increasing N expands the gap width.

After obtaining fundamental solitons, their nonlinear stability properties are investigated according to Vakhitov-Kolokolov stability criterion and by direct simulations. It is observed that direct simulations are in good agreement with Vakhitov-Kolokolov stability criterion.

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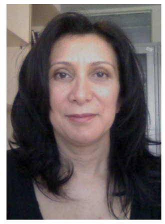

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