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Quantum Entanglement and Information Quantifier for Correlated and Uncorrelated Two-Mode Field State

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Abstract: In this article, we consider the time evolution of the quantum Fisher information in the context of the interaction between two-level atom and two-modes electromagnetic field. We discuss the correlation between the quantum Fisher information, its flow and quantum entanglement for the system under consideration. Analytic results under certain parametric conditions are obtained, by means of which we analyze the influence of different parameters on the von Neumann entropy and quantum Fisher information. We investigate different forms of the two-mode field states such as uncorrelated two-mode coherent states and two-mode squeezed vacuum states. We explore an interesting monotonic relation between the quantum Fisher information and nonlocal correlation behavior of the atom-two modes field interaction under the estimator parameter and field type effect during the time evolution

Keywords: quantum Fisher information, Fisher flow, correlated and uncorrelated two-mode states, entanglement

1 Introduction

Entanglement is the main part of quantum information and computation. In this way entanglement is a property of correlations between two or more quantum systems [1]. These correlations defy classical description and associated with intrinsically quantum phenomena. This nonlocal nature of entanglement has also been identified as an essential resource for many novel tasks such as quantum computation, quantum teleportation [2], quantum cryptography [3] and more recently, one-way quantum computation [4] and quantum metrology [5]. These quantum information tasks cannot be carried out by classical resources and they rely on entangled states. This recognition led to an intensive search for mathematical tools that would enable a proper quantification of this resource [6]. In particular, it is of primary importance to test whether a given quantum state is separable or entangled. For this quantification, several entanglement measures have been proposed such as concurrence [7,8], entanglement of formation [9,10], negativity [11,12], etc. 1). The current and new applications of the quantum Fisher information (**QFI**) in the new and important of quantum technologies field which is called quantum metrology. In this regard, quantum metrology is discussed for entangled coherent states, and gives an improved for the phase estimation and smallest variance in the phase parameter in comparison to NOON, BAT and optimal states. This work has modified to discuss the quantum metrology resources for two modes entangled spin-coherent states [13].

Parameter estimation is a significant pillar of different branches of science and technology, and developed new techniques in measurement for parameter sensitivity have often led to scientific breakthroughs and technological advancement. There is a great deal of work on phase estimation addressing the practical problems of state generation, loss, and decoherence [14,15,16,17,18,19]. Fisher information lies at the heart of a parameter estimation theory that was originally introduced by Fisher [20]. It provides in particular a bound to distinguish the members of a family of probability distributions. When quantum systems are involved, especially for problems in which the quantity of interest is not directly accessible, the optimal measurement may be found using tools from

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quantum estimation theory. The quantum version of the Cramér-Rao inequality has been established and the lower bound is imposed by QFI [21]. An abstract quantity that measures the maximum information about a parameter ϕ that can be extracted from a given measurement procedure. Since the mathematical treatment of the lower bound in physical problems has been clarified [22,23], the best resource for the phase estimation has been discussed [24,25]. Quantum communication theory including quantum estimation and quantum detection may predict a possibility to beat such a limit [26]. It is shown that, surprisingly, for states with high temperature, phase estimation is better for states with only classical correlations [27,28] than with entangled GHZ diagonal states [29].

It is well known that, the correlation between quantum entanglement and Fisher information **(FI)**, as we know about a certain parameter in a quantum state, has not been studied widely. However, there are some studies to quantify the pure state entanglement by using FI. In this regard, the entanglement evaluation with atomic classical Fisher information has been investigated [30]. It has been shown that entanglement of a two-level atom can be quantified by atomic FI and their marginal distribution. Furthermore, the correlation between the FI and quantum entanglement during the time evolution for a trapped ion in laser field. It is found that FI is an important tool to study single qubit dynamics as an indicator of entanglement under certain conditions [31]. Also, the time evolution of the QFI of a system whose the dynamics is described by the phase-damped model has studied [32]. It observed that there is an interesting monotonic relation between the QFI and nonlocal correlation behavior measured by the negativity depending on choosing the estimator parameter during the time evolution.

In this present article, our main interest is to investigate and discuss in detail the time evolution of the QFI, QFI flow and the entanglement between two-level atom and the input field initially in correlated and uncorrelated two-mode states.

The article will be outlined in the following order. The main concepts of the two-mode coherent states (TMCS) and two-mode squeezed vacuum states (TMSVS) will be presented in Sec. II. In section III, we introduce the model of the interaction between a two-level atom and two-mode electromagnetic field in the presence of nonlinear terms such as Kerr like medium and detuning parameter. In Sec. IV we evaluate and discuss the main results. Finally, we conclude our work in Sec. VI.

2 Correlated and Uncorrelated two-mode states

The quantum behavior of a two-level atom interacting with correlated two modes has been investigated [33]. The importance of correlated states such as two-mode pair coherent states [34,35,36] or two-mode squeeze vacuum state [37,38] of light lies in their close connection to two-photon nonlinear optical processes. Such correlated states play an important role in quantum mechanics and quantum information processing. For example, the dissipation of a two-mode squeezed vacuum state (**TMSVS**) in the single-mode amplitude damping channel has been investigated [39], where it is found that the outcome state is no more a pure state, but an entangled mixed state. Also, the quantum dense coding via a two-mode squeezed-vacuum state is studied [40]. It is shown that the decoding at a receiver side is performed by simultaneous measurement of two quadrature components of the two-mode state. where the information can be transmitted without error in the strong-squeezing limit because the two-mode squeezed-vacuum state is partially entangled.

2.1 Two mode squeezed vacuum states

The two-mode squeezed state is defined as [41]

$$|\Psi\rangle_{sq} = \hat{D}(\hat{a}_1,\alpha_1)\hat{D}(\hat{a}_2,\alpha_2)\hat{S}(r,\varphi)\,|0\rangle\,, \tag{1}$$

where $\hat{D}(\hat{a}_i,\alpha_i)$ is the common displacement operator

$$\hat{D}(\hat{a}_i,\alpha_i) = \exp\left(\alpha_i\hat{a}_i^{\dagger} - \alpha_i^{*}\hat{a}_i\right), \quad i=1,2 \tag{2}$$

and $\hat{S}(r,\varphi)$ is the two-mode squeezing operator

$$\hat{S}(r,\varphi) = \exp\left[r(-\hat{a}_1\hat{a}_2e^{-i\varphi} + \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}e^{i\varphi})\right]. \tag{3}$$

The expansion of equation (1) into number states is quite complicated; however, we would like to consider the simplest form of the two-mode squeezed state [41,42], namely the two-mode squeezed vacuum i.e., without any displacement ($\alpha_1=\alpha_2=0$):

$$TMSVS = \hat{S}(r,\varphi)\,|0\rangle = (\cosh r)^{-1}\sum_n \left(e^{i\varphi}\tanh r\right)^n |n,n\rangle\,, \tag{4}$$

$$P_{n_1,n_2}(0) = \frac{\left|e^{i\varphi}\tanh r\right|^{n_1+n_2}}{\cosh^2 r}\delta_{n_1,n_2}. \tag{5}$$

In the case of the two-mode squeezed vacuum, the double summation becomes over n due to correlation of the two modes which keeps the equal n numbers in every mode, then equation (5) is written as

$$P_n(0) = \frac{\left|e^{i\varphi}\tanh r\right|^{2n}}{\cosh^2 r}. \tag{6}$$

2.2 Two mode coherent states

We use uncorrelated two-mode coherent states (**TMCS**) as an example of the uncorrelated field states. The uncorrelated coherent state is given by

$$|\alpha_1,\alpha_2\rangle = \sum_{n_1,n_2=0}^{\infty} \mathscr{C}_{n_1,n_2}|n_1,n_2\rangle, \quad (7)$$

where the amplitude $\mathscr{C}_{n_1,n_2}$(assumed to be real) in equation (7) can be written as

$$\mathscr{C}_{n_1,n_2} = \exp\left[-\left(\frac{\bar{n}_1+\bar{n}_2}{2}\right)\right]\sqrt{\frac{(\bar{n}_1)^{n_1}(\bar{n}_2)^{n_2}}{n_1!n_2!}}, \quad (8)$$

$\bar{n_i} = |\alpha_i|^2,\ i = 1,2.$

In the next section, we shall use the TMSVS and TMCS given by Eqs. (4), (7) as an initial state of the input field in the interaction with two-level atom.

3 Model and its dynamics

We consider a system of a two-level atom interacting with two-modes of the electromagnetic field. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ denote the excited state and ground state of the atom, respectively. ω_A is the transition frequency between state $|\uparrow\rangle$ and $|\downarrow\rangle$, while ω_1 and ω_2 denote the frequencies of the two electromagnetic modes. The total Hamiltonian for this system in the rotating wave approximation (RWA) can be written as

$$H = \Delta S_z + \sum_{j=1}^{2}\left(\omega_j\hat{a}_j^\dagger\hat{a}_j + \chi_j\hat{a}_j^{\dagger 2}\hat{a}_j^2\right) + \hbar\lambda\left[\hat{A}S_+ + S_-\hat{A}^\dagger\right], \quad (9)$$

where we take $\hbar = 1$, $\hat{a}_j^\dagger$ and $\hat{a}_j$ are the photon creation and annihilation operators of the $j^{\underline{th}}$ mode. λ is the coupling parameter between the atom and two-mode field, while $\omega_j (j = 1,2)$ is the frequency of the $j^{\underline{th}}$ mode of the radiation field, $\hat{A} = \hat{a}_1\hat{a}_2 \otimes f(\hat{a}_1^\dagger\hat{a}_1, \hat{a}_2^\dagger\hat{a}_2)$ [43]. We denote by χ_j $(j = 1,2)$, the dispersive part of the third-order nonlinearity of the Kerr-like medium of $j^{\underline{th}}$ mode, and $f(\hat{a}_1^\dagger\hat{a}_1, \hat{a}_2^\dagger\hat{a}_2)$ represents an arbitrary intensity-dependent coupling. Also, $\hat{S}_\pm$ and S_z are the atomic spin operators defined by

$$\hat{S}_+ = |\uparrow\rangle\langle\downarrow|,\ \hat{S}_- = |\downarrow\rangle\langle\uparrow| \text{ and } \hat{S}_z = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right). \quad (10)$$

The above Hamiltonian can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}_{in}, \quad (11)$$

with

$$\hat{H}_0 = \omega_1\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{S}_z\right) + \omega_2\left(\hat{a}_2^\dagger\hat{a}_2 + \hat{S}_z\right), \quad (12)$$

which describes the free Hamiltonian. While the interaction Hamiltonian part can be written as

$$\hat{H}_{in} = \Delta\hat{S}_z + \chi_1\hat{a}_1^{\dagger 2}\hat{a}_1^2 + \chi_1\hat{a}_2^{\dagger 2}\hat{a}_2^2 + \lambda\left[\hat{A}\hat{S}_+ + \hat{S}_-\hat{A}^\dagger\right], \quad (13)$$

where the detuning $\Delta = \omega_A - \omega_1 - \omega_2$. For the two bases $|\uparrow, n_1, n_2\rangle$ and $|\downarrow, n_1+1, n_2+1\rangle$ the interaction Hamiltonian (13) can be transformed into the following form

$$\hat{H}_{in} = \begin{bmatrix} \frac{\Delta}{2} + \sum_{j=1}^{2}\chi_j n_j(n_j-1) & \lambda f\sqrt{(n_1+1)(n_2+1)} \\ \lambda f\sqrt{(n_1+1)(n_2+1)} & -\frac{\Delta}{2} + \sum_{j=1}^{2}\chi_j n_j(n_j+1) \end{bmatrix}. \quad (14)$$

The corresponding eigenvalues of the interaction Hamiltonian are given by

$$\mu_{1,2} = \lambda\left\{\Delta + \chi_1 n_1^2 + \chi_2 n_2^2 \pm \Omega_{n_1,n_2}\right\}, \quad (15)$$

where

$$\Omega_{n_1,n_2} = \sqrt{\delta_{n_1,n_2}^2 + f^2(n_1+1)(n_2+1)}, \quad (16)$$

$$\delta_{n_1,n_2} = \Delta - \chi_1 n_1 - \chi_2 n_2. \quad (17)$$

In this case, the time evolution operator can be written as

$$\hat{U}(n_1,n_2,t) = \exp(-H_{in}t) = \begin{bmatrix} U_{11}(n_1,n_2,t) & U_{12}(n_1,n_2,t) \\ U_{12}(n_1,n_2,t) & U_{22}(n_1,n_2,t) \end{bmatrix}, \quad (18)$$

where

$$U_{11}(n_1,n_2,t) = \frac{1}{2\Omega}\left\{(\Omega+\delta/2)e^{-i\mu_1 t} + (\Omega-\delta/2)e^{-i\mu_2 t}\right\}, \quad (19)$$

$$U_{22}(n_1,n_2,t) = \frac{1}{2\Omega}\left\{(\Omega-\delta/2)e^{-i\mu_1 t} + (\Omega+\delta/2)e^{-i\mu_2 t}\right\}, \quad (20)$$

$$U_{12}(n_1,n_2,t) = \frac{\lambda}{2\Omega}\sqrt{(n_1+1)(n_2+1)}\left(e^{-i\mu_1 t} - e^{-i\mu_2 t}\right),$$

where $f = f(n_1+1, n_2+1)$, $\Omega = \Omega_{n_1,n_2}$ and $\delta = \delta_{n_1,n_2}$.

The initial state is given by $|\psi(0)\rangle = |\psi_A(0)\rangle \otimes |\psi_{TMF}(0)\rangle$, where $|\psi_A(0)\rangle$ is the initial state of the two-level atom and $|\psi_{TMF}(0)\rangle$ is the initial state of the input two-mode field which will be TMCS (i.e. $|\psi_{TMF}(0)\rangle = |\alpha_1,\alpha_2\rangle$ given by Eq. (7) and *TMSVS* given by Eq. (4). The combined atom-field system state at $t = 0$, can be written as

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes |\psi_{TMF}(0)\rangle, \quad (21)$$

Once the matrix representation of $\hat{U}(t)$ is obtained, the density operator of the system at any time $t > 0$ as

$$\hat{\rho}(t) = \hat{U}(t)|\psi(0)\rangle\langle\psi(0)|\hat{U}^{\dagger}(t). \tag{22}$$

Furthermore, we introduce the reduced atomic density operator $\hat{\rho}^{A}(t)$ and the reduced field density operator $\hat{\rho}^{F}(t)$ by taking the trace of $\hat{\rho}(t)$ over the field states and over the atomic states, respectively,

$$\hat{\rho}^{A}(t) = Tr_{F}\left[\hat{\rho}(t)\right], \tag{23}$$

Then the matrix elements of the reduced atomic density operator are given by

$$\rho^{A}_{kl}(t) = \sum_{n_1,n_2=0} \langle k,n_1,n_2|\hat{\rho}(t)|l,n_1,n_2\rangle, \tag{24}$$

It is well known that the entanglement of the atom-field state can be measured in terms of the von Neumann entropy [44,45,46], which is generally defined for the reduced atomic density matrix $\hat{\rho}^{A}(t)$ as

$$S_V = -\mathrm{Tr}\left(\hat{\rho}^{A}\ln\hat{\rho}^{A}\right) = -\mu_1\ln\mu_1 - \mu_2\ln\mu_2, \tag{25}$$

where μ_1 and μ_2 are the eigenvalues of the atomic density matrix ρ^{A}.

Now, we turn the attention to the QFI which plays an essential role in quantum information processing and quantum metrology, where the highest precision of estimating an unknown parameter we may achieve is related to inverse of the QFI and is defined as

$$I_{QF} = \mathrm{Tr}\left[\rho(\phi)L^{2}\right], \tag{26}$$

where $\rho(\phi)$ is the density matrix of the system, ϕ is the parameter to be measured, and L is the quantum score (symmetric logarithmic derivative) which is defined by

$$\frac{\partial\rho(\phi)}{\partial\phi} = \frac{1}{2}\left[L\rho(\phi) + \rho(\phi)L\right]. \tag{27}$$

The Fisher information is related to the uncertainty of the measurement via the Gramer-Rao bound, $\Delta\phi \geq 1/\sqrt{F_Q}$. Here, the QFI-based parameter is assumed to be induced by a single-atom phase gate $U(\phi) :=|\downarrow\rangle\langle\downarrow| + \exp(i\phi)\,|\uparrow\rangle\langle\uparrow|$, acting on the two-level atom. To estimate the unknown parameter ϕ as precisely as possible, the optimal input state may be chosen as $|\psi(0)\rangle$, which maximizes the QFI of the output state $U(\phi)|\psi(0)\rangle$. After the phase gate operation and before the measurement performed, the qubit is coupled to a environment consisting of phase noise laser described by the master equation given above.

4 Numerical results and discussions

In this section, we present the link between the direction of each QFI, quantum entanglement and the sign of the QFI flow for a single qubit. By using the QFI, we investigate the problem of the parameter estimation in a two-level system interacting with the two-mod field initially in the TMCS and TMSVS. Furthermore, we introduce the QFI flow to characterize the progression of the quantum entanglement. The QFI flow, which is defined as the change rate, $I_{WF} := \partial I_{QF}/\partial t$, of the QFI.

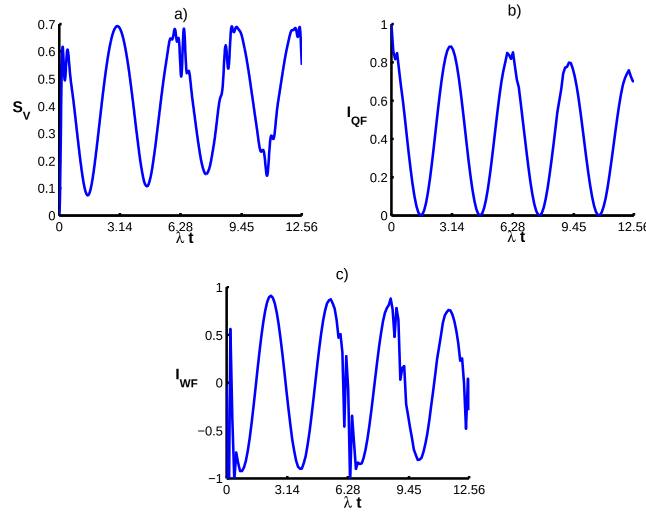

Fig. 1: The time evolution of the: (a) von Neumann entropy S_v (for $\phi = \pi/2$), (b) the QFI I_{QF} (solid line) and (c) the QFI flow I_{WF} for single qubit interacting with field initially in TMCS for $\bar{n}_1 = \bar{n}_2 = 10$, $\Delta/\lambda = \chi_1/\lambda = \chi_2/\lambda = 0$ and the intensity dependent function $f(n_1,n_2) = 1$.

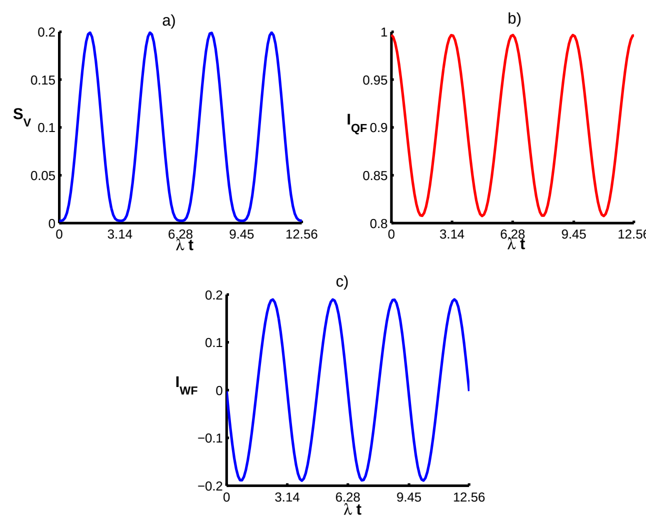

Fig. 2: The same as Fig.1 and the intensity dependent function $f(n_1,n_2) = 1\Big/\sqrt{(n_1+1)(n_2+1)}$.

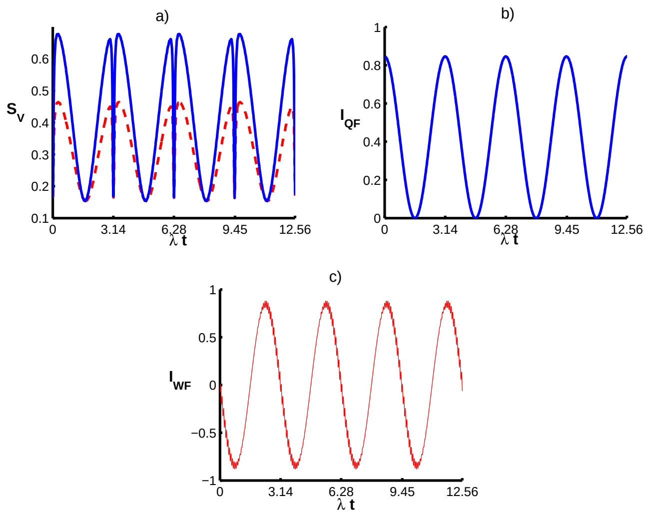

Fig. 3: The time evolution of the: (a) von Neumann entropy S_v ("dashed line" for $\phi = \pi/2$, "dashed-dotted line" for $\phi = \pi/4$), (b) the QFI I_{QF} and (c) the QFI flow I_{WF} for single qubit interacting with two-field initially in TMVS for $\bar{n} = \sinh^2 r = 10$, $\Delta/\lambda = \chi_1/\lambda = \chi_2/\lambda = 0$ and the intensity dependent function $f(n_1, n_2) = 1$.

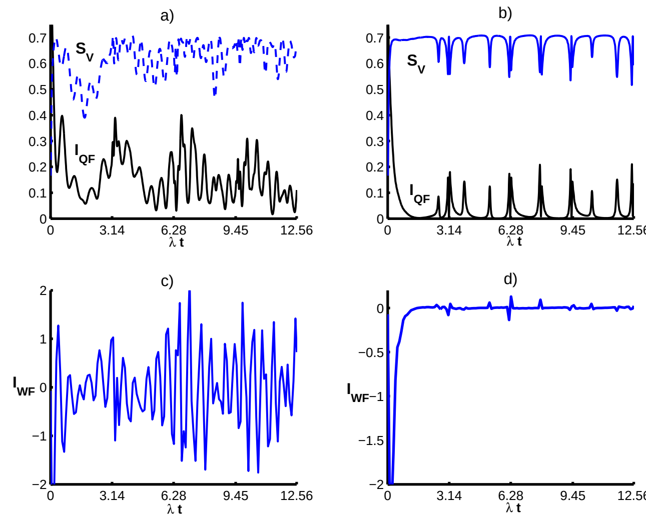

Fig. 4: The time evolution of the: (a) von Neumann entropy S_v (dashed line), the QFI I_{QF} (solid line) and (c) the QFI flow I_{WF} for single qubit interacting with two-field initially in TMVS for $\bar{n} = \sinh^2 r = 10$, $\Delta/\lambda = 10$, $\chi_1/\lambda = \chi_2/\lambda = 0$ and the intensity dependent function $f(n_1, n_2) = 1$. Figs (b,d) are the same as Figs. (a,c) but for $\Delta/\lambda = 0$, $\chi_1/\lambda = \chi_2/\lambda = 0.1$.

We point out the this quantity is feasible for improving and understanding the different branches of quantum metrology [13].

In Fig.1, the quantum entanglement measured by the von Neumann entropy S_v, QFI and QFI flow are plotted as a function of the scaled time (one unit of time is given by the inverse of the coupling constant λ). The input field is initially in TMCS and the effect of the detuning parameter and Kerr like medium is ignored. Moreover, the effect of the intensity dependent function is neglected (i.e. $f(n_1 + 1, n_2 + 1) = 1$). It is observed that the maximum value of the QFI is decrease as the scaled time goes on. There a monotonic correlation between the behavior of S_v and QFI during the time evolution. In the other hand, the QFI flow exhibits an adverse behavior with S_v and QFI.

Now, we would like to consider the effect of the intensity dependent function $f(n_1 + 1, n_2 + 1)$ on the dynamical behavior of the quantities S_v, I_{QF} and I_{WF}. In this case the intensity dependent function $f(n_1 + 1, n_2 + 1)$ have taken to be $1/\sqrt{(n_1 + 1)(n_2 + 1)}$ and the other parameters are the same as in Fig.1. It is interesting to see that S_v and I_{QF} have an inverse behavior and they satisfy the equation $S_V + I_{QF} = 1$. As seen from Fig. 1 (a) the system returns back to the separable state (i.e. $S_v = 0$), and the QFI takes it maximum value ($I_{QF} = 1$) at the periodic time $\lambda t = m\pi$ $(m = 0, 1, 2, ...)$. Instantaneously, the QFI flow drops to minimum value and there is a monotonic correlation between the quantum entanglement and the sign of QFI flow at the half of the periodic time $\lambda t = \left(m + \frac{1}{2}\right)\pi$. Interestingly, we see that the decreasing of quantum entanglement corresponding the positive QFI flow and the negative QFI flow corresponding the growth of the quantum entanglement. From the above discussion the QFI flow can be used an indicator of the quantum entanglement between two-level atom and two-mode coherent state field.

Now, we are in a position to discuss an important and practical situation of the interaction between the two-level atom and two-mode field. In this considered case, we assume that the input field initially in the TMSVS which have different tasks and applications in quantum information processing. As observed from Fig. 3(a) the von Neumann entropy S_v is affected by the changing of the phase shift ϕ from $\pi/2$ to $\pi/4$ where the amplitude of S_v is decreased. This is the main advantage of the QFI and its flow where they do not affected by any change of the estimator phase shift parameter ϕ. The system does not reach to the separable state during the time evolution. Also, there is a monotonic correlation between the dynamical behavior of QFI, S_v and the sign of the sign QFI flow.

Finally, figure 4 presents the effect of the detuning parameter and Kerr medium on the QFI, S_v and QFI flow.

It can be seen that the QFI and S_v has an monotonic behavior. Interestingly, during the time evolution. The high amount of the quantum entanglement between the two-mode field and two-level atom can be achived by the effect of the effect detuning parameter (see Fig 4(a)). Moreover the Kerr medium leads to maximize the S_v and minimize the QFI after initial time. On the other hand the sign of the QFI flow is negative according to the sudden increasing of S_v and sudden drop for QFI within certain small interval of the time and then it goes to zero value (see Fig 4(d)).

5 Conclusion

In summary, in the point of view the quantum Fisher information, we have investigated in detail the problem of parameter estimation of a two-level atom interacting with input two-mode field. The model was considered when the two-level atom is initially in the optimal state while the field is initially in two-mode coherent states and squeezed vacuum states. By numerical calculation, we considered the maximal QFI by choosing the optimal input state. We have shown that the QFI flow is useful to characterize the progression of quantum entanglement and dynamics of atom-field correlations under different cases of the intensity dependent function. The relationship between QFI, QFI flow and nonlocal correlation was demonstrated. Our results show that the interaction between the two-level atom and with two-mode field under the effect of the intensity dependent function, detuning parameter and Kerr like medium provides much substantial structure than the absence of the deuning parameter, Kerr like medium and intensity dependent function. In this way, it is found that, the variation of the atom-field entanglement and quantum Fisher is determined by the sign of the quantum Fisher flow during the time evolution. Also, we have found that the deuning parameter and Kerr like medium create a high amount of the atom-field entanglement and minimizing the QFI. More and deep future investigation is planned regarding the effect decoherence of the dynamics of the nonlocal correlation, QFI and QFI flow considering the cavity damping.

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