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On Ekeland's Variational Principle in Partial Metric Spaces

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Abstract: In this paper, lower semi-continuous functions are used to extend Ekeland's variational principle to the class of partial metric spaces. As consequences of our results, we obtain some fixed point theorems of Caristi and Clarke types.

Keywords: Ekeland's principle, fixed point theory, lower semi-continuity, partial metric space

1 Introduction

Ekeland formulated a variational principle, that is the foundation of modern variational calculus, having applications in many branches of Mathematics, including optimization and fixed point theory [16,17,18]. Indeed, this result has found applications in nonlinear analysis, since it entails the existence of approximate solutions of minimization problems for a lower semi-continuous function that is bounded from below on complete metric spaces. Also, Ekeland's variational principle is a fruitful tool in simplifying and unifying the proofs of already known theorems and has many generalizations, see Borwein and Zhu [11].

Furthermore, fixed point theory plays an increasingly important role in different fields of nonlinear functional analysis. Indeed, it has wide application areas, such as, physics, chemistry, biology, several branches of engineering, economics, etc. (see, e.g., [10,19,20,31,32,46,47,50,52,53] and references therein). One of the initial and pivotal results in this direction is the Banach contraction mapping principle [8]: Every contraction in a complete metric space has a unique fixed point. Due to necessity, analog of Banach contraction mapping principle is proved in various generalized metric spaces, such as, in quasi-metric spaces, fuzzy metric spaces, cone metric spaces, G-metric spaces, statistical metric spaces, b-metric spaces, partial metric spaces (see, e.g., [15,21,31,33,34,35,36,51]).

In this paper, we attract attention to the notion of partial metric space, introduced by Matthews [33] in 1992. The concept of partial metric space erased from needs of computer science, in particular, domain theory and semantics (see e.g. [19,20,30,32,46,47,50,52,53] and references therein). Roughly speaking, a partial metric space is distinguished from a metric space with the fact that the self-distance of a point need not to be zero. In the mentioned paper, Matthews [33] also proved the analog of Banach contraction mapping principle. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see, e.g., [1]-[7], [9,13], [22]-[29], [37]-[48] and references therein).

In this paper, due to the relevance of Ekeland's principle in the literature over the last decades, the authors believe that extending this principle to the class of partial metric spaces could be useful for developing various applications (see, e.g., [11], [49]). As consequences of our results, we obtain some fixed point theorems of Caristi and Clarke types.

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2 Preliminaries on partial metric spaces

Let $\mathbb{R}^+$ denote the set of all non-negative real numbers. A partial metric space is a pair (X,p) where X is a non-empty set and $p : X\times X \to \mathbb{R}^+$ is such that

(P_1) $p(x,y) = p(y,x)$ (symmetry);
(P_2) if $p(x,x) = p(x,y) = p(y,y)$, then $x = y$ (equality);
(P_3) $p(x,x) \le p(x,y)$ (small self-distances);
(P_4) $p(x,z)+p(y,y) \le p(x,y)+p(y,z)$ (triangle inequality);

for all $x,y,z \in X$. We will use the abbreviation PMS for the partial metric space (X,p).

Notice that, for a partial metric p on X, the function $d_p : X\times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X. Observe that each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x)+\varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$. Similarly, a closed p-ball is defined as

$$B_p[x,\varepsilon] = \{y \in X : p(x,y) \le p(x,x)+\varepsilon\}.$$

Definition 2.1. ([33]) Let (X,p) be a PMS. Then

(i) a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$;
(ii) a sequence $\{x_n\}$ in X is called Cauchy if and only if $\lim_{n,m\to+\infty} p(x_n,x_m)$ exists (and is finite);
(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

Example 2.1. ([33]) Consider $X = \mathbb{R}^+$ and define $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. Then (X,p) is a PMS. It is clear that p is not a (usual) metric.

Example 2.2. ([28]) Let (X,d) and (X,p) be a metric space and a partial metric space, respectively. The functions $\rho_i : X\times X \to \mathbb{R}^+$, $i \in \{1,2,3\}$, given by

$$\begin{aligned}\rho_1(x,y) &= d(x,y)+p(x,y),\\ \rho_2(x,y) &= d(x,y)+\max\{\omega(x),\omega(y)\},\\ \rho_3(x,y) &= d(x,y)+a\end{aligned}$$

define partial metrics on X, where $\omega : X \to \mathbb{R}^+$ is an arbitrary function and $a \ge 0$.

Example 2.3. ([33]) Let $X = \{[a,b] : a,b \in \mathbb{R},\ a \le b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ for all $[a,b],[c,d] \in X$. Then (X,p) is a PMS.

Example 2.4. ([33]) Let $X = [0,1]\cup[2,3]$ and define $p : X\times X \to \mathbb{R}^+$ by

$$p(x,y) = \begin{cases}\max\{x,y\} & \text{if } \{x,y\}\cap[2,3] \ne \emptyset,\\ |x-y| & \text{if } \{x,y\} \subset [0,1].\end{cases}$$

Then (X,p) is a complete PMS.

Lemma 2.1. ([33]) Let (X,p) be a PMS. Then

(i) a sequence $\{x_n\}$ in X is Cauchy if and only if $\{x_n\}$ is Cauchy in the metric space (X,d_p);
(ii) (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover, $\lim_{n\to+\infty} d_p(x,x_n) = 0$ if and only if

$$p(x,x) = \lim_{n\to+\infty} p(x,x_n) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

3 Main results

We start this section with the following definition and lemmas that will be used in the proof of the main theorem.

Definition 3.1. Let (X,p) be a PMS and $\phi : X \to \mathbb{R}^+$ be a given function. Then, ϕ is said to be a *lower semi-continuous (l.s.c)* function on X if

$$\lim_{n\to+\infty} p(x_n,x) = p(x,x) \Rightarrow \phi(x) \le \liminf_{n\to+\infty} \phi(x_n),$$

for every $x \in X$.

Lemma 3.1. ([1,25]) Let (X,p) be a PMS. Then

(i) if $p(x,y) = 0$, then $x = y$;
(ii) if $x \ne y$, then $p(x,y) > 0$.

Lemma 3.2. ([1,25]) Let (X,p) be a PMS and assume that $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to +\infty$, where $z \in X$ and $p(z,z) = 0$. Then

$$\lim_{n\to+\infty} p(x_n,y) = p(z,y),\ \text{for every } y \in X.$$

Now, we state and prove the following theorem.

Theorem 3.1. Let (X,p) be a complete PMS and $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function. Let $\varepsilon > 0$ and $x \in X$ be such that

$$\phi(x) \le \inf_{t\in X} \phi(t)+\varepsilon \quad \text{and} \quad \inf_{t\in X} p(x,t) < 1. \tag{1}$$

Then there exists some point $y \in X$ such that

$$\phi(y) \le \phi(x), \tag{2}$$

$$p(x,y) \le 1, \tag{3}$$

$$\forall z \in X \text{ with } z \ne y,\ \ \phi(z) > \phi(y) - \varepsilon p(y,z). \tag{4}$$

Proof. Let $x \in X$ be such that (1) holds. Define a sequence $\{x_n\}$ inductively, in the following way: for $n = 1$, take $x_1 := x$ so that $\phi(x_1) \le \phi(x)$ and $p(x,x_1) = p(x,x) \le 1$; for the other terms, assume that $x_n \in X$, with $\phi(x_n) \le \phi(x)$ and $p(x,x_n) \le 1$, is known and one of the following cases occurs:

(a) $\phi(x_n) - \phi(z) < \varepsilon p(x_n,z)$, for all $z \neq x_n$;
(b) there exists $z \neq x_n$ such that $\varepsilon p(x_n,z) \leq \phi(x_n) - \phi(z)$.

In case (a), if we take $y = x_n$, then (2)-(4) hold true trivially, since $\phi(y) = \phi(x_n) \leq \phi(x)$.
On the other hand, let S_n be the set of all $z \in X$ such that case (b) holds. Then $x_{n+1} \in S_n$ is chosen in a way that

$$\phi(x_{n+1}) - \inf_{t\in S_n} \phi(t) \leq \frac{1}{2}\left[\phi(x_n) - \inf_{t\in S_n} \phi(t)\right]. \tag{5}$$

Consequently, one has

$$\varepsilon p(x_n,x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1}), \quad \text{for all } n \in \mathbb{N} \tag{6}$$

and, by using the triangle inequality, one can obtain (for all $n \leq m$)

$$\begin{aligned}
&\varepsilon p(x_n,x_m)\\
&\leq \varepsilon[p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) - p(x_{n+1},x_{n+1})]\\
&+\varepsilon[p(x_{n+2},x_{n+3}) + p(x_{n+3},x_{n+4}) - p(x_{n+3},x_{n+3})]\\
&+\cdots+\varepsilon[p(x_{m-2},x_{m-1}) + p(x_{m-1},x_m) - p(x_{m-1},x_{m-1})]\\
&\leq \varepsilon \sum_{k=n}^{m-1} p(x_k,x_{k+1})\\
&\leq \sum_{k=n}^{m-1} (\phi(x_k) - \phi(x_{k+1})) \qquad (7)\\
&= \phi(x_n) - \phi(x_m).
\end{aligned}$$

By (6), the sequence $\{\phi(x_n)\}$ is non-increasing in $\mathbb{R}^+$ and bounded below by zero. Thus, the sequence $\{\phi(x_n)\}$ is convergent, which implies that the right hand side of (7) tends to zero, that is, $p(x_n,x_m)$ tends to zero as $n,m \to +\infty$, so $\{x_n\}$ is a Cauchy sequence in the complete partial metric space (X,p). By Lemma 2.1, $\{x_n\}$ is Cauchy in the metric space (X,d_p) (also, it is complete). Then, there exists $y \in X$ such that $\{x_n\}$ is convergent to y in (X,d_p). Again by Lemma 2.1, we get

$$p(y,y) = \lim_{n\to+\infty} p(x_n,y) = \lim_{n,m\to+\infty} p(x_n,x_m). \tag{8}$$

Since $\lim_{n,m\to+\infty} p(x_n,x_m) = 0$, therefore by (8) we have $p(y,y) = 0$.
We claim that y satisfies (2)-(4).
Due to (6), the sequence $\{\phi(x_n)\}$ is non-increasing, that is

$$\cdots \leq \phi(x_{n+1}) \leq \phi(x_n) \leq \cdots \leq \phi(x_1) \leq \phi(x),$$

then (2) holds.
The inequality (3) is obtained by taking $n = 1$ in (7) and by using (1). Indeed, we have

$$\begin{aligned}
\varepsilon p(x,x_m) &= \varepsilon p(x_1,x_m)\\
&\leq \phi(x) - \phi(x_m)\\
&\leq \phi(x) - \inf_{t\in X} \phi(t) \leq \varepsilon.
\end{aligned}$$

Hence, taking $m \to +\infty$ it follows that $p(x,y) \leq 1$.
The inequality (4) is observed by the method of reductio ad absurdum. Assume (4) is not true, then there is $z \in X$ with $z \neq y$ such that

$$\phi(z) \leq \phi(y) - \varepsilon p(y,z). \tag{9}$$

Since $p(y,z) > 0$, we have

$$\phi(z) < \phi(y). \tag{10}$$

By (7), we get

$$\phi(x_m) \leq \phi(x_n) - \varepsilon p(x_n,x_m), \quad \text{for all } n \leq m.$$

Then, taking $m \to +\infty$ in above inequality, one can obtain

$$\phi(y) \leq \liminf_{m\to+\infty} \phi(x_m) \leq \phi(x_n) - \varepsilon p(x_n,y). \tag{11}$$

From (P_4), we have

$$p(x_n,z) \leq p(x_n,y) + p(y,z) - p(y,y) = p(x_n,y) + p(y,z).$$

Next, using this inequality and (9), from (11) we get

$$\phi(z) \leq \phi(y) - \varepsilon p(y,z) \leq \phi(x_n) - \varepsilon p(x_n,z),$$

which implies that $z \in S_n$, for all $n \in \mathbb{N}$. Now, note that (5) can be written as

$$2\phi(x_{n+1}) - \phi(x_n) \leq \inf_{t\in S_n} \phi(t) \leq \phi(z).$$

Therefore, having in mind that $\{\phi(x_n)\}$ is a non-increasing sequence in $\mathbb{R}^+$, there exists $L \geq 0$ such that

$$\lim_{n\to+\infty} \phi(x_n) = L.$$

Letting $n \to +\infty$ in the previous inequality, then we get $L \leq \phi(z)$. On the other hand, since ϕ is l.s.c, then we have

$$\phi(y) \leq \liminf_{n\to+\infty} \phi(x_n) = L$$

and so we get $\phi(y) \leq \phi(z)$, that is a contradiction with respect to (10). □
Notice that if in Theorem 3.1 we do not assume that $\inf_{t\in X} p(x,t) < 1$, then we can (only) deduce that there exists $y \in X$ such that (2) and (4) hold true.
Building on Theorem 3.1, we give the following result.
Theorem 3.2. Let (X,p) be a complete PMS and $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function. Given $\varepsilon > 0$, then there exists $y \in X$ such that

$$\phi(y) \leq \inf_{t\in X} \phi(t) + \varepsilon, \tag{12}$$

$$\forall z \in X, \quad \phi(z) \geq \phi(y) - \varepsilon p(y,z). \tag{13}$$

Proof. The proof is clear. Indeed, recalling the fact that there is always some point x such that $\phi(x) \leq \inf_{t\in X} \phi(t) + \varepsilon$, then (12) and (13) follow from (2) and (4), respectively. □
Notice that Theorem 3.1 is stronger than Theorem 3.2. Precisely, the main difference lies in inequality (1), which gives the whereabouts of point x in X, and which has no counterpart in Theorem 3.2. Thus, Theorem 3.1 is said to be the strong statement, and Theorem 3.2 is said to be the weak statement.

4 Fixed point theorems

The significance of the results given in Section 3 will become clear as we proceed with the following applications of fixed points.

4.1 Caristi's fixed point theorem

The following theorem is an extension of the result of Caristi [12, Theorem 2.1]. We note that this theorem corresponds to [25, Theorem 5]. Here, we shorten the proof.

Theorem 4.1. Let (X,p) be a complete PMS and let $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function. Then any mapping $T : X \to X$ satisfying

$$p(x,Tx) \leq \phi(x) - \phi(Tx), \quad \text{for each } x \in X \tag{14}$$

has a fixed point in X.

Proof. We apply Theorem 3.2 (for $\varepsilon = \frac{1}{2}$) to the function ϕ satisfying (14) (T, verifying (14), is called a Caristi mapping on (X,p)). Then, there exists some point $y \in X$ such that

$$\forall t \in X, \quad \phi(t) \geq \phi(y) - \frac{1}{2}p(y,t).$$

This inequality holds also for $t = Ty$, therefore

$$\phi(y) - \phi(Ty) \leq \frac{1}{2}p(y,Ty).$$

Substituting $x = y$ in the inequality (14), one can get

$$p(y,Ty) \leq \phi(y) - \phi(Ty).$$

Comparing the last inequalities, we deduce that

$$p(y,Ty) \leq \frac{1}{2}p(y,Ty).$$

This holds unless $p(y,Ty) = 0$ and so by Lemma 3.1, we have $Ty = y$, that is, T has a fixed point. □

4.2 Clarke's fixed point theorem

In 1976, Clarke [14] extended the Banach contraction principle for some directional contractions (see condition (D) of Theorem 4.2) on closed convex subsets of Banach spaces.

Theorem 4.2. Let X be a closed convex subset of a Banach space and let $T : X \to X$ be a continuous mapping satisfying the following condition:

(D) there exists $k \in (0,1)$ such that corresponding to each $u \in X$, there exists $t \in (0,1]$ for which $\|T(u_t) - T(u)\| \leq k\|u_t - u\|$, where $u_t = tT(u) + (1-t)u$ describes the line segment from u to $T(u)$ as t runs from 0 to 1.

Then, T has a fixed point in X.

Proof. The main difference with the proof of Ekeland [18] is that here the proof is reposed on considering a partial metric (not a metric). First, we apply Theorem 3.2 to the functional $\varphi : X \to \mathbb{R}^+$ given by

$$\varphi(w) = \|w - T(w)\| + b,$$

for all $w \in X$, where $b > 0$ is arbitrary and $0 < \varepsilon < 1 - k$. Then, we define the partial metric $p : X \times X \to \mathbb{R}^+$ by

$$p(w,z) = \|w - z\| + b.$$

Clearly, p is not a metric since $p(w,w) = b > 0$. Moreover,

$$d_p(w,z) = 2\|w - z\|$$

and so (X,p) is a complete partial metric space.
Since $T : (X, \|\cdot\|) \to (X, \|\cdot\|)$ is continuous, then if $w_n \to w$ in $(X, \|\cdot\|)$, we have $T(w_n) \to T(w)$ in $(X, \|\cdot\|)$.
Note that $\varphi(w) = p(w, T(w))$. Now, let $w_n \to w$ in (X,p), then

$$\lim_{n\to+\infty} p(w_n,w) = p(w,w).$$

By definition of the partial metric p, we get that

$$\lim_{n\to+\infty} \|w_n - w\| = 0.$$

Therefore $\lim_{n\to+\infty} \|T(w_n) - T(w)\| = 0$. As a consequence, we have

$$\lim_{n\to+\infty} \|w_n - T(w_n)\| = \|w - T(w)\|,$$

that is,

$$\lim_{n\to+\infty} \varphi(w_n) = \varphi(w).$$

We conclude that φ is continuous and so is l.s.c in X. Due to Theorem 3.2, there exists some $y \in X$ such that

$$\forall w \in X, \ \varphi(w) \geq \varphi(y) - \varepsilon p(w,y)$$

that is,

$$\|w - T(w)\| \geq \|y - T(y)\| - \varepsilon(\|w - y\| + b). \tag{15}$$

By condition (D), there exist $k \in (0,1)$ and $t \in (0,1]$ such that

$$\|T(y_t) - T(y)\| \leq k\|y_t - y\| \leq kt\|y - T(y)\|.$$

Writing $w = y_t$ into the inequality (15), we get

$$\begin{aligned}
&\|y - T(y)\| \\
&\leq \|y_t - T(y_t)\| + \varepsilon(\|y_t - y\| + b) \\
&\leq \|y_t - T(y)\| + \|T(y) - T(y_t)\| + \varepsilon(t\|y - T(y)\| + b) \\
&\leq \|y_t - T(y)\| + kt\|y - T(y)\| + \varepsilon(t\|y - T(y)\| + b).
\end{aligned}$$

Now, since y_t belongs to the line segment $[y, T(y)]$, we have

$$\begin{aligned}
\|y - T(y)\| &= \|y - y_t\| + \|y_t - T(y)\| \\
&= t\|y - T(y)\| + \|y_t - T(y)\|.
\end{aligned}$$

It follows easily that

$$t\|y-T(y)\| \leq (k+\varepsilon)t\|y-T(y)\|+\varepsilon b,$$

for each $b>0$. Consequently, letting $b\to 0$, we derive that

$$t\|y-T(y)\| \leq (k+\varepsilon)t\|y-T(y)\|.$$

Since $t>0$, we divide by t to obtain

$$\|y-T(y)\| \leq (k+\varepsilon)\|y-T(y)\|,$$

which holds unless that $y=Ty$, as $k+\varepsilon<1$. Therefore, y is a fixed point of T. $\square$

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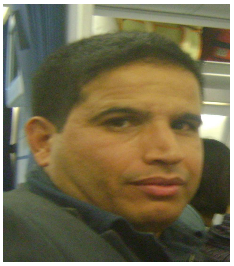

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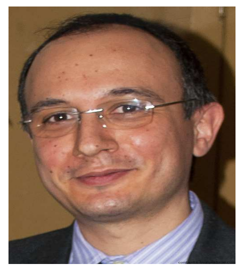

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