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An Impulsive Sturm-Liouville Problem with Boundary Conditions Containing Herglotz-Nevanlinna Type Functions

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Abstract: In this article, an impulsive Sturm–Liouville boundary value problem with boundary conditions contain **Herglotz–Nevanlinna type rational functions** of the spectral parameter is considered. It is shown that the coefficients of the problem are uniquely determined by either the Weyl function or by the Prüfer angle or by the classical spectral data consist of eigenvalues and norming constants.

Keywords: Inverse Problem, Parameter Dependent Boundary Condition, Herglotz-Nevanlinna Function, Transfer Condition.

1 Introduction

Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in mathematical physics, mechanics, electronics, geophysics and other branches of natural sciences. The first inverse problem of a regular Sturm-Liouville operator was brought out by Ambarzumyan in 1929 [1]. However, the most important uniqueness theorem for inverse Sturm-Liouville problem was proved by G. Borg in 1945 [2]. Borg's result has been generalized to various versions until today.

A large body of literature has built up, over the years, on problems of Sturm-Liouville type but where the boundary conditions depend on parameter. A boundary condition, rationally dependent on the spectral parameter, has the form

$$a(\lambda)y(1)+b(\lambda)y'(1)=0,$$

where $a(\lambda)$ and $b(\lambda)$ are polynomials. This equality, in the case when $\deg a(\lambda)=\deg b(\lambda)=1$, is said as affinely (or linearly) dependent boundary conditions. Walter [28] and Fulton [15] have extensive bibliographies and we also refer to Fulton for some physical applications. Inverse problems for some classes of differential operators depending linearly on the parameter were studied in various publications (see [4], [5], [9], [11], [18] and [21]). The more general cases of the polynomials $a(\lambda)$ and $b(\lambda)$ are more difficult to investigate. There are several papers about the spectral problems for differential operators with the boundary conditions rationally dependent on the spectral parameter. ([6]-[8], [10], [14], [20], [22], [25]-[27] and [29]). Binding et al investigated direct and inverse spectral theory for Sturm-Liouville problem, when $\dfrac{a(\lambda)}{b(\lambda)}$ is a rational function of Herglotz–Nevanlinna type such that

$$f(\lambda)=a\lambda+b-\sum_{k=1}^{N}\frac{f_k}{\lambda-g_k},$$

in one boundary condition in [6] and [7].

Spectral problems arising in mechanical engineering and having boundary conditions depending on the spectral parameter can be found in the classical textbook [12] of Collatz. Furthermore, these kinds of problems appear among others in connection with accoustic wave propagation in a rectangular duct with a uniform meanflow profile and walls with finite accoustic impedance [19].

Boundary value problems with transfer conditions inside the interval often appear in applied sciences. Spectral problems for differential operator with the transfer conditions have been studied in [3], [13], [17],

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[18] and [21]-[24] where further references and links to applications can be found.

The aim of this paper is to present various uniqueness theorems for an inverse Sturm–Liouville problem with eigenparameter-dependent-boundary conditions and transfer conditions. We are studied in two types of generalizations of classical Sturm-Liouville problems: First, both of the boundary conditions depend on λ by Herglotz–Nevanlinna functions. Second, we have two transfer conditions depending linearly on λ. The conditions of the considered problem have an important place in the class of parameter-dependent conditions. In this case, not only eigenvalues and eigenfunctions of the problem are real, but it is also possible to define ”norming constants” which play an important role in inverse spectral theory.

2 Preliminaries

Let us consider the boundary value problem L, generated by the regular Sturm–Liouville equation

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in \Omega = \bigcup_{i=0}^{2}(d_i, d_{i+1}) \tag{1}$$

subject to the boundary conditions

$$U_1(y) := y'(0) - f_1(\lambda)y(0) = 0 \tag{2}$$
$$U_2(y) := y'(1) - f_2(\lambda)y(1) = 0 \tag{3}$$

and two transfer conditions

$$\begin{cases} y(d_i+0) = \alpha_i y(d_i-0) \\ y'(d_i+0) = \alpha_i^{-1} y'(d_i-0) \\ \qquad - (\beta_i\lambda + \gamma_i) y(d_i-0), \ i = 1,2 \end{cases} \tag{4}$$

where λ is the spectral parameter; $q(x)$ is a real valued function in $L_2(0,1)$; α_i, β_i and γ_i are real numbers, $\alpha_i > 0$, $\beta_i > 0$, $d_0 = 0 < d_1 < d_2 < d_3 = 1$. We assume that $f_1(\lambda)$ and $f_2(\lambda)$ are rational functions of Herglotz–Nevanlinna type such that

$$f_j(\lambda) = a_j\lambda + b_j - \sum_{k=1}^{N_j} \frac{f_{jk}}{\lambda - g_{jk}}, \quad j = 1,2,$$

where a_j, b_j, f_{jk}, g_{jk} are real numbers, $a_1 < 0$, $f_{1k} < 0$, $a_2 > 0$, $f_{2k} > 0$, $g_{j1} < g_{j2} < ... < g_{jN_j}$. It should be noted that, if $f_j(\lambda) = \infty$ then the condition (2) and/or (3) are interpreted as the Dirichlet conditions $y(0) = y(1) = 0$.

Consider the space $H = L_2(0,1) \oplus \mathbb{C}^{N_1+1} \oplus \mathbb{C}^{N_2+1} \oplus \mathbb{C}^2$ and an element in H such that
$Y = (y(x), \mathbf{u}, \mathbf{v}, \mathbf{w})$, $\mathbf{u} = (u_1, u_2, ..., u_{N_1}, u_{N_1+1})$, $\mathbf{v} = (v_1, v_2, ..., v_{N_2}, v_{N_2+1})$, $\mathbf{w} = (w_1, w_2)$.
H is a Hilbert space with the inner product defined by

$$\begin{aligned}\langle Y, Y' \rangle := &\int_0^1 y(x)\overline{z(x)}dx - \sum_{k=1}^{N_1} \frac{u_k\overline{u'_k}}{f_{1k}} + \sum_{k=1}^{N_2} \frac{v_k\overline{v'_k}}{f_{2k}} + \\ &- \frac{u_{N_1+1}\overline{u'_{N_1+1}}}{a_1} + \frac{u_{N_2+1}\overline{u'_{N_2+1}}}{a_2} + \\ &+ \frac{\alpha_1 w_1\overline{w'_1}}{\beta_1} + \frac{\alpha_2 w_2\overline{w'_2}}{\beta_2}\end{aligned} \tag{5}$$

for $Y = (y(x), \mathbf{u}, \mathbf{v}, \mathbf{w})$, $Y' = (z(x), \mathbf{u}', \mathbf{v}', \mathbf{w}')$ in H. Here, $\overline{s}$ denotes complex conjugate of the component s.

Define the operator T on the domain

$D(T) = \{Y \in H$: i) $y(x)$ and $y'(x)$ are absolutely continuous in Ω, $\ell y \in L_2(0,1)$;
ii) $u_{N_1+1} = a_1 y(0)$, $v_{N_2+1} = a_2 y(1)$;
iii) $w_i = -\beta_i y(d_i - 0)$;
iv) $y(d_i+0) - \alpha_i y(d_i-0) = 0$, $i = 1,2\}$,
such that

$$T(Y) = (\ell y(x), \mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}) \tag{6}$$

where $\mathbf{Tu} = (Tu_i)$, $\mathbf{Tv} = (Tv_i)$, $\mathbf{Tw} = (Tw_i)$,

$$Tu_i = \begin{cases} g_{1i}u_i - f_{1i}y(0), & i = \overline{1,N_1} \\ y'(0) - b_1 y(0) - \sum_{k=1}^{N_1} u_k, & i = N_1 + 1 \end{cases} \tag{7}$$

$$Tv_i = \begin{cases} g_{2i}v_i - f_{2i}y(1), & i = \overline{1,N_2} \\ y'(1) - b_2 y(1) - \sum_{k=1}^{N_2} v_k, & i = N_2 + 1 \end{cases} \tag{8}$$

$$Tw_i = y'(d_i+0) - \alpha_i^{-1} y'(d_i-0) + \gamma_i y(d_i-0), \ i = 1,2 \tag{9}$$

The following theorem can be proven by using same methods in [7] or [24] .

Theorem 1. *The eigenvalues of the operator T and the problem L coincide.*

It is clear that $f_j(\lambda)$ can be written as follows:

$$f_j(\lambda) = \frac{a_j(\lambda)}{b_j(\lambda)} \tag{10}$$

where

$$a_j(\lambda) = (a_j\lambda + b_j)\prod_{k=1}^{N_j}(\lambda - g_{jk}) - \sum_{k=1}^{N_j} f_{jk} \prod_{i=1, i\neq k}^{N_j} (\lambda - g_{ji}), \tag{11}$$

$$b_j(\lambda) = \prod_{k=1}^{N_j}(\lambda - g_{jk}), \quad j = 1,2. \tag{12}$$

Let the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (1) under the initial conditions

$$\varphi(0,\lambda) = b_1(\lambda), \quad \varphi'(0,\lambda) = a_1(\lambda) \tag{13}$$
$$\psi(1,\lambda) = b_2(\lambda), \quad \psi'(1,\lambda) = a_2(\lambda) \tag{14}$$

and the conditions (4).

It can be proven that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ also satisfy the following equalities:

$$\varphi(x,\lambda)=\begin{cases}\varphi_1(x,\lambda), & x<d_1\\ \varphi_2(x,\lambda), & d_1<x<d_2\\ \varphi_3(x,\lambda), & x>d_2\end{cases} \quad (15)$$

$$\varphi'(x,\lambda)=\begin{cases}\varphi_1'(x,\lambda), & x<d_1\\ \varphi_2'(x,\lambda), & d_1<x<d_2\\ \varphi_3'(x,\lambda), & x>d_2\end{cases} \quad (16)$$

$$\varphi_1(x,\lambda)=a_1\lambda^{N_1+1/2}\sin\sqrt{\lambda}x+O\left(\lambda^{N_1}\exp\tau x\right)$$
$$\varphi_2(x,\lambda)=\frac{a_1\beta_1}{2}\lambda^{N_1+1}\left[\cos\sqrt{\lambda}x-\cos\sqrt{\lambda}\,(2d_1-x)\right]+O\left(\lambda^{N_1+1/2}\exp\tau x\right)$$
$$\varphi_3(x,\lambda)=-\frac{a_1\beta_1\beta_2}{4}\lambda^{N_1+3/2}\left[\sin\sqrt{\lambda}x-\sin\sqrt{\lambda}\,(2d_1-x)+-\sin\sqrt{\lambda}\,(2d_2-x)+\sin\sqrt{\lambda}\,(2d_2-2d_1-x)\right]+O\left(\lambda^{N_1+1}\exp\tau x\right)$$

$$\varphi_1'(x,\lambda)=a_1\lambda^{N_1+1}\cos\sqrt{\lambda}x+O\left(\lambda^{N_1+1/2}\exp\tau x\right)$$
$$\varphi_2'(x,\lambda)=-\frac{a_1\beta_1}{2}\lambda^{N_1+3/2}\left[\sin\sqrt{\lambda}x+\sin\sqrt{\lambda}\,(2d_1-x)\right]++O\left(\lambda^{N_1+1}\exp\tau x\right)$$
$$\varphi_3'(x,\lambda)=-\frac{a_1\beta_1\beta_2}{4}\lambda^{N_1+2}\left[\cos\sqrt{\lambda}x+\cos\sqrt{\lambda}\,(2d_1-x)++\cos\sqrt{\lambda}\,(2d_2-x)-\cos\sqrt{\lambda}\,(2d_2-2d_1-x)\right]++O\left(\lambda^{N_1+3/2}\exp\tau x\right)$$

$$\psi(x,\lambda)=\begin{cases}O\left(\lambda^{N_2+3/2}\exp\tau(1-x)\right), & x<d_1\\ O\left(\lambda^{N_2+1}\exp\tau(1-x)\right), & d_1<x<d_2\\ O\left(\lambda^{N_2+1/2}\exp\tau(1-x)\right), & x>d_2\end{cases} \quad (17)$$

$$\psi'(x,\lambda)=\begin{cases}O\left(\lambda^{N_2+2}\exp\tau(1-x)\right), & x<d_1\\ O\left(\lambda^{N_2+3/2}\exp\tau(1-x)\right), & d_1<x<d_2\\ O\left(\lambda^{N_2+1}\exp\tau(1-x)\right), & x>d_2\end{cases} \quad (18)$$

where $\tau=\left|Im\sqrt{\lambda}\right|$.

Consider the function

$$\Delta(\lambda):=W(\varphi,\psi) \quad (19)$$
$$=a_2(\lambda)\varphi(1,\lambda)-b_2(\lambda)\varphi'(1,\lambda)$$
$$=b_1(\lambda)\psi'(0,\lambda)-a_1(\lambda)\psi(0,\lambda)$$

and the sequence

$$\rho_n:=\int_0^1\varphi^2(x,\lambda_n)dx+ \quad (20)$$
$$-f_1'(\lambda_n)\varphi^2(0,\lambda_n)+f_2'(\lambda_n)\varphi^2(1,\lambda_n)$$
$$+\alpha_1\beta_1\varphi^2(d_1-0,\lambda_n)+\alpha_2\beta_2\varphi^2(d_2-0,\lambda_n).$$

$\Delta(\lambda)$ is an entire function and its zeros, namely $\{\lambda_n\}_{n\geq 0}$ are eigenvalues of L. Since $a_j(g_{jk})\neq 0$ and $b_j(g_{jk})=0$ for each $j\in\{1,2\}$ and $k\in\{1,2,...,N_j\}$, g_{jk} is an eigenvalue if and only if $\varphi(j-1,g_{jk})=0$, i.e., $\Delta(g_{jk})=0$.

Lemma 1.*i) The eigenvalues $\{\lambda_n\}_{n\geq 0}$ are real numbers. ii) The equality $\Delta'(\lambda_n)=\rho_n s_n$ is valid for all n, where*

$$s_n=\frac{\psi(0,\lambda_n)}{b_1(\lambda_n)}=\frac{\psi'(0,\lambda_n)}{a_1(\lambda_n)} \quad (21)$$

*Proof.*i) It is sufficient to prove that the eigenvalues of T are real. For Y in $D(T)$, we calculate using integration by part that

$$\langle TY,Y\rangle=\int_0^1\ell y(x)\overline{y}(x)dx-\sum_{k=1}^{N_1}\frac{Tu_k\overline{u}_k}{f_{1k}}-\frac{Tu_{N_1+1}\overline{u}_{N_1+1}}{a_1}+$$
$$+\sum_{k=1}^{N_2}\frac{Tv_k\overline{v}_k}{f_{2k}}+\frac{Tv_{N_2+1}\overline{v}_{N_2+1}}{a_2}+$$
$$+\frac{\alpha_1Tw_1\overline{w_1}}{\beta_1}+\frac{\alpha_2Tw_2\overline{w_2}}{\beta_2}$$
$$=\int_0^1\left(\left|y'(x)\right|^2+q(x)\left|y(x)\right|^2\right)dx+b_1\left|y(0)\right|^2-b_2\left|y(1)\right|^2$$
$$-\sum_{k=1}^{N_1}\frac{g_{1k}}{f_{1k}}\left|u_k\right|^2+\sum_{k=1}^{N_2}\frac{g_{2k}}{f_{2k}}\left|v_k\right|^2+$$
$$+2Re\left\{\sum_{k=1}^{N_1}y(0)\overline{u}_k-\sum_{k=1}^{N_2}y(1)\overline{v}_k\right\}+$$
$$-\alpha_1\gamma_1\left|y(d_1-0)\right|^2-\alpha_2\gamma_2\left|y(d_2-0)\right|^2$$

Therefore, it can be concluded that $\langle TY,Y\rangle$ is real for each Y in $D(T)$. This completes the proof of (i).

ii) Let $\lambda_n\neq g_{ik}$ and $\varphi(x,\lambda_n)$ be the eigenfunction which corresponds to the eigenvalue λ_n. The equation (1) can be written for $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$ as follows

$$-\psi''(x,\lambda)+q(x)\psi(x,\lambda)=\lambda\psi(x,\lambda),$$
$$-\varphi''(x,\lambda_n)+q(x)\varphi(x,\lambda_n)=\lambda_n\varphi(x,\lambda_n)$$

If these equations are (i): multiplied by $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$, respectively; (ii): subtracted from each other and (iii): integrated over the interval $[0,1]$, the following equality is obtained:

$$\int_0^1 d\left[\varphi'(x,\lambda_n)\,\psi(x,\lambda)-\psi'(x,\lambda)\,\varphi(x,\lambda_n)\right] \quad (22)$$

$$=(\lambda-\lambda_n)\int_0^1 \psi(x,\lambda)\,\varphi(x,\lambda_n)\,dx$$

The initial conditions (13), (14) and the transfer conditions (4) are used to get

$$\begin{aligned}\frac{b_1(\lambda)}{b_1(\lambda_n)}\frac{\Delta(\lambda)}{\lambda-\lambda_n} &= \int_0^1 \psi(x,\lambda)\,\varphi(x,\lambda_n)\,dx+\\ &-\frac{[f_1(\lambda)-f_1(\lambda_n)]}{\lambda-\lambda_n}\varphi(0,\lambda_n)\psi(0,\lambda)\\ &+\frac{[f_2(\lambda)-f_2(\lambda_n)]}{\lambda-\lambda_n}\varphi(1,\lambda_n)\psi(0,\lambda)+\\ &+\alpha_1\beta_1\,\psi(d_1-0,\lambda)\,\varphi(d_1-0,\lambda_n)\\ &+\alpha_2\beta_2\,\psi(d_2-0,\lambda)\,\varphi(d_2-0,\lambda_n)\end{aligned}$$

Letting limit as $\lambda\to\lambda_n$ it can be obtained that

$$\Delta'(\lambda_n)=\rho_n s_n \quad (23)$$

If g_{1k} and/or g_{2k} are the eigenvalues, $\varphi(0,g_{1k})=0$ and/or $\varphi(1,g_{2k})=0$, so we see validity of (ii).

It is concluded from Lemma-1 that all eigenvalues of L are simple zeros of $\Delta(\lambda)$.

3 Uniqueness Theorems

Together with L, consider the problem $\widetilde{L}$:.

$$\widetilde{\ell}y:=-y''+\widetilde{q}(x)y=\lambda y,\quad x\in\widetilde{\Omega}=\bigcup_{i=0}^{2}(\widetilde{d}_i,\widetilde{d}_{i+1}) \quad (24)$$

$$\widetilde{U}_1(y):=y'(0)-\widetilde{f}_1(\lambda)y(0)=0 \quad (25)$$
$$\widetilde{U}_2(y):=y'(1)-\widetilde{f}_2(\lambda)y(1)=0 \quad (26)$$

$$\left\{\begin{array}{l} y(\widetilde{d}_i+0)=\widetilde{\alpha}_i y(\widetilde{d}_i-0)\\ y'(\widetilde{d}_i+0)=\widetilde{\alpha}_i^{-1}y'(\widetilde{d}_i-0)\\ \qquad -\left(\widetilde{\beta}_i\lambda+\widetilde{\gamma}_i\right)y(\widetilde{d}_i-0),\ i=1,2\end{array}\right. \quad (27)$$

It is assumed in what follows that if a certain symbol s denotes an object related to L, then the corresponding symbol $\widetilde{s}$ denotes the analogous object related to $\widetilde{L}$.

We consider three statements of the inverse problem of the reconstruction of the boundary-value problem L; i) by the Weyl function; ii) by Prüfer's angle; iii) by the spectral data $\{\lambda_n,\rho_n\}_{n\geq 0}$ and $\{\lambda_n,\mu_n\}_{n\geq 0}$.

3.1 By the Weyl Function

Denote

$$m(\lambda):=\frac{\psi(0,\lambda)}{\Delta(\lambda)} \quad (28)$$

The function $m(\lambda)$ is called Weyl function of the boundary value problem L. It is clear that $m(\lambda)$ is a meromorphic function with poles in $\{\lambda_n\}_{n\geq 0}$.

Let $s(x,\lambda)$ and $c(x,\lambda)$ be the solutions of (1) that satisfy the conditions

$$s(0,\lambda)=c'(0,\lambda)=0;\ \ c(0,\lambda)=s'(0,\lambda)=1 \quad (29)$$

and (4). The following equalities are valid:

$$\varphi(x,\lambda)=a_1(\lambda)s(x,\lambda)+b_1(\lambda)c(x,\lambda) \quad (30)$$
$$\frac{\psi(x,\lambda)}{\Delta(\lambda)}=\frac{1}{b_1(\lambda)}[s(x,\lambda)+m(\lambda)\varphi(x,\lambda)] \quad (31)$$

Theorem 2.*If $m(\lambda)=\widetilde{m}(\lambda)$ and $f_1(\lambda)=\widetilde{f}_1(\lambda)$ then $L=\widetilde{L}$, i.e., $q(x)=\widetilde{q}(x)$, almost everywhere in Ω; $f_2(\lambda)=\widetilde{f}_2(\lambda)$ and $(\alpha_i,\beta_i,\gamma_i)=\left(\widetilde{\alpha}_i,\widetilde{\beta}_i,\widetilde{\gamma}_i\right)$, $i=1,2.$*

*Proof.*Let us define the functions $P_1(x,\lambda)$ and $P_2(x,\lambda)$ as follows,

$$P_1(x,\lambda)=\varphi(x,\lambda)\frac{\widetilde{\psi}'(x,\lambda)}{\widetilde{\Delta}(\lambda)}-\tilde{\varphi}'(x,\lambda)\frac{\psi(x,\lambda)}{\Delta(\lambda)} \quad (32)$$

$$P_2(x,\lambda)=\tilde{\varphi}(x,\lambda)\frac{\psi(x,\lambda)}{\Delta(\lambda)}-\varphi(x,\lambda)\frac{\widetilde{\psi}(x,\lambda)}{\widetilde{\Delta}(\lambda)} \quad (33)$$

Since $f_1(\lambda)\equiv\widetilde{f}_1(\lambda)$, $a_1=\widetilde{a}_1$, $b_1=\widetilde{b}_1$, $f_{1k}=\widetilde{f}_{1k}$, $g_{1k}=\widetilde{g}_{1k}$, so $a_1(\lambda)\equiv\widetilde{a}_1(\lambda)$ and $b_1(\lambda)\equiv\widetilde{b}_1(\lambda)$. From the hypothesis $m(\lambda)=\widetilde{m}(\lambda)$ and the equalities (30)-(33), we get

$$P_1(x,\lambda)=\widetilde{s}'(x,\lambda)c(x,\lambda)-s(x,\lambda)\widetilde{c}'(x,\lambda)$$
$$P_2(x,\lambda)=s(x,\lambda)\widetilde{c}(x,\lambda)-\widetilde{s}(x,\lambda)c(x,\lambda)$$

Therefore $P_1(x,\lambda)$ and $P_2(x,\lambda)$ are entire functions of λ. Denote

$$G_\delta=\left\{\lambda:\lambda=k^2,\ \left|k-\sqrt{\lambda_n}\right|>\delta,\ n=0,1,2,...\right\}$$

and

$$\widetilde{G}_\delta=\left\{\lambda:\lambda=k^2,\ \left|k-\sqrt{\widetilde{\lambda}_n}\right|>\delta,\ n=0,1,2,...\right\}$$

where δ is sufficiently small number. One can easily show by using (15)-(19), (32) and (33) that the relations

$$|P_1(x,\lambda)|\leq C_\delta,\ |P_2(x,\lambda)|\leq C_\delta\,|\lambda|^{-1/2} \quad (34)$$

hold for $\lambda\in G_\delta\cap\widetilde{G}_\delta$. Thus, $P_1(x,\lambda)$ is a function, namely $A(x)$, depends only on x and $P_2(x,\lambda)=0$. Use (32) and (33) again to take

$$\varphi(x,\lambda)=A(x)\widetilde{\varphi}(x,\lambda),\ \frac{\psi(x,\lambda)}{\Delta(\lambda)}=A(x)\frac{\widetilde{\psi}(x,\lambda)}{\widetilde{\Delta}(\lambda)} \quad (35)$$

Since $W\left[\varphi(x,\lambda),\dfrac{\psi(x,\lambda)}{\Delta(\lambda)}\right]=1$ and similarly $W\left[\widetilde{\varphi}(x,\lambda),\dfrac{\widetilde{\psi}(x,\lambda)}{\widetilde{\Delta}(\lambda)}\right]=1$, then $A^2(x)=1$ and so $\varphi^2(x,\lambda)=\widetilde{\varphi}^2(x,\lambda)$. From the asymptotic formula (15) we have $d_1=\widetilde{d}_1$ and $d_2=\widetilde{d}_2$. Suppose that $A(x)=-1$ for some x. In this case, we contradict using (15) to the assumptions $a_1<0$, $\beta_1>0$ and $\beta_2>0$. Hence $A(x)=1$ and

$$\varphi(x,\lambda)=\widetilde{\varphi}(x,\lambda) \text{ and } \frac{\psi'(x,\lambda)}{\psi(x,\lambda)}=\frac{\widetilde{\psi}'(x,\lambda)}{\widetilde{\psi}(x,\lambda)} \tag{36}$$

It can be obtained from (1), (4), (10) and (14) that $q(x)=\widetilde{q}(x)$, a.e. in Ω; $f_2(\lambda)=\widetilde{f}_2(\lambda)$ and $(a_i,\beta_i,\gamma_i)_{i=1,2}=\left(\widetilde{a}_i,\widetilde{\beta}_i,\widetilde{\gamma}_i\right)_{i=1,2}$.

3.2 By the Prüfer angle

Denote

$$\phi(x,\lambda):=\begin{cases}\cot^{-1}\dfrac{\psi'(x,\lambda)}{\psi(x,\lambda)} & \text{if } \psi(x,\lambda)\neq 0,\\[2mm] \tan^{-1}\dfrac{\psi(x,\lambda)}{\psi'(x,\lambda)} & \text{if } \psi'(x,\lambda)\neq 0,\end{cases} \tag{37}$$

The function $\phi(0,\lambda)$ is called Prüfer angle. It can be calculated that the function $\Phi(x,\lambda):=\cot\phi(x,\lambda)$ is the solution of the following initial value problem:

$$\frac{d}{dx}\Phi(x,\lambda)+\Phi^2(x,\lambda)=q(x)-\lambda$$
$$\Phi(1,\lambda)=f_2(\lambda)$$

Theorem 3.*If $\phi(0,\lambda)=\widetilde{\phi}(0,\lambda)$ and $f_1(\lambda)=\widetilde{f}_1(\lambda)$, $L=\widetilde{L}$; i.e. the Prüfer angle $\phi(0,\lambda)$ and the coefficient $f_1(\lambda)$ together determine uniquely the problem L.*

*Proof.*It is obvious from (28) and (37) that the equality $m(\lambda)[a_1(\lambda)-b_1(\lambda)\cot\phi(0,\lambda)]=1$ holds. Therefore, under the hypothesis of the theorem $m(\lambda)=\widetilde{m}(\lambda)$. This completes the proof.

3.3 By the norming constants

Now, we turn the notations in the structure of the operator T and make a relation between the norm of an eigenvector on the space H and the sequence ρ_n, defined above. For an element $Y=(y(x),\mathbf{u},\mathbf{v},\mathbf{w})$ in H, the norm of Y is defined by $\|Y\|^2:=\langle Y,Y\rangle$. From (5), we get

$$\|Y\|^2=\int_0^1|y(x)|^2dx-\sum_{k=1}^{N_1}\frac{|u_k|^2}{f_{1k}}+\sum_{k=1}^{N_2}\frac{|v_k|^2}{f_{2k}} -\frac{|u_{N_1+1}|^2}{a_1}+\frac{|u_{N_2+1}|^2}{a_2}+\frac{\alpha_1|w_1|^2}{\beta_1}+\frac{\alpha_2|w_2|^2}{\beta_2}. \tag{38}$$

Lemma 2.*Let λ_n be an eigenvalue of T (or the problem L) and Y_n eigenvector for λ_n. Then, the equality $\|Y_n\|^2=\rho_n$ is valid.*

*Proof.*Let $\lambda_n\neq g_{jk}$. The case $\lambda_n=g_{jk}$ requires minor modification in the following proof.

Using the structure of $D(T)$ and the equalities (5)-(9), a direct calculation yields

$$\|Y_n\|^2=\int_0^1\varphi^2(x,\lambda_n)dx-\sum_{k=1}^{N_1}\frac{|u_k|^2}{f_{1k}}+\sum_{k=1}^{N_2}\frac{|v_k|^2}{f_{2k}}+ -\frac{|u_{N_1+1}|^2}{a_1}+\frac{|u_{N_2+1}|^2}{a_2}+\frac{\alpha_1|w_1|^2}{\beta_1}+\frac{\alpha_2|w_2|^2}{\beta_2}$$

$$=\int_0^1\varphi^2(x,\lambda_n)dx-\sum_{k=1}^{N_1}\frac{|u_k|^2}{f_{1k}}+\sum_{k=1}^{N_2}\frac{|v_k|^2}{f_{2k}}+$$
$$-a_1\varphi^2(0,\lambda_n)+a_2\varphi^2(1,\lambda_n)+$$
$$+\alpha_1\beta_1\varphi^2(d_1-0,\lambda_n)+\alpha_2\beta_2\varphi^2(d_2-0,\lambda_n)$$

$$=\int_0^1\varphi^2(x,\lambda_n)dx-\varphi^2(0,\lambda_n)\sum_{k=1}^{N_1}\frac{f_{1k}}{[g_{1k}-\lambda_n]^2}+$$
$$+\varphi^2(1,\lambda_n)\sum_{k=1}^{N_2}\frac{f_{2k}}{[g_{2k}-\lambda_n]^2}+$$
$$-a_1\varphi^2(0,\lambda_n)+a_2\varphi^2(1,\lambda_n)+$$
$$+\alpha_1\beta_1\varphi^2(d_1-0,\lambda_n)+\alpha_2\beta_2\varphi^2(d_2-0,\lambda_n).$$

$$=\int_0^1\varphi^2(x,\lambda_n)dx-\varphi^2(0,\lambda_n)\left\{a_1+\sum_{k=1}^{N_1}\frac{f_{1k}}{[g_{1k}-\lambda_n]^2}\right\}+$$
$$+\varphi^2(1,\lambda_n)\left\{a_2+\sum_{k=1}^{N_2}\frac{f_{2k}}{[g_{2k}-\lambda_n]^2}\right\}+$$
$$+\alpha_1\beta_1\varphi^2(d_1-0,\lambda_n)+\alpha_2\beta_2\varphi^2(d_2-0,\lambda_n).$$

$$=\int_0^1\varphi^2(x,\lambda_n)dx-f_1'(\lambda_n)\varphi^2(0,\lambda_n)+f_2'(\lambda_n)\varphi^2(1,\lambda_n)+$$
$$+\alpha_1\beta_1\varphi^2(d_1-0,\lambda_n)+\alpha_2\beta_2\varphi^2(d_2-0,\lambda_n)=\rho_n.$$

Theorem 4.*If $\{\lambda_n,\rho_n\}_{n\geq 0}=\left\{\widetilde{\lambda}_n,\widetilde{\rho}_n\right\}_{n\geq 0}$ and $f_1(\lambda)=\widetilde{f}_1(\lambda)$ then $L=\widetilde{L}$; i.e. the spectral data $\{\lambda_n,\rho_n\}_{n\geq 0}$ and the coefficient $f_1(\lambda)$ together determine uniquely the problem L.*

*Proof.*Recall that, $a_1(\lambda)=\widetilde{a}_1(\lambda)$ and $b_1(\lambda)=\widetilde{b}_1(\lambda)$ when $f_1(\lambda)=\widetilde{f}_1(\lambda)$. Denote $\Gamma_n=\{\mu : |\mu|=\left(\sqrt{\lambda_n}+\varepsilon\right)^2\}$, where ε is sufficiently small number. Consider the contour integral

$F_n(\lambda)=\int\limits_{\Gamma_n}\frac{m(\mu)}{(\mu-\lambda)}d\mu, \quad \lambda\in int\Gamma_n$. From (15)-(19), it can be calculated that, $|m(\mu)|\leq C|\mu|^{-N_1-1}$. Therefore, $\lim\limits_{n\to\infty}F_n(\lambda)=0$. According to Lemma1, we obtain

$$m(\lambda)=-\sum_{n=0}^{\infty}Res\left\{\frac{m(\mu)}{(\mu-\lambda)},\lambda_n\right\} \tag{39}$$

$$=\sum_{n=0}^{\infty}\frac{\psi(0,\lambda_n)}{(\lambda-\lambda_n)\Delta'(\lambda_n)} \tag{40}$$

$$=\sum_{n=0}^{\infty}\frac{b_1(\lambda_n)}{\rho_n(\lambda-\lambda_n)} \tag{41}$$

Consequently, if $\lambda_n=\widetilde{\lambda}_n$, $\rho_n=\widetilde{\rho}_n$ for all n and $f_1(\lambda)=\widetilde{f}_1(\lambda)$ then $m(\lambda)=\widetilde{m}(\lambda)$. Hence, Theorem2 yields $L=\widetilde{L}$.

3.4 By two given spectra

We consider the boundary value problem L_1 with the condition $y(0,\lambda)=0$ instead of (2) in L. Let $\{\eta_n^2\}_{n\geq 0}$ be the eigenvalues of the problem L_1. It is obvious that η_n are zeros of $\Delta_1(\eta):=\psi(0,\eta)$.

Theorem 5.*If* $\{\lambda_n,\eta_n\}_{n\geq 0}=\left\{\widetilde{\lambda}_n,\widetilde{\eta}_n\right\}_{n\geq 0}$ *and* $f_1(\lambda)=\widetilde{f}_1(\lambda)$ *then* $L=\widetilde{L}$.

*Proof.*The functions $\Delta(\lambda)$ and $\Delta_1(\eta)$ which are entire of order $\frac{1}{2}$, can be represented by Hadamard's factorization theorem as follows

$$\Delta(\lambda)=C\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right) \tag{42}$$

$$\Delta_1(\eta)=C_1\prod_{n=0}^{\infty}\left(1-\frac{\eta}{\eta_n}\right), \tag{43}$$

where C and C_1 are constants which depend only on $\{\lambda_n\}$ and $\{\eta_n\}$, respectively. Therefore, $\Delta(\lambda)\equiv\widetilde{\Delta}(\lambda)$ and $\Delta_1(\eta)\equiv\widetilde{\Delta}_1(\eta)$, when $\lambda_n=\widetilde{\lambda}_n$ and $\eta_n=\widetilde{\eta}_n$ for all n. Consequently, the equality (28) yields $m(\lambda)\equiv\widetilde{m}(\lambda)$. Hence, the proof is completed by Theorem2.

4 Conclusion

The aim of this paper is to give uniqueness theorems for an inverse Sturm–Liouville problem with eigenparameter-dependent-boundary conditions and transfer conditions. We are studied in two types of generalizations of classical Sturm-Liouville problems: First, both of the boundary conditions depend on λ by Herglotz–Nevanlinna type functions; second, we have two transfer conditions depending linearly on λ. We prove that, if the coefficient $f_1(\lambda)$ in the first boundary condition is known, the other coefficients of the boundary value problem L can be uniquely determined by each of the following:
i) The Weyl function $M(\lambda)$;
ii) The Prüfer angle $\phi(0,\lambda)$;
iii) The sequence $\{\lambda_n,\rho_n\}$ consists of eigenvalues and norming constants;
iv) The sequence $\{\lambda_n,\eta_n\}$ consists of two given spectra.

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