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Duality in Minimax Fractional Programming Problem Involving Nonsmooth Generalized (F,α,ρ,d)-Convexity

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Abstract: In this paper, we discuss nondifferentiable minimax fractional programming problem where the involved functions are locally Lipschitz. Furthermore, weak, strong and strict converse duality theorems are proved in the setting of Mond-Weir type dual under the assumption of generalized (F,α,ρ,d)-convexity.

Keywords: Nondifferentiable minimax fractional programming problem, Generalized (F,α,ρ,d)-convexity, Duality

1 Introduction

Throughout our discussion R^n will denote the n-dimensional Euclidean space unless otherwise mentioned. In nonlinear optimization problems, where minimization and maximization process are performed together, are called minimax (minmax) problems. Frequently, problems of this type arise in many areas like game theory, Chebychev approximation, economics, financial planning and facility location [7]. The optimization problems in which the objective function is a ratio of two functions are commonly known as fractional programming problems. In past few years, many authors have shown interest in the field of minimax fractional programming problems.

It is known that minimax fractional programming problems often arise in management science and in particular in financial planning where objective functions in the optimization problems involve ratios such as cost or profit in time, return on capital, earnings per share. Minimax fractional problems also come to light in discrete rational approximation where the Chebychev norm is used. These minimax problems deal with finitely many ratios.

Recently there has been an increasing interest in developing optimality conditions and duality relations for minimax fractional programming problems. As for their earlier differentiable counterparts, optimality conditions and duality relations have been established under various kinds of generalized convexity assumptions. See for example [1,2,3,4,8,10,11,12,13,14,15,21,22].

In [19], Schmitendorf obtained the necessary and sufficient optimality conditions for generalized minimax programming problem under the condition of convexity. Later, Tanimoto [20] applied the optimality conditions of [19] to define a dual problem and derived the duality theorems for minimax programming problems which are considered by Schmitendorf. Bector and Bhatia [2] relaxed the convexity assumptions in the sufficient optimality condition in [19] and also employed the optimality conditions to construct several dual models which involve pseudo-convex and quasi-convex functions, and derived weak and strong duality theorems. Yadav and Mukherjee [22] construct two types of dual problems for (convex) differentiable fractional minimax programming and derived appropriate duality theorems. In [4], Chandra and Kumar pointed out that the formulation of Yadav and Mukherjee [22] has some omissions and inconsistencies and they constructed two modified dual problems and proved duality theorems for (convex) differentiable fractional minimax programming.

Liu and Wu [14,15] derived the sufficient optimality conditions and duality theorems for the minimax fractional programming in the framework of invexity and (F,α,ρ,d)-convex functions. Liang et al. [12,13]

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introduced a unified formulation of generalized convexity, which was called (F,α,ρ,d)-convex and obtained some corresponding optimality conditions and duality results for the single objective fractional problems and multiobjective problems. For multiobjective fractional programming problem Liu and Feng [16] introduced a new concept of generalized (F,θ,ρ,d)-convexity about the Clarkes generalized gradient. They also established optimality conditions and duality results using the concept of generalized (F,θ,ρ,d)-convexity. Ahmad and Husain [1] established appropriate duality theorems for a class of nondifferentiable minimax fractional programming problems involving (F,α,ρ,d)-pseudo convex function. Recently, Mishra and Rautela [17] derived Karush-Kuhn-Tucker type sufficient optimality conditions and duality theorems for nondifferentiable minimax fractional programming problem under the assumptions of generalized α-type I invex which defined in the setting of Clarke subdifferential functions.

Motivated by the work of Liu and Feng [16] and Mishra and Rautela [17], in this paper, we extend the earlier work of Ahmad and Husain [1] to the nonsmooth case. The paper is organized as follow: Section 2 is devoted to some definitions and notations. In Section 3, we discuss weak, strong and strict converse duality theorems in the setting of Mond-Weir type dual for a class of nondifferentiable minimax fractional programming problems using generalized (F,α,ρ,d)-convexity type assumptions.

2 Preliminaries

We begin with the following definitions and Lemmas that will be needed in the sequel. Let X be a nonempty open subset of R^n. Then, we recall the following:

Definition 1.*A function is said to Lipschitz near $x \in X$ iff for some $K > 0$*
$| f(y) - f(z) | \leq K \| y - z \|$ *for all* y,z within a neighbourhood of x.

We say that $f : X \to R$ is locally Lipschitz on X if it is Lipschitz near any point of X.

Definition 2.*If $f : X \to R$ is locally Lipschitz at $x \in X$, the generalized derivative (in the sence of Clarke [6]) of f at $x \in X$ in the direction $v \in R^n$, denote by $f^0(x;v)$, is given by*

$$f^0(x;v) = \lim_{\lambda \to 0} \sup_{y \to x} \frac{f(y+\lambda v) - f(y)}{\lambda},$$

Definition 3.. *The Clarkes generalized gradient of f at $x \in X$, denoted by $\partial f(x)$ and defined as follows:*

$$\partial f(x) = max\{\xi \in R^n : f^0(x;v) \geq \xi^T v \ \text{ for all } v \in R^n\}.$$

It follows that, for any $v \in R^n$

$$f^0(x;v) = max\{\xi^T v : xi \in \partial f(x)\}.$$

Lemma 1.*Let $\phi_1, \phi_2 : X \to R$ be Lipschitz near x. If $\phi_1 x \geq 0$, $\phi_2 x > 0$ and if ϕ_1, ϕ_2 are regular at x, then*

$$\partial \left(\frac{\phi_1}{\phi_2} \right) = \frac{\phi_2(x)\partial\phi_1(x) - \phi_1(x)\partial\phi_2(x)}{[\phi_2(x)]^2}.$$

Let $f,g : R^n \times R^m \to R$ and $h : R^n \to R^p$ are locally Lipschitz functions. Let A and B be $n \times n$ positive semi-definite matrices. Suppose that Y is a compact subset of R^m. Consider the following nondifferentiable minimax fractional problem:

$$\inf_{x \in R^n} \sup_{y \in Y} \frac{f(x,y) + \langle x,Ax \rangle^{\frac{1}{2}}}{g(x,y) - \langle x,Bx \rangle^{\frac{1}{2}}} \quad \text{subject to } h(x) \leq 0.$$

where $\langle .,. \rangle$ denotes the inner product in Euclidean space. This problem is non-differentiable programming problem if either A or B is nonzero. If A and B are null matrices, the problem (P) is a minimax fractional programming problem. We denote by $\Im_P$ the set of all feasible solutions of (P) and by R^n_+ the positive orthant of R^n. For each $(x,y) \in R^n \times R^m$ define

$$\phi(x,y) = \frac{f(x,y) + \langle x,Ax \rangle^{\frac{1}{2}}}{g(x,y) - \langle x,Bx \rangle^{\frac{1}{2}}}$$

Assume that for each $(x,y) \in \Im_p \times Y$, $f(x,y) + \langle x,Ax \rangle^{\frac{1}{2}} \geq 0$ and $g(x,y) - \langle x,Bx \rangle^{\frac{1}{2}} > 0$. Denote

$$\overline{Y} = \left\{ \overline{y} \in Y : \frac{f(x,\overline{y}) + \langle x,Ax \rangle^{\frac{1}{2}}}{g(x,\overline{y}) - \langle x,Bx \rangle^{\frac{1}{2}}} = \sup_{y \in Y} \frac{f(x,y) + \langle x,Ax \rangle^{\frac{1}{2}}}{g(x,y) - \langle x,Bx \rangle^{\frac{1}{2}}} \right\}$$

$$J = \{1,2,...p\},\ J(x) = \{j \in J : h_j(x) = 0\}.$$

Let K be a triplet such that

$K(x) = \{(s,t,\overline{y}) \in N \times R^s_+ \times R^{ms} : 1 \leq s \leq n+1,\ t = (t_1,t_2,...,t_s) \in R^s_+\}$

with $\sum_{i=1}^{s} t_1 = 1$ and $\overline{y} = (\overline{y_1}, \overline{y_2}, ..., \overline{y_s})$ and $\overline{y_i} \in \overline{Y}(x)$, $\forall\ i = 1,2,...,s$.

Since f and g are continuous differentiable, and Y is a compact subset of R^m, it follows that for each $x^* \in \Im_p$, $\overline{Y}(x^*) \neq \phi$. Thus for any $\overline{y_i} \in \overline{Y}(x^*)$, we have a positive constant $k_0 = \phi(x^*, \overline{y_i})$ we shall need the following generalized Schwarz inequality in our discussions:

$$\langle x,Av \rangle \leq \langle x,Ax \rangle^{\frac{1}{2}} \langle v,Av \rangle^{\frac{1}{2}} \text{ for some } x,v \in R^n \qquad (1)$$

the equality holds when $Ax=\lambda Av$ for some $\lambda \geq 0$. Hence if $\langle v,Av\rangle^{\frac{1}{2}} \leq 1$, we have $\langle x,Av\rangle \leq \langle x,Ax\rangle^{\frac{1}{2}}$.

Definition 4. *A functional $F: X\times X\times R^n \to R$ where $X\subseteq R^n$ is said to be sub-linear if for $(x,x_0)\in X\times X$*

(i) $F(x,x_0;a_1+a_2) \leq F(x,x_0;a_1)+F(x,x_0;a_2)$ *for all* $a_1,a_2\in R^n$
(ii) $F(x,x_0;\alpha a)=\alpha F(x,x_0;a)$ *for all* $\alpha\in R,\ \alpha\geq 0$ *and* $a\in R^n$.

Based upon the concept of sublinear functional, we recall a unified formulation of generalized convexity [i.e., (F,α,ρ,d)-convexity] where the involved functions are locally Lipschitz given in Liu and Feng [16] as follows.

Definition 5. *Let $F: X\times X\times R^n\to R$ be a sublinear functional, let the function $\zeta: X\to R$ be locally Lipschitz at $x_0\in X$, $\alpha: X\times X\to R_+\backslash\{0\}$, $\rho\in R$ and $d: X\times X\to R$. The function ζ is said to be (F,α,ρ,d)-convexity at x_0 if $\zeta(x)-\zeta(x_0)\geq F(x,x_0;\alpha(x,x_0)\zeta)+\rho d^2(x,x_0),\ \forall\,\zeta\in\partial\zeta(x_0)$.*

The function ζ is said to be (F,α,ρ,d)-convex over X if for all $x_0\in X$, it is (F,α,ρ,d)-convex at x_0. In particular, ζ is said to be strongly (F,α,ρ,d)-convex or (F,ρ)-convex if $\rho>0$ or $\rho=0$, respectevely.

Special Cases. From Definition 5, there are the following special cases:

(i) If $\alpha(x,x_0)=1$, for all $x,x_0\in X$, then the (F,α,ρ,d)-convexity is the (F,ρ)-convexity defined in [5].
(ii) If $F(x,x_0;\alpha(x,x_0)\zeta)=\zeta'\eta(x,x_0)$ for a certain map $\eta: X\times X\to R^n$, then, the (F,α,ρ,d)-convexity is the ρ-invexity of [9].
(iii) If *zeta* is continuous differentiable at x_0, then we obtain (F,α,ρ,d)-type convexity [12].
(iv) If $\rho=0$ or $d(x,x_0)=0$ for all $x,x_0\in X$ and if $F(x,x_0;\alpha(x,x_0)\zeta)=\zeta'\eta(x,x_0)$ for a certain map $\eta: X\times X\to R^n$, then the (F,α,ρ,d)-convexity reduces to the invexity [18].

Definition 6. *Let $F: X\times X\times R^n\to R$ be a sublinear functional, let the function $\zeta: X\to R$ be locally Lipschitz at $x_0\in X$, $\alpha: X\times X\to R_+\backslash\{0\}$, $\rho\in R$ and $d: X\times X\to R$. The function ζ is said to be (F,α,ρ,d)-pseudoconvex at x_0, if $\zeta(x)<\zeta(x_0)\implies F(x,x_0;\alpha(x,x_0)\zeta)<-\rho d^2(x,x_0),\ \forall\,\zeta\in\partial\zeta(x_0)$.*

Further, ζ is said to be strictly (F,α,ρ,d)-pseudoconvex at x_0, if $F(x,x_0;\alpha(x,x_0)\zeta)\geq-\rho d^2(x,x_0)\implies\zeta(x)>\zeta(x_0)\ \ \forall\,\zeta\in\partial\zeta(x_0)$. The following result from [9] is needed in the sequel.

Lemma 2. *Let x^* be an optimal solution for (P) satisfying $\langle x^*,Ax^*\rangle>0$, $\langle x^*,Bx^*\rangle>0$ and $\partial h_j(x^*)$, $j\in J(x^*)$ are linearly independent. Then there exist $(s,t^*,\bar{y})\in K(x^*), u,v\in R^n$ and $\mu^*\in R^p_+$ such that*

$$0\in\sum_{i=1}^{s}t_i^*(\partial f(x^*,\bar{y_i})+Au-k_0(\partial g(x^*,\bar{y_i})-Bv))+\partial\langle\mu^*,h(x^*)\rangle \qquad (2)$$

$$f(x^*,\bar{y_i})+\langle x^*,Ax^*\rangle^{\frac{1}{2}}-k_0\left(g(x^*,\bar{y_i})-\langle x^*,Bx^*\rangle^{\frac{1}{2}}\right)=0,\quad i=1,2,\ldots,s, \qquad (3)$$

$$\langle\mu^*,h(x^*)\rangle=0 \qquad (4)$$

$$t_i^*\in R^s_+ \text{ with } \sum_{i=1}^{s}t_i^*=1 \qquad (5)$$

$$\langle u,Au\rangle\leq 1,\quad\langle v,Bv\rangle\leq 1 \qquad (6)$$

$$\langle x^*,Au\rangle=\langle x^*,Ax^*\rangle^{\frac{1}{2}},\quad \langle x^*,Bv\rangle=\langle x^*,Bx^*\rangle^{\frac{1}{2}}. \qquad (7)$$

It should be noted that both the matrices A and B are positive definite at the solution x_0 in the above lemma. If one of $\langle Ax^*,x^*\rangle$ and $\langle Bx^*,x^*\rangle$ is zero, or both A and B are singular at x_0, then for $(s,t^*,\bar{y})\in K(x^*)$, we can take

$$Z_{\bar{y}}(x^*)=\{z\in R^n:\langle\zeta_j,z\rangle\leq 0\ \forall\ \zeta_j\in\partial h_j(x^*),j\in J(x^*)\}$$

with any one of the following $(i)-(iii)$ holds for all $v\in\partial f(x^*,\bar{y_i}),\vartheta\in\partial g(x^*,\bar{y_i})$:

(i) $\langle Ax^*,x^*\rangle>0$, $\langle Bx^*,x^*\rangle=0 \implies$

$$\left\langle\sum_{i=1}^{s}t_i^*v+\frac{Ax^*}{\langle Ax^*,x^*\rangle^{\frac{1}{2}}}-k_0\vartheta,z)\right\rangle+\left\langle(k_0^2B)z,z\right\rangle^{\frac{1}{2}}<0$$

(ii) $\langle Ax^*,x^*\rangle=0$, $\langle Bx^*,x^*\rangle>0 \implies$

$$\left\langle\sum_{i=1}^{s}t_i^*\left(v-k_0\left(\vartheta-\frac{Bx^*}{\langle Bx^*,x^*\rangle^12}\right)\right),z\right\rangle+\langle Bz,z\rangle^{\frac{1}{2}}<0$$

(iii) $\langle Ax^*,x^*\rangle=0$, $\langle Bx^*,x^*\rangle=0 \implies$

$$\left\langle\sum_{i=1}^{s}t_i^*(v-k_0\vartheta),z\right\rangle+\langle(k_0B)z,z\rangle^{\frac{1}{2}}+\langle Bz,z\rangle^{\frac{1}{2}}<0.$$

If we take the condition $Z_{\bar{y}}(x^*)=\phi$ in Lemma 2, then the result of Lemma 2 still holds.

3 Duality model

We now recast the necessary condition in Lemma 2 in the following form:

Lemma 3. *Let x^* be an optimal solution for (P). Assume that $\partial g_j(x^*), j\in J(x^*)$ are linearly independent. Then there exist $(s,t^*,\bar{y})\in K$ and $\mu^*\in R^p_+$ such that*

$$0\in\partial\left(\frac{\sum_{i=1}^{s}t_i^*(f(x^*,\bar{y_i})+\langle x^*,Au\rangle)+\langle\mu^*,h(x^*)\rangle}{\sum_{i=1}^{s}t_i^*(g(x^*,\bar{y_i})-\langle x^*,Bv\rangle)}\right) \qquad (8),$$

$$\langle \mu^*, h(x^*)\rangle = 0, \quad (9)$$

$$\langle u, Au\rangle \le 1,\ \langle v, Bv\rangle \le 1,\ \langle x^*, Ax^*\rangle^{\frac{1}{2}} = \langle x^*, Au\rangle,\ \langle x^*, Bx^*\rangle^{\frac{1}{2}} \le \langle x^*, Bv\rangle, \quad (10)$$

$$\mu^* \in R_+^p,\ t_i^* \ge 0 \text{ with } \sum_{i=1}^{s} t_i^*,\ y_i \in Y(x^*)\ \ i = 1,2,...,s. \quad (11)$$

Now, we consider the following Mond-Weir type dual for (P):

$$\textbf{(D)} \max_{(s,t,\bar{y})\in K} \sup_{(z,\mu,u,v)\in H(s,t,\bar{y})} \frac{\sum_{i=1}^{s} t_i^*(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle \mu,h(z)\rangle}{\sum_{i=1}^{s} t_i^*(g(z,\bar{y_i})-\langle z,Bv\rangle)},$$

subject to

$$0 \in \partial\left(\frac{\sum_{i=1}^{s} t_i^*(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle \mu,h(z)\rangle}{\sum_{i=1}^{s} t_i^*(g(z,\bar{y_i})-\langle z,Bv\rangle)}\right), \quad (12)$$

$$\langle u, Au\rangle \le 1, \quad \langle v, Bv\rangle \le 1,$$

$$\langle z, Az\rangle^{\frac{1}{2}} = \langle z, Au\rangle, \quad \langle z, Bz\rangle^{\frac{1}{2}} \le \langle z, Bv\rangle, \quad (13)$$

where $H(s,t,\bar{y}) \in K$ denotes the set of $(z,\mu,u,v) \in R^n \times R_+^p \times R^n \times R^n$ satisfying (12) (13). For a triplet $(s,t,\bar{y}) \in K$, if the set $H(s,t,\bar{y})$ is empty, then we define the supremum over it to be -∞. In this section, we denote

$$\psi(.) = \left[\sum_{i=1}^{s} t_i^*(g(z,\bar{y_i})-\langle z,Bv\rangle)\right]\left[\sum_{i=1}^{s} t_i^*(f(.,\bar{y_i})+\langle .,Au\rangle)+\langle \mu,h(.)\rangle\right] - \left[\sum_{i=1}^{s} t_i^*(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle \mu,h(z)\rangle\right]\left[\sum_{i=1}^{s} t_i^*(g(.,\bar{y_i})-\langle .,Bv\rangle)\right].$$

Suppose that $\sum_{i=1}^{s} t_i^*(f(z,\bar{y_i})+\langle z,Au\rangle) + \langle \mu,h(z)\rangle \ge 0$, $\sum_{i=1}^{s} t_i^*(\langle z,Bv\rangle - g(z,\bar{y_i})) < 0$ and regular for all $(s,t^*,\bar{y}) \in K(z)$, $(z,\mu,u,v) \in H(s,t^*,\bar{y})$.

Theorem 1.*(Weak duality). Let $x \in \Im_p$ be a feasible solution for (P) and let $(z,\mu,k,u,v,s,t,\bar{y})$ be a feasible solution for (D). Suppose that $\psi(.)$ is (F,α,ρ,d)-pseudoconvex at z , and the inequality $\frac{\rho}{\alpha(x,z)} \ge 0$, hold. Then*

$$\sup_{y\in Y} \frac{f(x,y)+\langle x,Ax\rangle^{\frac{1}{2}}}{g(x,y)-\langle x,Bx\rangle^{\frac{1}{2}}} \ge \frac{\sum_{i=1}^{s} t_i(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle \mu,h(z)\rangle}{\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)}.$$

*Proof.*Assume to the contrary that

$$\sup_{y\in Y} \frac{f(x,y)+\langle x,Ax\rangle^{\frac{1}{2}}}{g(x,y)-\langle x,Bx\rangle^{\frac{1}{2}}} < \frac{\sum_{i=1}^{s} t_i(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle \mu,h(z)\rangle}{\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)},$$

for all $y \in Y$. If we replace y by $\bar{y_i}$ in the above inequality and sum up after multiplying by t_i, then we have

$$\left[\sum_{i=1}^{s} t_i f(x,\bar{y_i})+\langle x,Ax\rangle^{\frac{1}{2}}\right]\left[\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)\right] < \left[\sum_{i=1}^{s} t_i(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle \mu,h(z)\rangle\right]\left[\sum_{i=1}^{s} t_i\left(g(x,\bar{y_i})-\langle x,Bx\rangle^{\frac{1}{2}}\right)\right].$$

Using the generalized Schwartz inequality and (13), we get

$$\psi(x) \le \left[\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)\right]\left[\sum_{i=1}^{s} t_i\left(f(x,\bar{y_i})+\langle x,Ax\rangle^{\frac{1}{2}}\right)+\langle \mu,h(x)\rangle\right] - \left[\sum_{i=1}^{s} t_i(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle \mu,h(z)\rangle\right]\left[\sum_{i=1}^{s} t_i\left(g(x,\bar{y_i})-\langle x,Bx\rangle^{\frac{1}{2}}\right)\right] < \langle \mu,h(x)\rangle \times \left[\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)\right].$$

Since $\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle) > 0$ and $\langle \mu,h(z)\rangle \le 0$, it follows that $\psi(x) < 0 = \psi(z)$.

As is $\psi(x)$ is (F,α,ρ,d)-pseudoconvex at z. Therefore

$$F(x,z;\alpha(x,z)\xi) < -\rho d^2(x,z), \ \ \forall\, \xi \in \partial\psi(z),$$

which yeilds

$$F(x,z;\alpha(x,z)\{[\sum_{i=1}^{s} t_i(g(z,\bar{y_i})- < z,Bv >)]\partial[\sum_{i=1}^{s} t_i(f(z,\bar{y_i})+ < z,Au >)+ < \mu,h(z) >] - [\sum_{i=1}^{s} t_i(f(z,\bar{y_i})+ < z,Au >)+ < \mu,h(z) >]\partial[\sum_{i=1}^{s} t_i(\{g(z,\bar{y_i})- < z,Bv >\})]\}) < -\rho d^2(x,z))$$

On multiplying the above inequality by $\frac{1}{\alpha(x,z)\left[\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)\right]^2}$, using the sublinearity of F and Lemma 1, we have

$$F\left[x,z;\partial\left[\frac{\sum_{i=1}^{s} t_i(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle\mu,h(z)\rangle}{\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)}\right]\right] < -\frac{\rho d^2(x,z)}{\left[\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)\right]^2},$$

Using the fact that $\frac{\rho}{\alpha(x,z)} \geq 0$, we have

$$F\left[x,z;\partial\left[\frac{\sum_{i=1}^{s} t_i(f(z,\bar{y_i})+\langle z,Au\rangle)+\langle\mu,h(z)\rangle}{\sum_{i=1}^{s} t_i(g(z,\bar{y_i})-\langle z,Bv\rangle)}\right]\right] < 0 \quad (14)$$

which contradicts the dual constraints (12), since $F(x,z;0)=0$. Hence the theorem is proved.

Theorem 2.*(Strong Duality). Assume that $\bar{x}$ is an optimal solution for (P) and $\bar{x}$ satisfies a constraints qualification for (P). Then there exist $(\bar{s},\bar{t},\bar{y}^*) \in K(\bar{x})$ and $(\bar{x},\bar{\mu},\bar{u},\bar{v}) \in H(\bar{s},\bar{t},\bar{y}^*)$ such that $(\bar{x},\bar{\mu},\bar{u},\bar{v},\bar{s},\bar{t},\bar{y}^*)$ is feasible for (D). If, in addition, the hypothesis of theorem 3.2 holds for feasible points $(z,\mu,k,u,v,s,t,\bar{y})$, then $(x^*,\mu^*,k^*,u^*,v^*,s^*,t^*,y^*)$ is an optimal solution for (D) and the problem (P) and (D) have the same optimal values.*

*Proof.*By Lemma 3, there exist $(s^*,t^*,\bar{y^*}) \in K(x^*)$ and $(x^*,\mu^*,u^*,v^*) \in \underline{H}(s^*,t^*,y^*)$ such that $(x^*,\mu^*,k^*,u^*,v^*,s^*,t^*,y^*)$ is a feasible for (D) and the two objective values are equal. The optimality of this feasible solution for (D) follows from Theorem 1.

Theorem 3.*(Strict Converse Duality). Let $\bar{x}$ be optimal solution for (P) and let $(\bar{z},\bar{\mu},\bar{k},\bar{u},\bar{v},\bar{s},\bar{t},\bar{y}^*)$ be optimal solution for (D). Assume that the hypothesis of Theorem 2 is fulfilled. Further, assume that $\psi(.)$ is strictly (F,α,ρ,d)-pseudoconvex at $\bar{z}$, and the inequality $\frac{\rho}{\alpha(x,z)} \geq 0$, hold. Then, $\bar{x}=\bar{z}$; that is, $\bar{z}$ is an optimal solution for (P).*

*Proof.*Suppose on the contrary that $\bar{x}\neq\bar{z}$. From Theorem 2, we know that there exist $\bar{s},\bar{t},\bar{y}^*) \in K(\bar{x})$ and $(\bar{x},\bar{\mu},\bar{u},\bar{v}) \in H(\bar{s},\bar{t},\bar{y}^*)$ such that $(\bar{x},\bar{\mu},\bar{u},\bar{v},\bar{s},\bar{t},\bar{y}^*)$ is a feasible for (D) with the optimal value

$$\sup_{y\in Y}\frac{f(\bar{x},y)+\langle\bar{x},A\bar{x}\rangle^{\frac{1}{2}}}{g(\bar{x},y)-\langle\bar{x},B\bar{x}\rangle^{\frac{1}{2}}} = \frac{\sum_{i=1}^{s} t_i\left(f(\bar{z},\bar{y_i})+\langle\bar{z},A\bar{u}\rangle\right)+\langle\bar{\mu},h(\bar{z})\rangle}{\sum_{i=1}^{s} t_i\left(g(\bar{z},\bar{y_i})-\langle\bar{z},B\bar{v}\rangle\right)}.$$

On the other hand, since $(\bar{z},\bar{\mu},\bar{u},\bar{v},\bar{s},\bar{t},\bar{y}^*)$ is feasible for (D), it follows that

$$0\in\partial\left[\frac{\sum_{i=1}^{s} t_i\left(f(\bar{z},\bar{y_i})+\langle\bar{z},A\bar{u}\rangle\right)+\langle\bar{\mu},h(\bar{z})\rangle}{\sum_{i=1}^{s} t_i\left(g(\bar{z},\bar{y_i})-\langle\bar{z},B\bar{v}\rangle\right)}\right].$$

the above inequality along with the sublinearity of F and $\frac{\rho}{\alpha(x,z)} \geq 0$ implies

$$F\left[\bar{x},\bar{z};\partial\left[\frac{\sum_{i=1}^{s} t_i(f(\bar{z},\bar{y_i})+\langle\bar{z},A\bar{u}\rangle)+\langle\bar{\mu},h(\bar{z})\rangle}{\sum_{i=1}^{s} t_i(g(\bar{z},\bar{y_i})-\langle\bar{z},B\bar{v}\rangle)}\right]\right] = 0 \geq \frac{\rho d^2(\bar{x},\bar{z})}{\alpha(\bar{x},\bar{z})}$$

which together with the sublinearity of F and yields

$$F\left[\bar{x},\bar{z};\alpha(\bar{x},\bar{z})\partial\left[\frac{\sum_{i=1}^{s} t_i(f(\bar{z},\bar{y_i})+\langle\bar{z},A\bar{u}\rangle)+\langle\bar{\mu},h(\bar{z})\rangle}{\sum_{i=1}^{s} t_i(g(\bar{z},\bar{y_i})-\langle\bar{z},B\bar{v}\rangle)}\right]\right] \geq -\rho d^2(\bar{x},\bar{z})$$

Using the strict (F,α,ρ,d)-pseudoconvex of $\psi(.)$, we get $\psi(\bar{x}) > \psi(\bar{z})$. Since $\psi(\bar{z})=0$, then we have $\psi(\bar{x})>0$, that is

$$\left[\sum_{i=1}^{s} t_i\left(g(\bar{z},\bar{y_i})-\langle\bar{z},B\bar{v}\rangle\right)\right]\left[\sum_{i=1}^{s} t_i\left(f(\bar{z},\bar{y_i})+\langle\bar{z},A\bar{u}\rangle\right)+\langle\bar{\mu},h(\bar{z})\rangle\right] > \left[\sum_{i=1}^{s} t_i\left(f(\bar{z},\bar{y_i})+\langle\bar{z},A\bar{u}\rangle\right)+\langle\bar{\mu},h(\bar{z})\rangle\right]\left[\sum_{i=1}^{s} t_i\left(g(\bar{z},\bar{y_i})-\langle\bar{z},B\bar{v}\rangle\right)\right] \quad (15)$$

From (1), (13), (15) and $\langle\bar{\mu},h(\bar{x})\rangle \leq 0$ imply

$$\sup_{y\in Y}\frac{f(\bar{x},y)+\langle\bar{x},A\bar{x}\rangle^{\frac{1}{2}}}{g(\bar{x},y)-\langle\bar{x},B\bar{x}\rangle^{\frac{1}{2}}} > \frac{\sum_{i=1}^{s} t_i\left(f(\bar{z},\bar{y}_i^*)+\langle\bar{z},A\bar{u}\rangle\right)+\langle\bar{\mu},h(\bar{z})\rangle}{\sum_{i=1}^{s} t_i\left(g(\bar{z},\bar{y}_i^*)-\langle\bar{z},B\bar{v}\rangle\right)}.$$

Thus, we have a contradiction. Hence the theorem is proved.

4 Conclusion

The notion of generalized (F,α,ρ,d)-convexity is adopted, which includes many other generalized convexity concepts in mathematical programming as special cases. This concept is appropriate to discuss the weak, strong and strict converse duality theorems for a higher order dual (ND) of a nondifferentiable minimax fractional programming problem (NP). The results of this paper can be discussed by formulating a unified higher order dual involving support functions. Frequently, problems of this type arise in many areas and may have a lot of applications in game theory, Chebychev approximation, economics, financial planning and facility location [10].

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