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Series Solution for Porous Medium Equation Arising in Fingero-Imbibition Phenomenon during Oil Recovery Process

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Abstract: In this paper, we consider the Cauchy problem for a porous medium equation in Fingero-imbibition phenomena arising in double phase flow through porous media during oil recovery process. Fingero-imbibition is a physical phenomenon which represents the simultaneous occurrence of two special phenomenon viz. fingering and imbibitions in a porous media. We obtain a series solution of porous medium equation by using Adomian decomposition Method (ADM)

Keywords: Porous medium equation, Adomian decomposition method
AMS Classification: 81U30, 76S05

1 Introduction

Fingero-Imbibition is the simultaneous occurrence of two special phenomena viz. fingering and imbibitions. If a porous medium filled with some fluid is brought in to contact with another fluid which preferentially wets the medium then there is a spontaneous flow of the wetting fluid in to the medium and a counter-flow of the resident fluid from the medium. This phenomenon is called imbibitions again when a fluid contained in a porous medium is displaced by another of lesser viscosity, instead of a regular displacement of the whole front, perturberance may occur that shoot through the porous medium at relatively great speeds. These perturberances are called fingers. The phenomena of finger- imbibitions and the flow of two immiscible fluids through porous media have gained considerable current interest due to their frequent occurrence in problems of petroleum technology.

Investigation of oil recovery by spontaneous imbibition has been a highly active area of research in multiphase flow in porous media .Morrow and Mason [3] provide an excellent review of the recent developments of spontaneous imbibition. The rate of nonwetting phase recovery by capillary imbibition depends on pore geometry, matrix permeability, relative permeability, interfacial tension, size, shape and boundary conditions of the matrix. The shape factor concept was first introduced by Warren and Root [7] to the mathematical model of fractured reservoirs to relate the rate of oil recovery with the flow geometry and average rock matrix pressure. Yildiz et al. [8] presented experimental work on oil recovery by spontaneous water imbibition.

The mathematical models of physical phenomenon of Fingero-imbibition generally get nonlinear partial differential equations. To find analytical solution of such nonlinear partial differential equations is not easy. There are some methods to obtain approximate solution of this kind of equations. Some of them are linearization of the equation, perturbation and numerical methods. In the beginning of the 1980s, Adomian aimed to find new method which is called Adomian Decomposition Method. This method is very useful to find series solution of ordinary differential equations, partial differential equations, integral equations, and also this method avoids linearization of the problem and unnecessary assumptions. In this paper, we obtain a series solution of porous medium equation of fingero- imbibition phenomena which has one nonlinear term by using ADM.

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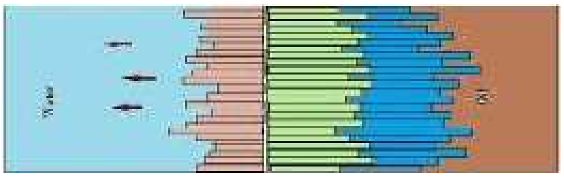

Fig. 1: Schematic diagram of Fingero-Imbibition phenomena

2 Statement and Mathematical Formulation of the problem

During secondary oil recovery, water is injected into the oil reservoir through one well that displaces the oil from the reservoir through a neighboring well. Fingero-imbibition phenomenon occurs at the common interface where injected water meets the oil formatted region. This phenomena occurs in single direction i.e. in x-direction only provided $0 < x < L$which occurs due to some external force after imbibitions phenomena which happens upto small distance $0 < x < l$.

Assuming that Darcys law is valid in the case being investigated, we may write the basic flow equations governing fingero-imbibitions phenomenon as

$$V_w = -\frac{k_w}{\mu_w}.K.\frac{\partial p_w}{\partial x} \tag{1}$$

$$V_o = -\frac{k_o}{\mu_o}.K.\frac{\partial p_o}{\partial x} \tag{2}$$

$$V_p = -V_o \tag{3}$$

$$p_c = p_o - p_w \tag{4}$$

$$\phi\frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} \tag{5}$$

Where V_w and V_o are velocities,k_w and k_o are relative permeabilities;μ_w and μ_o are kinemetic viscosities (which are constants) of the wetting phase and non-wetting phase respectively;ϕ and K are the porosity and permeability of homogeneous medium. The coordinate x is measured along the axis of the cylindrical medium, the origin being located at the imbibition face $x = 0$.

It may be mentioned that the statistical treatment of fingers (Verma [6]) is formally identical to the Buckley Leveret description of two immiscible fluids flow and the displacing phase saturation is defined by the average cross-sectional area occupied by fingers. Combining equation (1) to (5), we get

$$\phi\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[K\frac{k_w k_o}{k_w\mu_o + k_o\mu_w}\frac{dp_c}{dw}\frac{\partial S_x}{\partial t}\right] \tag{6}$$

An analytical expression due to (verma [6]) for relationship between the relative permeability, phase saturation and capillary pressure for Fingero-imbibition phenomena is given by

$$k_w = S_w^3, p_c = -\beta S_w \tag{7}$$

(β is constant)
Negative sign shows the direction of saturation is opposite to capillary pressure.
Simplifying equation (6) and substituting the value of p_c and k_w in equation (6) we get

$$\phi\frac{\partial S_w}{\partial t} - \frac{\beta k}{\mu w}\frac{\partial}{\partial x}\left[S_w^3\frac{\partial S_w}{\partial x}\right] = 0 \tag{8}$$

where k_w and k_0 are function of S_w only (Scheidegger [5]) and the negative sign of the second term is due to the fact that p_c is decreasing function of S_w This is the non linear differential equation of the saturation together with initial condition

$$S_w(x,0) = S(x)$$
$$and Dirichlit's equation is$$
$$S_w(0,T) = S_1(T), S_w(1,T) = S_2(T) \tag{9}$$

The equation (8) together with (9) is the desired governing non-linear partial differential equation describing the linear counter current Fingero-imbibition phenomenon with capillary pressure.

3 Adomian Decomposition Method Theoretic Approach

Equation (8) can be rewritten in the dimensionless form by considering

$$X = \frac{x}{L}, T = \frac{\beta K}{\phi L^2\mu_w}t$$

Substituting this value in equation (8), we get

$$\frac{\partial S_w}{\partial T} - \frac{\partial}{\partial X}\left[S_w^3\frac{\partial S_w}{\partial X}\right] = 0 \tag{10}$$

Equation (10) can be written in operator form $L_T S_w$ as

$$L_T S_w(X,T) = L_X\left(NS_w(X,T)\right) \tag{11}$$

We assume the integration inverse operatorsL_T^{-1} (inverse of L_T),L_X^{-1} (inverse of L_X) exist and are defined of the form:

$$L_T^{-1} = \int(.)dT, L_X^{-1}(.) = \int_0^x(.)dX \tag{12}$$

Then operating both sides of equation (11) with the inverse operators (12), it gives

$$S_w(X,T) = S_{w,o}(X) + L_T^{-1}\left[L_X\left(NS_w(X,T)\right)\right] \tag{13}$$

where $NS_w = S_w^3 \frac{\partial S_w}{\partial X}$ and $S_{(w.o)}$ can be solved subjected to the corresponding initial condition (9) and it gives:

$$S_{w,o}(X) = S(X) \tag{14}$$

By ADM [1], we can write the solution in series form

$$S_w(X,T) = \Sigma_{n=0}^{\infty} S_{wn}(X,T) \tag{15}$$

and

$$NS_w(X,T) = \Sigma_{n=0}^{\infty} A_n(X,T) \tag{16}$$

Where $S_{w0}, S_{w1}, S_{w2}, \ldots\ldots\ldots$ are saturations of different fingers at any X,for any time $T > 0$,where $A_n's$ are the Adomians special polynomials are to be determined.
To find the solution in T-direction, we solves the recursive relations
$S_{w,o} = S_w(X,0), S_{n+1} = L_T^{-1}[L_X(A_n)], n \geq 0$ and the Adomian polynomial, namely

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\Sigma_{n=0}^{\infty}\lambda^n S_{w,n}\right)\right]_{\lambda=0}, n \geq 0 \tag{17}$$

The convergence of this series is studied in [2].
Absolute errors have defined as $|S_w(X,T) - \sigma_n(X,T)|$ in Tables 1 and 2. The exact solution be $S_w(X,T)$ and $sigma_n(X,T)$ be the nth partial sum.

$$\sigma_n(X,T) = \Sigma_{k=0}^{n} S_{w,k}(X,T) \tag{18}$$

$$S_w(X,T) = \lim_{n\to\infty} \sigma_n(X,T) \tag{19}$$

Application

The Exact solution to equation (10) in closed form [4] as follows

$$S_w(X,T) = \frac{X^{2/3}}{\left(1 - \frac{10}{3}T\right)^{1/3}} \tag{20}$$

With initial condition $S_w(X,0) = X^{\frac{2}{3}}$ For the solution of the equation, we use the recursive relation given by (17) to obtain the terms of the decomposition series (18)
In this case

$$\begin{aligned} S_{w,0} &= X^{2/3}, A_0 = \frac{2}{3}X^{5/3} \\ S_{w,1} &= \frac{10}{9}X^{2/3}T, A_1 = \frac{80}{27}X^{5/3}T \\ S_{w,2} &= \frac{200}{81}X^{2/3}T^2, A_2 = \frac{28000}{243}X^{5/3}T^2 \\ S_{w,3} &= \frac{14000}{2187}X^{2/3}T^3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{21}$$

Table 1: Satuartion Vs Time (Keeping Distance fixed)

X=0.1 LTA	X=0.1 ADM	X=0.2 LTA	X=0.2 ADM	X=0.3 LTA
.21544	.21544	.22287	.21789	.23083
.34199	.34199	.35378	.34588	.36642
.44814	.44814	.46359	.45323	.48015
.54288	.54288	.56160	.54905	.58166
.62996	.62996	.65168	.63712	.67495
x=0.3 ADM	X=0.4 LTA	X=0.4 ADM	X=0.5 LTA	X=0.5 ADM
.22045	.23938	.22314	.24858	.22595
.34995	.37999	.35421	.39461	.35868
.45856	.49793	.46415	.51708	.47001
.55551	.60320	.56228	.62640	.56937
.64461	.69995	.65346	.71687	.66070

In this manner the rest of the terms of the decomposition series can be calculated using Maple software.
Substituting these individual terms in (18), we obtain

$$\begin{aligned} \sigma_n(X,T) =& S_{w,0}(X,T) + S_{w,1}(X,T) + S_{w,2}(X,T) \\ &+ S_{w,3}(X,T) + \ldots\ldots \\ =& X^{2/3} + \tfrac{10}{9}X^{2/3}T + \tfrac{200}{81}X^{2/3}T \\ &+ \frac{14000}{2187}X^{2/3}T^3 + \ldots\ldots \end{aligned} \tag{22}$$

Which gives the exact solution as obtained in (20) in the closed form. This result can be verified through substitution. Programming for finding the solution using Maple Software has been given in Appendix.
It is interesting to note that $S_w(X,T)$ in (19) has the following asymptotic behaviour

$$\lim_{T\to\infty} S_w(X,T) = 0 \tag{23}$$

and the flux satisfies

$$\lim_{T\to\infty} S_w\left(S_w(X)\right)_X = 0 \tag{24}$$

Equations (23) represents the approximate solution of Saturation of injected water for Fingero- imbibition phenomena during oil recovery process at any distance X and for any $T > 0$.We can use the boundary condition if we proced X-direction it gives constant solution if $S_1(T)$ and $S_2(T)$ are constant and it gives trivial solution if $S_1(T)$ and $S_2(T)$ are zero.

Table 2: Absolute error (A.E.)$|S_w(X,T) - \sigma_n(X,T)|$ considering appx. up to 4 terms

X=0.1	X=0.2	X=0.3	X=0.4	X=0.5
0.0	.00498	.01038	.o1624	.02263
0.0	.0079	.01647	.02578	.03593
0.0	.01036	.02159	.03378	.04707
0.0	.01255	.02615	.04092	.05703
0.0	.01456	.03034	.04749	.06617

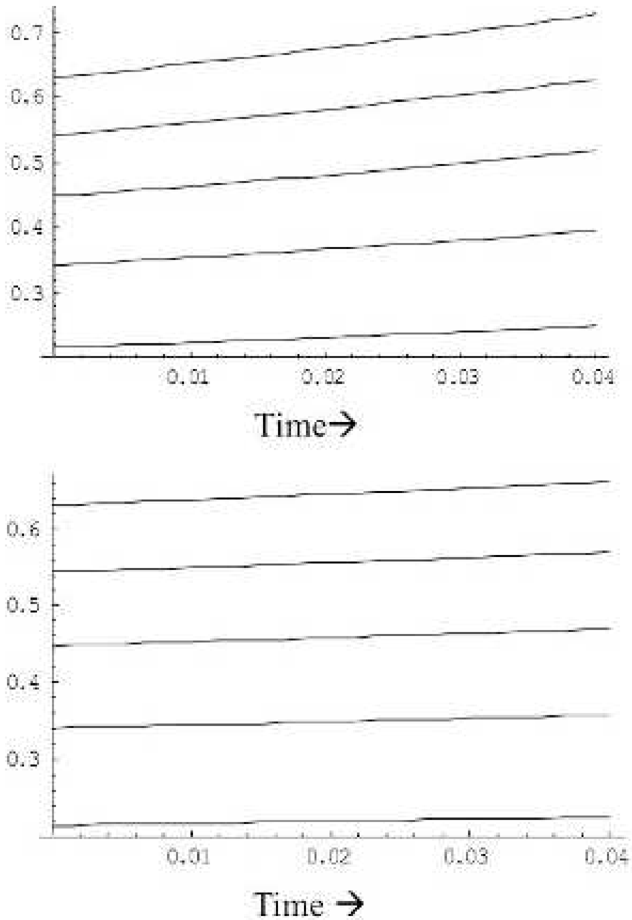

Fig. 2: Saturation Vs.Time(keeping distance fixed)

4 Conclusion

In this paper, we obtained the series solution of porous medium equation. We found that theapproximate solution of the problem is very close to the exact solution of the problem. Thisshows us that no matter how is the nonlinear term the partial differential equation have; wecan find series solution of the partial differential equation without linearization of them.Tables 1 and 2 to show that the series solution converges very rapidly to the exact solution.

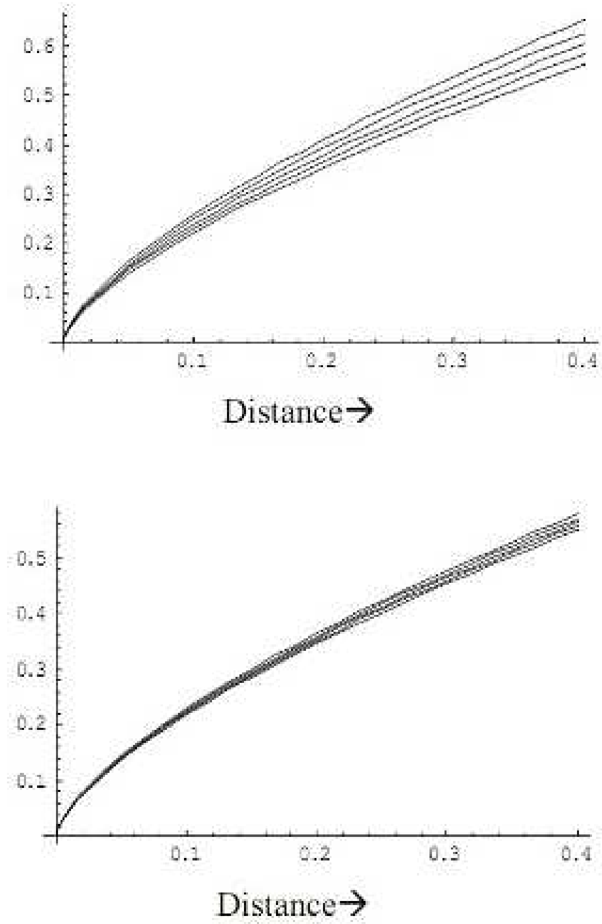

Fig. 3: Saturation vs.Distance (keeping time fixed)

Appendix

Maple Programme for finding Adomian polynomials

```
with(PDETools): declare(Sw(x)): N:=5;
ADM1:=i->convert(subs(lambda=0,
value(1/i!*Diff
(F(Sum(lambdak̂*sw[k],k=0..i)),
lambdai))),diff);  A0:=F(sw[0]);   for i
from 1 to N do A||i:=ADM1(i); od;   for
i from 0 to N do convert(1/4*diff
(subs(seq(sw[k]=Sw[k](x),k=0..N),
expand(subs(F(sw
[0])=sw[0]4̂,A||i))),x),diff); od;
```

Maple Program for finding the Solution

```
with(PDEtools): declare((u,Sw)(x,t));
KN:=5; ADM1:=n->convert(subs(lambda=0,
```

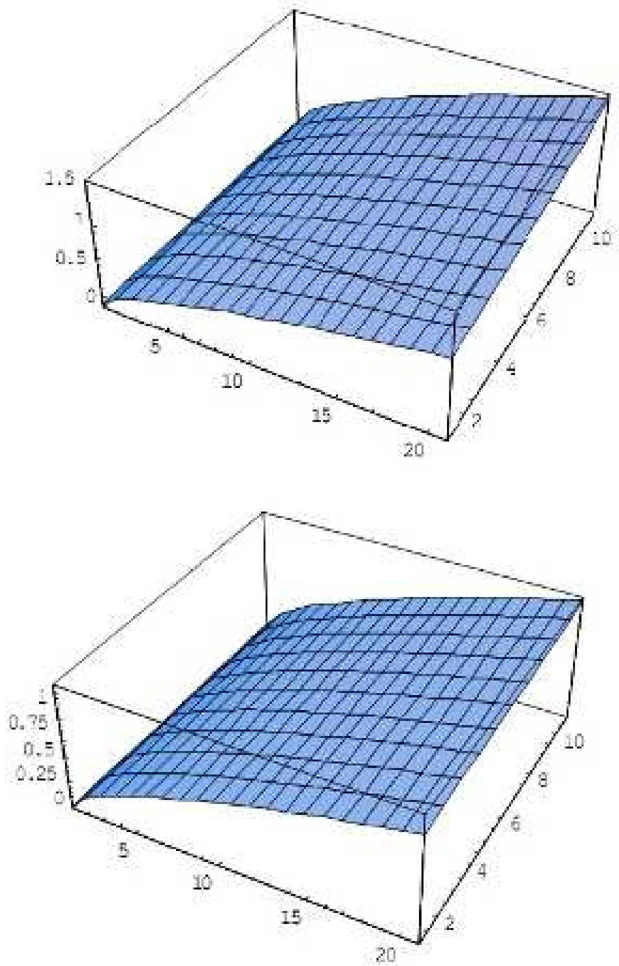

Fig. 4: Saturation vs. Distance and Time)

```
value(1/n!*Diff(F(Sum(
lambda^i * U[i],i = 0..n)),lambda n))),diff);
A0[0]:=F(U[0]);   for n from 1 to KN do
A0[n]:=ADM1(n);
od: for n from 0 to KN do
A[n]:=convert(1/4*diff(subs(seq(U[i]=
Sw[i](x,t),i=0..KN),
expand(subs(F(U[0])=
U[0]^4,A0[n]))),x),diff); od;  Dt := Sw-
> diff(Sw(x,t),t);
NL  :=  Sw-  >  (Sw^3)  *  diff(Sw(x,t),x);
Dx:=Sw->diff(Sw(x,t),x); PDE1:=Sw-
>Dt(Sw)=Dx(NL(Sw)); PDE1(Sw);
IC1:=[Sw(x,0)=f1(x)];f1 := x- > (x^(2/3));
LI:=Sw->Int(Sw(x,t),t=0..t); LI(u);
tr1:=Sw+(f1(x)); KN:=3;
Eq1:=LI(lhs(PDE1(Sw)))=
map(LI,rhs(PDE1(Sw)));
Eq2:=expand(subs(lhs(Eq1)=tr1,Eq1));
trN:=LI(Dx(NL(Sw)))=Int(add(Dx(A[i]),
i=0..KN),t=0..t);
Eq3:=lhs(Eq2)=value(subs(trN,rhs(Eq2)));
```

Maple Programme

```
Apr[0]:=Sw[0](x,t)=select(has,lhs(Eq3),x);
AprK:=Sw[k+1](x,t)=Int(Dx(AD[k]),
t=0..t); for i from 0 to KN do
Apr[i+1]:=expand
(value(subs(seq(Apr[m],m=0..i),
subs(seq(W[m]=Sw[m],m=0..i),
subs(k=i,AD[i]=A[i],AprK)))));
od; trSol:=seq(Apr[i],i=0..KN);
Sol:=value(subs(trSol, Sw=
add(Sw[j](x,t),j=0..KN)));
Sol1:=expand((rhs(Sol)+(x^(2/3))));
```

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