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Selfadjoint Extentions of a First Order Differential Operator

Z. I. Ismailov[1], *M. Sertbaş*[1,*] *and E. Otkun Çevik*[2]

[1] Department of Mathematics, Science Faculty, Karadeniz Technical University, Trabzon, Turkey
[2] Karadeniz Technical University, Institute of Natural Sciences, 61080 Trabzon, Turkey

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Abstract: In this work, all selfadjoint extensions of the minimal operator generated by linear singular multipoint symmetric differential expression $l = (l_-, l_1, \ldots, l_n, l_+)$, $l_\mp = i\frac{d}{dt} + A_\mp, l_k = i\frac{d}{dt} + A_k$, where the coefficients $A_\mp,\ A_k$ are selfadjoint operators in separable Hilbert spaces $H_\mp,\ H_k, k = 1, \ldots, n,\ n \in \mathbb{N}$ respectively, are researched in the direct sum of Hilbert spaces of vector-functions

$$L_2(H_-, (-\infty, a)) \oplus L_2(H_1, (a_1, b_1)) \oplus \ldots \oplus L_2(H_n, (a_n, b_n)) \oplus L_2(H_+, (b, +\infty))$$

$-\infty < a < a_1 < b_1 < \ldots < a_n < b_n < b < +\infty$. Also, the structure of the spectrum of these extensions is investigated.

Keywords: Selfadjoint extension, multipoint differential operators, spectrum.

1 Introduction

The general theory of selfadjoint extensions of a linear densely defined closed symmetric operator in a Hilbert space with equal deficiency indices was led by J. von Neuman in 1929-1930 [12]. Application of this theory to the two-point differential operators and survey of their spectral theory have been studying even in these days (see [3, 6, 10, 13, 17]).

Although the first studies of the theory multipoint differential operators were performed at the beginning of twentieth century, most of them which are about the investigation of the theory and application to spectral problems, have been seen since 1950 ([4, 7–9, 11, 16]). F. Shou-Zhong analyzed the characterizations of all selfadjoint extensions in terms of the domain of adjoint differential operator for singular symmetric minimal operator, which is generated by a differential expression $\sum_{k=0}^{n} p(\cdot)D^k$ with singularity of coefficients in endpoints of the finitely many subintervals of a finite interval in the scalar case [14].

It is well-known that the selfadjoint extension theory which is based on the GKN (Glazmann-Krein-Naimark) Theory [10] is already applied for any number of intervals, finite or infinite and any order expressions (see [4]).

In this work in section 2, by the method of J.W. Calkin-M.L. Gorbachuk (see [2, 6, 13]), all selfadjoint extensions of the minimal operator generated by a singular multipoint symmetric differential operator of first order are described in the direct sum of Hilbert space

$$L_2(H_-, (-\infty, a)) \oplus L_2(H_1, (a_1, b_1)) \oplus \ldots \oplus L_2(H_n, (a_n, b_n)) \oplus L_2(H_+, (b, +\infty)),$$

where $-\infty < a < a_1 < b_1 < \ldots < a_n < b_n < b < +\infty$ in terms of boundary values. In section 3, the spectrum of such extensions is examined.

2 Description of selfadjoint extensions

Let $H_\mp, H_k,\ k = 1, \ldots, n, n \in \mathbb{N}$ be separable Hilbert spaces, $-\infty < a < a_1 < b_1 < \ldots < a_n < b_n < b < +\infty$ and

* Corresponding author e-mail: m.erolsertbas@gmail.com

$dimH_- = dimH_+ \leq +\infty$. In the Hilbert space

$$L_2 := L_2(H_-,(-\infty,a)) \oplus L_2(H_1,(a_1,b_1)) \oplus \dots \oplus L_2(H_n,(a_n,b_n)) \oplus L_2(H_+,(b,+\infty))$$

of vector-functions we consider the linear multipoint differential expression

$$l(u) = \begin{cases} iu'_- + A_-u_-, & -\infty < t < a \\ iu'_k + A_ku_k, & a_k < t < b_k,\ k=1,\dots,n \\ iu'_+ + A_+u_+, & b < t < +\infty, \end{cases}$$

where $u = (u_-,u_1,\dots,u_n,u_+), A_\mp : D(A_\mp) \subset H_\mp \to H_\mp$, $A_k : D(A_k) \subset H_k \to H_k$, $k = 1,\dots,n$ are linear selfadjoint operators. In the linear manifolds $D(A_\mp) \subset H_\mp$, $D(A_k) \subset H_k$ inner products can be introduced like that

$$(f_\mp,g_\mp)_{\mp,+1} := (A_\mp f_\mp, A_\mp g_\mp)_{H_\mp} + (f_\mp,g_\mp)_{H_\mp},$$
$$f_\mp,g_\mp \in D(A_\mp),$$
$$(f_k,g_k)_{k,+1} := (A_kf_k,A_kg_k)_{H_k} + (f_k,g_k)_{H_k},$$
$$f_k,g_k \in D(A_k),\ k=1,\dots,n.$$

Each of $D(A_\mp)$ and $D(A_k)$, $k = 1,\dots,n$ is a Hilbert space under the positive norm $\|\cdot\|_{\mp,+1}$ and $\|\cdot\|_{k,+1}$ with respect to the Hilbert space $H_\mp$ and H_k. It is denoted by $H_{\mp,+1}$ and $H_{k,+1}$. Denote $H_{\mp,-1}$ and $H_{k,-1}$ Hilbert spaces with the negative norm. It is clear that the operators $A_\mp$ and A_k are continuous and that its adjoint operators $\tilde{A}_\mp : H_\mp \to H_{\mp,-1}$ and $\tilde{A}_k : H_k \to H_{k,-1}$ is extensions of the operators $A_\mp$ and A_k respectively. On the other hand, $\tilde{A}_\mp : D(\tilde{A}_\mp) = H_\mp \subset H_{\mp,-1} \to H_{\mp,-1}$ and $\tilde{A}_k : D(\tilde{A}_k) = H_k \subset H_{k,-1} \to H_{k,-1}$ are linear selfadjoint operators. In the direct sum L_2, it is defined as

$$\tilde{l}(u) = (\tilde{l}_-(u_-),\tilde{l}_1(u_1),\dots,\tilde{l}_n(u_n),\tilde{l}_+(u_+)), \qquad (1)$$

where $u = (u_-,u_1,\dots,u_n,u_+)$ and $\tilde{l}_k(u_\mp) = iu'_\mp + \tilde{A}_\mp u_\mp$, $\tilde{l}_k(u_k) = iu'_k + \tilde{A}_ku_k$, $k = 1,\dots,n$.

The minimal L_{-0} (L_{k0} and L_{+0}) and maximal L_- (L_k and L_+) operators generated by differential expression $\tilde{l}_-$ ($\tilde{l}_k$ and $\tilde{l}_+$) in $L_2(H_-,(-\infty,a))$ ($L_2(H_k,(a_k,b_k))$ and $L_2(H_-,(b,+\infty))$) have been investigation in [5].

The operators $L_0 = L_{-0} \oplus L_{10} \oplus \dots \oplus L_{n0} \oplus L_{+0}$ and $L = L_- \oplus L_1 \oplus \dots \oplus L_n \oplus L_+$ in the space L_2 are called minimal and maximal (multipoint) operators generated by the differential expression (1), respectively. Note that the operator L_0 is symmetric and $L_0^* = L$ in L_2. On the other hand, it is clear that, $m(L_{-0}) = 0$, $n(L_{k0}) = dimH_k$, $m(L_{+0}) = dimH_+$, $n(L_{-0}) = dimH_-$, $m(L_{k0}) = dimH_k$, $n(L_{+0}) = 0$, $k = 1,\dots,n$.

Since $dimH_- = dimH_+ \leq +\infty$, then $m(L_0) = n(L_0) = dimH_- + \sum_{k=1}^{n} dimH_k > 0$. Hence, the minimal operator L_0 has a selfadjoint extension [12]. For example, because $dimH_- = dimH_+$, there exists a unitary operator $V : H_- \to H_+$ [18] and the differential expression $\tilde{l}(u)$ with the boundary condition $Vu_-(a) = u_+(b)$, $u(a_k) = u(b_k)$, $k = 1,\dots,n$ is a selfadjoint operator in L_2.

All selfadjoint extensions of the minimal operator L_0 in L_2 in terms of the boundary values are described.

Note that space of boundary values has an important role in the theory of selfadjoint extensions of linear symmetric differential operators [6,13].

Let $B : D(B) \subset \mathscr{H} \to \mathscr{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathscr{H}$, having equal finite or infinite deficiency indices. A triplet $(\mathfrak{H},\gamma_1,\gamma_2)$, where $\mathfrak{H}$ is a Hilbert space, γ_1 and γ_2 are linear mappings of $D(B^*)$ into $\mathfrak{H}$, is called a space of boundary values for the operator B if for any $f,g \in D(B^*)$,

$$(B^*f,g)_{\mathscr{H}} - (f,B^*g)_{\mathscr{H}} = (\gamma_1(f),\gamma_2(g))_{\mathfrak{H}} - (\gamma_2(f),\gamma_1(g))_{\mathfrak{H}},$$

while for any $F,G \in \mathfrak{H}$, there exists an element $f \in D(B^*)$, such that $\gamma_1(f) = F$ and $\gamma_2(f) = G$.

Now we give some notations for convenience as follows

$$L_2(1,0,1) : = L_2(H_-,(-\infty,a)) \oplus 0_1 \oplus \dots \oplus 0_n \oplus L_2(H_+,(b,+\infty))$$
$$L_2(0,1_k,0) : = 0_- \oplus 0_1 \oplus \dots \oplus L_2(H_k,(a_k,b_k)) \oplus 0_{k+1} \oplus \dots \oplus 0_-, \quad k=1,\dots,n,$$

where $0_\mp := 0_{H_\mp}$ and $0_k := 0_{H_k}$ are zero vectors. Note that any symmetric operator with equal deficiency indices has at least one space of boundary values [6].

Firstly, note that the following proposition which validity of this claim can be easily proved.

Lemma 2.1. Let $dimH_- = dimH_+ \leq +\infty$ and $V : H_- \to H_+$ be a unitary operator such that $H_+ = VH_-$. The triplet (H_+,γ_1,γ_2), where

$$\gamma_1 : D((L_{-0} \oplus 0_1 \oplus \dots \oplus 0_n \oplus L_{+0})^*) \to H_+,$$
$$\gamma_1(u) = \frac{1}{i\sqrt{2}}(Vu_-(a) + u_+(b)),$$
$$\gamma_2 : D((L_{-0} \oplus 0_1 \oplus \dots \oplus 0_n \oplus L_{+0})^*) \to H_+,$$
$$\gamma_2(u) = \frac{1}{\sqrt{2}}(Vu_-(a) - u_+(b)),$$
$$u = (u_-,u_1,\dots,u_n,u_+) \in D((L_{-0} \oplus 0_1 \oplus \dots \oplus 0_n \oplus L_{+0})^*)$$

is a space of boundary values of the minimal operator $L_{-0} \oplus 0_1 \oplus \dots \oplus 0_n \oplus L_{+0}$ in the direct sum $L_2(H,(-\infty,a)) \oplus 0_1 \oplus \dots \oplus 0_n \oplus L_2(H_+,(b,+\infty))$.

Proof. For arbitrary $u = (u_-,u_1,\dots u_n,u_+)$ and $v = (v_-,v_1,\dots,v_n,v_+)$ from

$D((L_{-0}\oplus 0_1\oplus\ldots\oplus 0_n\oplus L_{+0})^*)$ the validity of the equality

$$(Lu,v)_{L_2(1,0,1)}-(u,Lv)_{L_2(1,0,1)} = (\gamma_1(u),\gamma_2(v))_{H_+}-(\gamma_2(u),\gamma_1(v))_{H_+}$$

can be easily verified. Now for any given elements $f,g\in H_+$, we will find the function $u=(u_-,u_1,\ldots,u_n,u_+)\in D((L_{-0}\oplus 0_1\oplus\ldots\oplus 0_n\oplus L_{+0})^*)$ such that

$$\gamma_1(u)=\frac{1}{i\sqrt{2}}(Vu_-(a)+u_+(b))=f \quad \text{and}$$
$$\gamma_2(u)=\frac{1}{\sqrt{2}}(Vu_-(a)-u_+(b))=g$$

that is,

$$Vu_-(a)=(if+g)/\sqrt{2} \quad \text{and} \quad u_+(b)=(if-g)/\sqrt{2}.$$

Since $V:H_-\to H_+$ is an isomorphism, so that we can choose the functions $u_-(t),u_+(t)$ in the following form

$$u_-(t)=\int_{-\infty}^{t}e^{s-a}dsV^*(if+g)/\sqrt{2}, \quad \text{with} \quad t<a;$$
$$u_k(t)=\quad 0_k, \quad \text{with} \quad a_k<t<b_k, k=1,\ldots,n;$$
$$u_+(t)=\int_{t}^{\infty}e^{b-t}ds(if-g)/\sqrt{2} \quad \text{with} \quad t>b$$

then it is clear that $(u_-,u_1,\ldots,u_n,u_+)\in D((L_{10}\oplus 0_1\oplus\ldots 0_n\oplus L_{+0})^*)$ and $\gamma_1(u)=f$, $\gamma_2(u)=g$.□

Furthermore, using the result which is obtained in [5] the next assertion is proved.

Lemma 2.2. The triplet $\left(H_k,\gamma_1^{(k)},\gamma_2^{(k)}\right)$,

$$\gamma_1^{(k)},\gamma_2^{(k)}:D((L_{k0})^*)\to H_k,$$
$$\gamma_1^{(k)}(u_k)=\frac{1}{i\sqrt{2}}(u_k(a_k)+u_k(b_k)),$$
$$\gamma_2^{(k)}(u_k)=\frac{1}{\sqrt{2}}(u_k(a_k)-u_k(b_k)),\ u_k\in D((L_{k0})^*)$$

is a space of boundary values of the minimal operator L_{k0} in the Hilbert space $L_2(0,1_k,0)$, $k=1,\ldots,n$.

The following result can be easily established.

Lemma 2.3. Every selfadjoint extension of L_0 in L_2 Hilbert space is a direct sum of selfadjoint extensions of the minimal operator $L_{-0}\oplus 0_1\oplus\ldots\oplus 0_n\oplus L_{+0}$ in $L_2(1,0,1)$ and minimal operators $0_-\oplus 0_1\oplus\ldots\oplus L_{k0}\oplus 0_{k+1}\oplus\ldots\oplus 0_+$ in $L_2(0,1_k,0)$, $k=1,\ldots,n$.

Finally, using the method in [6] the following result can be deduced.

Theorem 2.4. If $\tilde{L}$ is a selfadjoint extension of the minimal operator L_0 in L_2, then it generates by differential expression (1) and boundary conditions

$$u_+(b)=W_0u_-(a),$$
$$u_k(b_k)=W_ku_k(a_k), k=1,\ldots,n$$

where $W_0:H_-\to H_+$ and $W_k:H_k\to H_k$, $k=1,\ldots,n$ are unitary operators. Moreover, the unitary operators W_0,W_k are determined uniquely by the extension $\tilde{L}$; i.e. $\tilde{L}:=L_W=L_{W_0}\oplus L_{W_1}\oplus\ldots\oplus L_{W_n}$, $W=(W_0,W_1,\ldots,W_n)$ and vice versa.

3 The spectrum of the selfadjoint extensions

In this section the structure of the spectrum of the selfadjoint extension L_W in L_2 will be investigated. In this case from Lemma 2.4 it is clear that

$$L_W=L_{W_0}\oplus L_{W_1}\oplus\ldots\oplus L_{W_n},$$

where L_{W_0} and L_{W_k}, $k=1,\ldots,n$ are selfadjoint extensions of the minimal operators $L_0(1,0,1)=L_{10}\oplus 0_1\oplus\ldots\oplus 0_n\oplus L_{+0}$ and $L_0(0,1_k,0)=0_-\oplus 0_1\oplus\ldots\oplus L_{k0}\oplus 0_{k+1}\oplus\ldots\oplus 0_+$ in the Hilbert spaces $L_2(1,0,1)$ and $L_2(0,1_k,0)$, respectively.

First, we have to prove the following result.

Theorem 3.1. The point spectrum of any selfadjoint extension L_{W_0} in the Hilbert space $L_2(1,0,1)$ is empty; i.e.,

$$\sigma_p(L_{W_0})=\emptyset.$$

Proof. Let us consider the following problem for the spectrum of the selfadjoint extension L_{W_0} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$,

$$L_{W_0}u=\lambda u, \quad u=(u_-,0_1,\ldots,0_n,u_+)\in L_2(1,0,1);$$

that is,

$$\tilde{l}_-(u_-)=iu_-'+\tilde{A}_-u_-=\lambda u_-,u_-\in L_2(H_-,(-\infty,a)),$$
$$\tilde{l}_+(u_+)=iu_+'+\tilde{A}_+u_+=\lambda u_+,$$
$$u_+\in L_2(H_+,(b,+\infty)),\lambda\in\mathbb{R},$$
$$u(b)=W_0u_-(a).$$

The general solution of this problem is

$$u_-(\lambda;t)=e^{i(\tilde{A}_--\lambda)(t-a)}f_-^*, \quad t<a,$$
$$u_+(\lambda;t)=e^{i(\tilde{A}_+-\lambda)(t-b)}f_+^*, \quad t>b,$$
$$f_+^*=W_0f_-^*, \quad f_-^*,\in H_-,\ f_+^*\in H_+.$$

It is clear that for the $f_-^*\neq 0$, $f_+^*\neq 0$ the functions $u_-(\lambda;.)\notin L_2(H_-,(-\infty,a)),u_+(\lambda;.)\notin L_2(H_+,(b,+\infty))$. Therefore for every isometric isomorphism W_0 we have $\sigma_p(L_{W_0})=\emptyset$. □

Since residual spectrum of any selfadjoint operator in any Hilbert space is empty, it is sufficient to investigate the continuous spectrum of the selfadjoint extensions L_{W_0} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$.

Theorem 3.2. The continuous spectrum of any selfadjoint extension L_{W_0} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$ is $\sigma_c(L_{W_0})=\mathbb{R}$.

Proof. Firstly, we search for the resolvent operator of the extension L_{W_0} generated by the differential expression $(\tilde{l}_-, 0_1, \dots, 0_n, \tilde{l}_+)$ and the boundary condition

$$u_+(b) = W_0 u_-(a)$$

in the Hilbert space $L_2(1,0,1)$; i.e.

$$\begin{aligned}
&\tilde{l}_-(u_-) = iu'_- + \tilde{A}_- u_- = \lambda u_- + f_-,\\
&u_-, f_- \in L_2(H_-, (-\infty, a)),\\
&\tilde{l}_+(u_+) = iu'_+ + \tilde{A}_+ u_+ = \lambda u_+ + f_+,\\
&u_+, f_+ \in L_2(H_+, (b, +\infty)),\\
&\lambda \in \mathbb{C}, \quad \lambda_i = Im\lambda > 0\\
&u_+(b) = W_0 u_-(a) \qquad (2)
\end{aligned}$$

Now, we will show that the following function

$$u(\lambda;t) = (u_-(\lambda;t), 0_1, \dots, 0_n, u_+(\lambda;t)),$$

where

$$\begin{aligned}
&u_-(\lambda;t) = e^{i(\tilde{A}_- - \lambda)(t-a)} f_-^* + i\int_t^a e^{i(\tilde{A}_- - \lambda)(t-s)} f_-(s)ds,\\
&t < a,\\
&u_+(\lambda;t) = i\int_t^\infty \mathrm{e}^{i(\tilde{A}_+ - \lambda)(t-s)} f_+(s)ds, \quad t > b,\\
&f_-^* = W_0^* \left(i\int_b^\infty \mathrm{e}^{i(\tilde{A}_+ - \lambda)(t-s)(b-s)} f_+(s)ds \right)
\end{aligned}$$

is a solution of the boundary value problem (2) in the Hilbert space $L_2(1,0,1)$. It is sufficient to show that

$$u_-(\lambda;t) \in L_2(H_-, (-\infty, a)),$$
$$u_+(\lambda;t) \in L_2(H_+, (b, +\infty))$$

for $\lambda_i > 0$. Indeed, in this case

$$\begin{aligned}
&\|f_-^*\|_{H_-}^2 = \left\| \int_b^\infty \mathrm{e}^{i(\tilde{A}_+ - \lambda)(b-s)} f_+(s)ds \right\|_{H_+}^2\\
&\le \left(\int_b^\infty e^{\lambda_i(b-s)} \|f_+(s)\|_{H_+} ds \right)^2\\
&\le \left(\int_b^\infty e^{2\lambda_i(b-s)} ds \right) \left(\int_b^\infty \|f_+(s)\|_{H_+}^2 ds \right)\\
&= \frac{1}{2\lambda_i} \|f_+\|_{L_2(H_+, (b, +\infty))}^2 < \infty,
\end{aligned}$$

$$\begin{aligned}
&\|e^{i(\tilde{A}_- - \lambda)(t-a)} f_-^*\|_{L_2(H_-, (-\infty, a))}^2\\
&= \|e^{-i\lambda(t-a)} f_-^*\|_{L_2(H_-, (-\infty, a))}^2\\
&= \int_{-\infty}^a \|e^{-i\lambda(t-a)} f_-^*\|_{H_-}^2 dt\\
&= \int_{-\infty}^a e^{2\lambda_i(t-a)} dt \|f_-^*\|_{H_-}^2 = \frac{1}{2\lambda_i} \|f_-^*\|_{H_-}^2 < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\left\| i\int_t^a e^{i(\tilde{A}_- - \lambda)(t-s)} f_-(s)ds \right\|_{L_2(H_-, (-\infty, a))}^2\\
&\le \int_{-\infty}^a \left(\int_t^a e^{\lambda_i(t-s)} \|f_-(s)\|_{H_-} ds \right)^2 dt\\
&\le \int_{-\infty}^a \left(\int_t^a e^{\lambda_i(t-s)} ds \right) \left(\int_t^a e^{\lambda_i(t-s)} \|f_-(s)\|^2 ds \right) dt\\
&= \frac{1}{\lambda_i} \int_{-\infty}^a \int_t^a e^{\lambda_i(t-s)} \|f_-(s)\|^2 ds dt\\
&= \frac{1}{\lambda_i} \int_{-\infty}^a \left(\int_{-\infty}^s e^{\lambda_i(t-s)} \|f_-(s)\|^2 dt \right) ds\\
&= \frac{1}{\lambda_i} \int_{-\infty}^a \left(\int_{-\infty}^s e^{\lambda_i(t-s)} dt \right) \|f_-(s)\|^2 ds\\
&= \frac{1}{\lambda_i^2} \int_{-\infty}^a \|f_-(s)\|^2 ds = \frac{1}{\lambda_i^2} \|f_-\|_{L_2(H_-, (-\infty, a))}^2 < \infty.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left\| i\int_t^\infty e^{i(\tilde{A}_+ - \lambda)(t-s)} f_+(s)ds \right\|_{L_2(H_+, (b, +\infty))}^2\\
&\le \int_b^\infty \left(\int_t^\infty e^{\lambda_i(t-s)} \|f_+(s)\|_{H_+} ds \right)^2 dt\\
&\le \int_b^\infty \left(\int_t^\infty e^{\lambda_i(t-s)} ds \right) \left(\int_t^\infty e^{\lambda_i(t-s)} \|f_+(s)\|^2 ds \right) dt\\
&= \frac{1}{\lambda_i} \int_b^\infty \left(\int_t^\infty e^{\lambda_i(t-s)} \|f_+(s)\|^2 ds \right) dt\\
&= \frac{1}{\lambda_i} \int_b^\infty \left(\int_b^s e^{\lambda_i(t-s)} \|f_+(s)\|^2 dt \right) ds\\
&= \frac{1}{\lambda_i} \int_b^\infty \left(\int_b^s e^{\lambda_i(t-s)} dt \right) \|f_+(s)\|^2 ds\\
&= \frac{1}{\lambda_i^2} \int_b^\infty (1 - e^{\lambda_i(b-s)}) \|f_+(s)\|^2 ds\\
&\le \frac{1}{\lambda_i^2} \|f_+\|_{L_2(H_+, (b, +\infty))}^2 < \infty.
\end{aligned}$$

Above calculations imply that $u_-(\lambda;t) \in L_2(H_-, (-\infty, a))$, $u_+(\lambda;t) \in L_2(H_+, (b, +\infty))$ for $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0$. On the other hand, one can easily verify that $u(\lambda;t) = (u_-(\lambda;t), 0_1, \dots, 0_n, u_+(\lambda;t))$ is a solution of the boundary value problem (2).

When $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda < 0$ is true solution of the boundary value problem,

$$\begin{aligned}
&L_{W_0} u = \lambda u + f, u = (u_-, 0_1, \dots, 0_n, u_+),\\
&f = (f_-, 0_1, \dots, 0_n, f_+) \in L_2(1,0,1)\\
&u_+(b) = W_0 u_-(a),
\end{aligned}$$

where W_0 is a unitary operator from H_- to H_+, is in the form $u(\lambda;t) = (u_-(\lambda;t), 0_1, \dots, 0_n, u_+(\lambda;t))$,

$$\begin{cases} u_-(\lambda;t) = -i\int_{-\infty}^{t} e^{i(\tilde{A}_- - \lambda)(t-s)} f_-(s)ds, \ \ t<a \\ u_+(\lambda;t) = e^{i(\tilde{A}_+ - \lambda)(t-b)} f_+^* - i\int_b^t e^{i(\tilde{A}_+ - \lambda)(t-s)} f_+(s)ds, \\ \ t>b, \end{cases}$$

$$f_+^* = W_0\left(-i\int_{-\infty}^{a} e^{i(\tilde{A}_- - \lambda)(a-s)} f_-(s)ds\right).$$

First, we prove that $u(\lambda;t) \in L_2(1,0,1)$. In this case,

$$\begin{aligned}
&\|u_-(\lambda;t)\|^2_{L_2(H_-,(-\infty,a))} \\
&= \int_{-\infty}^{a}\left\|-i\int_{-\infty}^{t} e^{i(\tilde{A}_- - \lambda)(t-s)} f_-(s)ds\right\|^2_{H_-} dt \\
&\leq \int_{-\infty}^{a}\left(\int_{-\infty}^{t} e^{\lambda_i(t-s)}ds\right)\left(\int_{-\infty}^{t} e^{\lambda_i(t-s)}\|f_-(s)\|^2_{H_-}ds\right)dt \\
&= \frac{1}{|\lambda_i|}\int_{-\infty}^{a}\int_{-\infty}^{t} e^{\lambda_i(t-s)}\|f_-(s)\|^2_{H_-}dsdt \\
&= \frac{1}{|\lambda_i|}\int_{-\infty}^{a}\left(\int_{s}^{a} e^{\lambda_i(t-s)}\|f_-(s)\|^2_{H_-}dt\right)ds \\
&= \frac{1}{|\lambda_i|}\int_{-\infty}^{a}\left(e^{\lambda_i(t-s)}dt\right)\|f_-(s)\|^2_{H_-}ds \\
&= \frac{1}{|\lambda_i|^2}\int_{-\infty}^{a}(1-e^{\lambda_i(a-s)})\|f_-(s)\|^2_{H_-}ds \\
&\leq \frac{1}{|\lambda_i|^2}\|f_-\|^2_{L_2(H_-,(-\infty,a))} < \infty,
\end{aligned}$$

$$\begin{aligned}
&\|f_+^*\|^2_{H_+} = \left\|\int_{-\infty}^{a} e^{i(\tilde{A}_- - \lambda)(a-s)} f_-(s)ds\right\|^2_{H_-} \\
&\leq \left(\int_{-\infty}^{a} e^{\lambda_i(a-s)}\|f_-(s)\|_{H_-}ds\right)^2 \\
&\leq \left(\int_{-\infty}^{a} e^{2\lambda_i(a-s)}ds\right)\left(\int_{-\infty}^{a}\|f_-(s)\|^2_{H_-}ds\right) \\
&= \frac{1}{2|\lambda_i|}\|f_-\|^2_{L_2(H_-,(-\infty,a))} < \infty,
\end{aligned}$$

$$\begin{aligned}
&\|e^{i(\tilde{A}_+ - \lambda)(t-b)} f_+^*\|^2_{L_2(H_+,(b,+\infty))} \\
&\leq \int_b^{\infty} e^{2\lambda_i(t-b)}dt\|f_+^*\|^2_{H_+} = \frac{1}{2|\lambda_i|}\|f_+^*\|^2_{H_+} \\
&\leq \frac{1}{4|\lambda_i|^2}\|f_-\|^2_{L_2(H_+,(b,+\infty))} < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\int_b^t e^{i(\tilde{A}_+ - \lambda)(t-s)} f_+(s)ds\right\|^2_{L_2(H_+,(b,+\infty))} \\
&\leq \int_b^{\infty}\left(\int_b^t e^{\lambda_i(t-s)}\|f_+(s)\|_{H_+}ds\right)^2 dt \\
&\leq \int_b^{\infty}\left(\int_b^t e^{\lambda_i(t-s)}ds\right)\left(\int_b^t e^{\lambda_i(t-s)}\|f_+(s)\|^2_{H_+}ds\right)dt \\
&= \int_b^{\infty}\left(\frac{1}{\lambda_i}(1-e^{\lambda_i(t-b)})\right)\left(\int_b^t e^{\lambda_i(t-s)}\|f_+(s)\|^2_{H_+}ds\right)dt \\
&\leq \frac{1}{|\lambda_i|}\int_b^{\infty}\left(\int_b^t e^{\lambda_i(t-b)}\|f_+(s)\|^2_{H_+}ds\right)dt \\
&= \frac{1}{|\lambda_i|}\int_b^{\infty}\left(\int_s^{\infty} e^{\lambda_i(t-s)}\|f_+(s)\|^2_{H_+}dt\right)ds \\
&= \frac{1}{|\lambda_i|}\int_b^{\infty}\left(\int_s^{b} e^{\lambda_i(t-s)}dt\right)\|f_+(s)\|^2_{H_+}ds \\
&= \frac{1}{|\lambda_i|^2}\|f_+\|^2_{L_2(H_+,(b,+\infty))} < \infty.
\end{aligned}$$

The above simple calculations show that $u_-(\lambda;\cdot) \in L_2(H_-,(-\infty,a))$, $u_+(\lambda;\cdot) \in L_2(H_+,(b,+\infty))$; i.e. $u(\lambda;\cdot) = (u_-(\lambda;\cdot),0_1,\ldots,0_n,u_+(\lambda,\cdot)) \in L_2(1,0,1)$ in case $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda < 0$. On the other hand it can be verified that the function $u(\lambda;\cdot)$ satisfies the equation $L_{W_0}u = \lambda u(\lambda;\cdot) + f$ and $u_+(b) = W_0 u_-(a)$.

Hence, the following result has been proved that for the resolvent set $\rho(L_{W_0})$

$$\rho(L_{W_0}) \supset \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}.$$

Now, we will study continuous spectrum $\sigma_c(L_{W_0})$ of the extension L_{W_0}. For $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0$, norm of the resolvent operator $R_\lambda(L_{W_0})$ of the L_{W_0} is of the form

$$\begin{aligned}
&\|R_\lambda(L_{W_0})f(t)\|^2_{L_2(1,0,1)} \\
&= \left\|e^{i(\tilde{A}_- - \lambda)(t-a)} f_-^* + i\int_t^a e^{i(\tilde{A}_- - \lambda)(t-s)} f_-(s)ds\right\|^2_{L_2(H_-,(-\infty,a))} \\
&+ \left\|i\int_t^{\infty} e^{i(\tilde{A}_+ - \lambda)(t-s)} f_+(s)ds\right\|^2_{L_2(H_+,(b,+\infty))}, \\
&f = (f_-,0_1,\ldots,0_n,f_+) \in L_2(1,0,1).
\end{aligned}$$

Then, it is clear that for any $f = (f_-,0_1,\ldots,0_n,f_+) \in L_2(1,0,1)$ the following inequality is true.

$$\|R_\lambda(L_{W_0})f(t)\|^2_{L_2} \geq \left\|i\int_t^{\infty} e^{i(\tilde{A}_+ - \lambda)(t-s)} f_+(s)ds\right\|^2_{L_2(H_+,(b,+\infty))}.$$

The vector functions $f^*(\lambda;t)$ which is of the form $f^*(\lambda;t) = (0_-,0_1,\ldots,0_n,e^{i(\tilde{A}_+ - \bar{\lambda})t}f_+)$, $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0$, $f_+ \in H_+$ belong to $L_2(1,0,1)$. Indeed,

$$\|f^*(\lambda;t)\|^2_{L_2} = \int_b^\infty \|e^{i(\tilde{A}_+ - \bar{\lambda})t} f_+\|^2_{H_+} dt$$
$$= \int_b^\infty e^{-2\lambda_i t} dt \|f_+\|^2_{H_+} = \frac{1}{2\lambda_i} e^{-2\lambda_i b} \|f_+\|^2_{H_+} < \infty.$$

For such functions $f^*(\lambda;\cdot)$, we have

$$\|R_\lambda(L_{W_0}) f^*(\lambda;t)\|^2_{L_2(H_+,(b,+\infty))}$$
$$\geq \left\| i \int_t^\infty e^{i(\tilde{A}_+ - \lambda)(t-s)} e^{i(\tilde{A}_+ - \bar{\lambda})s} f_+ ds \right\|^2_{L_2(H_+,(b,+\infty))}$$
$$= \left\| \int_t^\infty e^{-i\lambda t} e^{-2\lambda_i s} e^{i\tilde{A}_+ t} f_+ ds \right\|^2_{L_2(H_+,(b,+\infty))}$$
$$= \left\| e^{-i\lambda t} e^{i\tilde{A}_+ t} \int_t^\infty e^{-2\lambda_i s} f_+ ds \right\|^2_{L_2(H_+,(b,+\infty))}$$
$$= \|e^{-i\lambda t} \int_t^\infty e^{-2\lambda_i s} ds\|^2_{L_2(H_+,(b,+\infty))} \|f_+\|^2_{H_+}$$
$$= \frac{1}{4\lambda_i^2} \int_b^\infty e^{-2\lambda_i t} dt \|f_3\|^2_H = \frac{1}{8\lambda_i^3} e^{-2\lambda_i b} \|f_+\|^2_{H_+}.$$

From this we get

$$\|R_\lambda(L_{W_0}) f^*(\lambda;\cdot)\|_{L_2} \geq \frac{e^{-\lambda_i b}}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}} \|f_+\|_{H_+}$$
$$= \frac{1}{2\lambda_i} \|f^*(\lambda;\cdot)\|_{L_2}$$

i.e. for $\lambda_i = Im\lambda > 0$ and $f_+ \neq 0$

$$\frac{\|R_\lambda(L_{W_0}) f^*(\lambda;\cdot)\|_{L_2}}{\|f^*(\lambda;\cdot)\|_{L_2}} \geq \frac{1}{2\lambda_i}.$$

is valid. On the other, hand it is clear that

$$\|R_\lambda(L_{W_0})\| \geq \frac{\|R_\lambda(L_{W_0}) f^*(\lambda;\cdot)\|_{L_2}}{\|f^*(\lambda;\cdot)\|_{L_2}}, \quad f_+ \neq 0.$$

Consequently, we have

$$\|R_\lambda(L_{W_0})\| \geq \frac{1}{2\lambda_i} \quad \text{for} \quad \lambda \in \mathbb{C}, \ \lambda_i = Im\lambda > 0.$$

□

Furthermore, the spectrum of selfadjoint extensions of the minimal operator $L_0(0,1_k,0), k = 1,\ldots,n$ will be investigated.

Theorem 3.3. For all $k = 1,\ldots,n,\ n \in \mathbb{N}$ the spectrum of the selfadjoint extension L_{W_k} of the minimal operator $L_0(0,1_k,0)$ in the Hilbert space $L_2(0,1_k,0)$ is of the form

$$\sigma(L_{W_k}) = \Big\{\lambda \in \mathbb{R}\colon \lambda = \frac{1}{b_k - a_k} \arg\mu + \frac{2m\pi}{b_k - a_k},\ m \in \mathbb{Z},$$
$$\mu \in \sigma(W_k^* e^{i\tilde{A}_k(b_k - a_k)}),\ 0 \leq \arg\mu < 2\pi\Big\}$$

Proof. The general solution of the following problem to spectrum of the selfadjoint extension L_{W_k} for any $k = 1,\ldots,n$,

$$\tilde{l}_k(u_k) = iu_k' + \tilde{A}_k u_k = \lambda u_k + f_k, \quad u_k, f_k \in L_2(H_k,(a_k,b_k))$$
$$u_k(b_k) = W_k u_k(a_k), \quad \lambda \in \mathbb{R}$$

is of the form

$$u_k(t) = e^{i(\tilde{A}_k - \lambda)(t - a_k)} f_k^* + \int_{a_k}^t e^{i(\tilde{A}_k - \lambda)(t-s)} f_k(s) ds,$$
$$a_k < t < b_k,$$
$$(e^{i\lambda(b_k - a_k)} - W_k^* e^{i\tilde{A}_k(b_2 - a_2)}) f_k^*$$
$$= W_k^* e^{i\lambda(b_k - a_k)} \int_{a_k}^{b_k} e^{i(\tilde{A}_k - \lambda)(b_k - s)} f_k(s) ds$$

This implies that $\lambda \in \sigma(L_{W_k})$ if and only if λ is a solution of the equation $e^{i\lambda(b_k - a_k)} = \mu$, where $\mu \in \sigma(W_k^* e^{i\tilde{A}_k(b_k - a_k)})$. We obtain that

$$\lambda = \frac{1}{b_k - a_k} \arg\mu + \frac{2m\pi}{b_k - a_k}, \quad m \in \mathbb{Z},\ \mu \in \sigma(W_k^* e^{i\tilde{A}_k(b_k - a_k)}).$$

□

Theorem 3.4. Spectrum $\sigma(L_W)$ of any selfadjoint extension $L_W = L_{W_0} \oplus L_{W_1} \oplus \ldots \oplus L_{W_k}$ coincides with $\mathbb{R}$.

Proof. Validity of this assertion is a simple result of the following claim. If $S_k, k = 1,\ldots,m, m \in \mathbb{N}$ are linear closed operators in any Hilbert spaces $\mathfrak{H}_k$, by using [19] we have

$$\sigma_p\left(\bigoplus_{k=1}^m S_k\right) = \bigcup_{k=1}^m \sigma_p(S_k),$$
$$\sigma_c\left(\bigoplus_{k=1}^m S_k\right) = \left(\bigcup_{k=1}^m \sigma_p(S_k)\right)^c \cap \left(\bigcup_{k=1}^m \sigma_r(S_k)\right)^c$$
$$\cap \left(\bigcup_{k=1}^m \sigma_c(S_k)\right).$$

Thus,the proof is completed by using last equalities, Theorem 3.2 and Theorem 3.3. □

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Zameddin I. Ismailov received the PhD degree in Mathematical Analysis in Institute of Mathematics of NASU in Kiev. His research interests are functional analysis, operator theory and applied mathematics. He is referee of mathematical journals.

Meltem Sertbaş is Assistant Professor at Karadeniz Technical University. She received the PhD degree in "Mathematics" at Karadeniz Technical University. Her main research interests are operator theory, differential operators and its applications.

Elif O. Çevik is PhD student in Institute of Natural Sciences of Karadeniz Technical University. Her main interests are functional analysis, operator theory and spectral theory of operators.