Applied Mathematics & Information Sciences
An International Journal

The maximum norm analysis of an overlapping Schwarz method for parabolic quasi-variational inequalities related to impulse control problem with the mixed boundary conditions

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Received: 4 Apr. 2012; Revised 24 Jul. 2012; Accepted 18 Aug. 2012
Published online: 1 Jan. 2013

Abstract: In this paper we provide a maximum norm analysis of an overlapping Schwarz method on non-matching grids for parabolic quasi-variational inequalities related to impulse control problem with the mixed boundary conditions. The parabolic quasi-variational inequalities are transformed into elliptic coercive quasi variational inequalities. Than we provide that the discretization on every sub-domain converges in uniform norm. Furthermore a result of asymptotic behavior in uniform norm is given.

Keywords: Domain Decomposition, Geometrical Convergence, Parabolic Quasi-variational Inequalities, Impulse Control, Asymptotic Behavior.

1. Introduction

Schwarz method has been invented by Herman Amandus Schwarz in 1890. This method has been used to solve the stationary or evolutionary boundary value problems on domains which consists of two or more overlapping sub-domains (see [1, 3, 15, 19, 20, 23, 24]). The solution is approximated by an infinite sequence of functions which results from solving a sequence of stationary or evolutionary boundary value problems in each of the sub-domain. In this work we provide a maximum norm analysis of an overlapping Schwarz method on non-matching grids for parabolic quasi-variational inequalities related to impulse control problem with respect to the mixed boundary conditions.

We consider the following evolutionary inequality: find $u \in L^2\left(0,T;H_0^1\left(\Omega\right)\right)$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u - f \leq 0, \text{ in } \Sigma \\ u - Mu \leq 0 \quad Mu \geq 0, \\ \left(\dfrac{\partial u}{\partial t} - \Delta u - f\right)(u - Mu) = 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma/\Gamma_0, \\ u\left(x,0\right) = u_0 \text{ in } \Omega \end{cases},$$

where Σ is a set in $\mathbb{R} \times \mathbb{R}^2$ defined as $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ and f is a regular function.

The symbol $(.,.)$ stands for the inner product in L^2.

f is a regular functions satisfying

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$$f \in L^2(0,T,L^\infty(\Omega)) \cap C^1\left(0,T,H^{-1}(\Omega)\right) \text{ and } f \geq 0. \tag{1}$$

We specify the following notations.

$$\|\cdot\|_{L^2(\Omega)} = \|\cdot\|_2\,,\ \|\cdot\|_1 = \|\cdot\|_{H_0^1(\Omega)}\,,\ \|\cdot\|_{L^\infty(\Omega\cup\Gamma)} = \|\cdot\|_\infty\,,$$

M is an operator given by

$$Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x+\xi)$$

where $k > 0$ and $\xi \geq 0$ means that $\xi = (\xi_1, \xi_2)$ with $\xi_1,\ \xi_2 \geq 0$,and Γ_0 is the part of the boundary defined by:

$$\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall \xi > 0,\ x + \xi \notin \bar{\Omega}\right\}.$$

Finally, $\dfrac{\partial u}{\partial \eta} = \nabla u.\overrightarrow{\eta}$, such that $\overrightarrow{\eta}$ is the normal vector. The symbol $(.,.)_\Omega$ stands for the inner product in $L^2(\Omega)$, $(.,.)_{\Gamma_0}$ stands for the inner product in $L^2(\Gamma_0)$.

A great deal of work has been done since now three decades on questions of existence and uniqueness for the discrete solution of parabolic and elliptic variational inequalities and quasi-variational. However, very much remains to be done on the numerical analysis side, especially the error estimates for them in uniform norm (cf., e.g.[6,7], [9-16]) and the asymptotic behavior in uniform norm for parabolic variational inequalities (cf., e.g.[4,5]). The existence and uniqueness and regularity of both the continuous and the discrete solution have been intensively studied and had already been dealt with in the past years, (cf., e.g.[13,14] [20]) for details.

In a recent work (see [3]), exploiting the above arguments, we analyzed the finite element approximation for the coercive problem and derived the following error estimate for elliptic quasi-variational inequalities

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^3\,, \tag{2}$$

with C a constant independent of both h and k.

In the previous our work [4,5] , we established firstly the existence and uniqueness of weak solutions of parabolic variational inequalities. Then we transformed the parabolic quasi-variational inequalities related to impulse control problem into a coercive elliptic quasi-variational inequalities, and we proposed a new iterative discrete algorithm to show the existence and uniqueness of the discrete solution, and we gave a simple proof to asymptotic behavior in L^∞-norm using the theta time scheme combined with a finite element spatial approximation.

Also, we analyzed the theta-scheme with respect to the t-variable combined with a finite element spatial approximation for the evolutionary variational inequalities and quasi-variational inequalities with an obstacle defined as an impulse control problem [4,5] and we derived the following asymptotic behavior, for $\theta \geq \dfrac{1}{2}$

$$\left\|u_h^{\theta,p} - u^\infty\right\|_\infty \leq C\left[h^2 |\log h|^2 + \left(\tfrac{1}{1+\beta\theta\Delta t}\right)^p\right], \tag{3}$$

and for the second case $0 \leq \theta < \dfrac{1}{2}$

$$\left\|u_h^{\theta,p} - u^\infty\right\|_\infty \leq C\left[h^2 |\log h|^2 + \left(\tfrac{2h^2}{2h^2+\beta\theta(1-2\theta)}\right)^p\right], \tag{4}$$

where $\rho(A)$ is the spectral radios of the elliptic operator A and $u_h^{\theta,p}$, the discrete solution calculated at the moment-end $T = p\Delta t$ for an index of the time discretization $k = 1, ..., p$, and u^∞, the asymptotic continuous solution.

and it can be seen in the previous our estimates in (1.3), (1.4) $\left(\dfrac{1}{1+\beta\theta\Delta t}\right)^p$, $\left(\dfrac{2}{2+\beta\theta(1-2\theta)\rho(A)}\right)^p$ tend to 0 when p approach to infinity. Therefore, we get the previous estimates for elliptic case defined in (1.2).

Our main concern in the present paper is to extend the above result for the parabolic quasi-variational which in turn can be transformed into system of coercive elliptic quasi variational according the step of the time discretization . Therefore the all results which introduced in this paper remain true, where we have introduced a new approach for the theta time scheme combined with a finite element spatial approximation of an overlapping Schwarz method on non-matching grids for the parabolic quasi-variational inequalities related to impulse control problem. We consider a domain which the union of two overlapping subdomains, where each sub-domain has its own generated triangulation. The grid points on the sub-domain boundaries need not much the grid points from the other subdomain. Under a discrete maximum principle [cf. 9], we show that the discretization on each sub-domain converges quasi-optimally in the L^∞-norm . For that purpose, further to the above arguments, our main tool is a discrete L^∞-stability property with respect the obstacle defined as an impulse control problem, the right-hand side and the mixed boundary conditions.

The outline of the paper is as follows. In Section 2, we lay down some notations and assumptions needed through out the paper and state both the continuous and discrete parabolic quasi variational inequalities. In section 3, we state the continuous alternating Schwarz sequence for parabolic quasi-variational inequalities and define their respective the theta scheme combined with a finite element counterparts in the context of overlapping grids. Then, we prove the L^∞-stability analysis of the θ-scheme for P.V.I. Finally in Section 4, we associate with the discrete P.V.I problem a fixed point mapping and we use that in proving the existence of a unique discrete solution, In section 5 the geometrical convergence is established using the new iterative discrete algorithm stands in theta scheme. Than an L^∞-asymptotic behavior estimate for each sub-domain is derived in uniform norm.

2. The Schwarz method for the parabolic Quasi-variational inequalities.

We begin by down some definitions and classical results related to Quasi-variational inequalities.

2.1. Parabolic Quasi-variational inequalities.

Let Ω be a convex domain in $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$. We consider the following obstacle problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u - f \leq 0, \text{ in } \Sigma, \\ u - Mu \leq 0 \quad Mu \geq 0, \\ \left(\frac{\partial u}{\partial t} - \Delta u - f\right)(u - Mu) = 0 \text{ in } \Omega, , \\ \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma/\Gamma_0, \\ u(x,0) = u_0 \text{ in } \Omega \end{cases} \quad (5)$$

M is an operator given by

$$Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x+\xi). \quad (6)$$

with M a regular operator given by

$$Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x+\xi)$$

where k is a positive number and $\xi \geq 0$ means that $\xi = (\xi_1, \xi_2)$ with $\xi_1, \xi_2 \geq 0$ and satisfying

$$Mu \in L^2\left(0, T, W^{2,\infty}(\Omega)\right), \quad (7)$$

and we know by [21] M is satisfying some proprieties as

M is concave; i.e., $\forall u, v \in C(\Omega), \exists \delta > 0:$

$$M(\delta u + (1-\delta)v) \geq \delta M(u) + (1-\delta)M(v), \quad (8)$$

and also it is satisfying

$$\forall \eta \in \mathbb{R}, M(u+\eta) = M(u) + \eta, \quad (9)$$

$f(.)$ is a regular function which satisfy

$$f \in L^2(0, T, L^\infty(\Omega)) \cap C^1\left(0, T, H^{-1}(\Omega)\right), \quad (10)$$

and φ is a regular function in Γ_0

Theorem 1.*[cf. 22] The problem (5) has an unique solution $u \in L^2\left(0, T; H^1(\Omega)\right)$. Moreover we have*

$$\begin{cases} u \in L^2\left(0, T; H^1(\Omega)\right), \\ \frac{\partial u}{\partial t} \in L^2\left(0, T; H^{-1}(\Omega)\right) \end{cases}. \quad (11)$$

After applying the Green formula, the (5) can be transformed to the following parabolic quasi-variational inequalities

$$\begin{cases} \left(\frac{\partial u}{\partial t}, v - u\right)_\Omega + a(u, v-u) - (f, v-u)_\Omega - \\ -(\varphi, v-u)_{\Gamma_0} \geq 0, \quad \text{on } \Omega, \ v \in H^1(\Omega) \\ u - Mu \leq 0 \quad Mu \geq 0, \\ Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x+\xi), \\ \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma/\Gamma_0, \\ u(x,0) = u_0 \text{ in } \Omega \end{cases}. \quad (12)$$

Thus, it can be easily deduced that

$$a(u,v) = \int_\Omega \nabla u.\nabla v dx = (\nabla u, \nabla v)_\Omega,$$

$$(f, v)_\Omega = \int_\Omega f.v dx$$

and

$$(\varphi, v)_{\Gamma_0} = \int_{\Gamma_0} \varphi.v d\sigma.$$

Lemma 1.*[cf. 22]Under the previous hypotheses we have the following inequality*

$$\|Mu - M\tilde{u}\|_{L^\infty(\Omega)} \leq \|u - \tilde{u}\|_{L^\infty(\Omega)} \quad (13)$$

Let $(M\xi, \varphi)$, $\left(M\tilde{\xi}, \tilde{\varphi}\right)$ be a pair of data, and $\xi = \sigma(M\xi, \varphi)$, $\tilde{\xi} = \sigma\left(M\tilde{\xi}, \tilde{\varphi}\right)$ be the corresponding solutions to the following parabolic quasi-variational inequalities (PQVI):

$$b(\xi, v-\xi) \geq (f, v-\xi)_\Omega +$$
$$+(\varphi, (v-\xi))_{\Gamma_0}, \forall v \in L^2\left(0, T; H_0^1(\Omega)\right)$$

and

$$b\left(\tilde{\xi}, v-\tilde{\xi}\right) \geq \left(f, v-\tilde{\xi}\right)_\Omega +$$
$$+(\tilde{\varphi}, (v-\xi))_{\Gamma_0}, \forall v \in L^2\left(0, T; H_0^1(\Omega)\right),$$

where

$$\left(\frac{\partial \xi}{\partial t}, (v-\xi)\right)_{\Omega} + a(\xi, v-\xi) = b(\xi, v-\xi)$$

Then, the following comparison result holds.

Lemma 2. *Under the previous hypotheses and the notation..*

$$\text{If } \varphi \geq \tilde{\varphi}. \text{ Then } \sigma(M\xi, \varphi) \geq \sigma\left(M\tilde{\xi}, \tilde{\varphi}\right).$$

Proof. Let $v = \min\left(0, \xi - \tilde{\xi}\right)$. In the region where v is negative ($v < 0$), we have

$$\xi \leq \tilde{\xi} \leq M\tilde{\xi} \leq M\xi,$$

which means that the obstacle is not active for u. So, for that v, we have

$$b(\xi, v) = (f, v)_{\Omega} + (\varphi, v)_{\Gamma_0}, \tag{14}$$

$$\tilde{\xi} + v \leq M\tilde{\xi} \tag{15}$$

so

$$b\left(\tilde{\xi}, v\right) \geq (f, v)_{\Omega} + (\varphi, v)_{\Gamma_0}. \tag{16}$$

Subtracting (14) and (16) from each other, we obtain

$$b\left(\xi - \tilde{\xi}, v\right) \geq 0. \tag{17}$$

But

$$b(v, v) = b\left(\xi - \tilde{\xi}, v\right) = -b\left(\tilde{\xi} - \xi, v\right) \leq 0 \tag{18}$$

so

$$v = 0$$

and consequently,

$$\xi \geq \tilde{\xi}$$

which completes the proof.

Proposition 1. *[cf. 3]Under the previous hypotheses, we have the following inequality*

$$\|u - \tilde{u}\|_{L^{\infty}(\Omega_i)} \leq \|Mu - M\tilde{u}\|_{L^{\infty}(\Omega_i)} +$$

$$+ \|\varphi - \tilde{\varphi}\|_{L^{\infty}(\partial\Omega_i \cap \Omega_j)}, \text{ such that } i \neq j, \; i, j = 1, 2.$$

3. The discrete parabolic quasi-variational inequalities

3.1. The space discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, ..., m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$ where M_j is a summit of the considered triangulation. We introduce the following discrete spaces V^h of finite element

$$V_h = \begin{cases} v_h \in L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right), \\ \text{such that } v_h \mid_K \in P_1, \; K \in \tau_h, \\ v_h \leq r_h M(v_h), \; v_h(.,0) = v_{h0} \text{ in } \Omega, \\ \dfrac{\partial v_h}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } v_h = 0 \text{ in } \Gamma/\Gamma_0 \end{cases} \tag{19}$$

We consider r_h be the usual interpolation operator defined by

$$\begin{aligned} & v_h \in L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right), \\ & r_h v = \sum_{i=1}^{m(h)} v(M_i)\, \varphi_i(x). \end{aligned} \tag{20}$$

The discrete maximum principle assumption (dmp)[cf. 9]: We assume the matrix whose coefficients $a(\varphi_i, \varphi_j)$ are $M-$matrix. For convenience in all the sequels, C will be a generic constant independent on h.

We discretize in space, i.e., that we approach the space H_0^1 by a space discretization of finite dimensional $V^h \subset H_0^1$. In a second step, we discretize the problem with respect to time using the θ-scheme. Therefore, we search a sequence of elements $u_h^n \in V^h$ which approaches $u^n(t_n)$, $t_n = n\Delta t$, with initial data $u_h^0 = u_{0h}$. Now we apply the θ-Scheme on the following to the semi-discrete approximation for $v_h \in V^h$

$$\begin{cases} \left(\frac{\partial u_h}{\partial t}, v_h - u_h\right)_{\Omega} + a(u_h, v_h - u_h) \geq \\ \geq (f, v_h - u_h)_{\Omega} + (\varphi, (v_h - u_h))_{\Gamma_0}. \end{cases} \tag{21}$$

3.2. The time discretization

Now, we discretize the problem (21) with respect to time by using the theta-scheme. Therefore, we search a sequence of elements $u_h^k \in V_h$ which approaches $u_h^i(t_k)$, $t_k = k\Delta t$, with initial data $u_h^0 = u_{0h}$.

Thus we have, for any $\theta \in [0,1]$ and $k = 1, ..., p$

$$\left(u_h^k - u_h^{k-1}, v_h - u_h^{\theta,k}\right)_\Omega + (\Delta t)\, a\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right) \geq$$

$$\geq (\Delta t)\left[\left(f^{\,\theta,k}, v_h - u_h^{\theta,k}\right)_\Omega + \left(\varphi^{\theta,k}, \left(v_h - u_h^{\theta,k}\right)\right)_{\Gamma_0}\right], \tag{22}$$

where

$$u_h^{\theta,k} = \theta u_h^k + (1-\theta)\, u_h^{k-1}$$

and

$$f^{\,\theta,k} = \theta f^k + (1-\theta)\, f^{k-1}. \tag{23}$$

$$\varphi^{\,\theta,k} = \theta \varphi^k + (1-\theta)\, \varphi^{k-1}. \tag{24}$$

By multiplying and dividing by θ and by adding

$$\left(\frac{u_h^{k-1}}{\theta \Delta t}, v_h - u_h^{\theta,k}\right)$$

to both parties of the inequalities (22), we get for $v_h \in V_h$

$$\begin{cases} \left(\dfrac{u_h^{\theta,k}}{\theta \Delta t}, v_h - u_h^{\theta,k}\right)_\Omega + a\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right) \geq \\ \geq \left(f^{\,\theta,k} + \dfrac{u_h^{k-1}}{\theta \Delta t}, v_h - u_h^{\theta,k}\right)_\Omega + \\ + \left(\varphi^{\theta,k}, v_h - u_h^{\theta,k}\right)_{\Gamma_0}. \end{cases} \tag{25}$$

We have

$$u_h^{\theta,k} = \theta u_h^k + (1-\theta)\, u_h^{k-1}$$

$$\leq \theta r_h M u_h^k + (1-\theta)\, r_h M u_h^{k-1},$$

using the concavity of $r_h M$ we get

$$u_h^{\theta,k} \leq \theta r_h M u_h^k + (1-\theta)\, r_h M u_h^{k-1}$$

$$\leq r_h M(\theta u_h^k + (1-\theta)\, u_h^{k-1})$$

$$\leq r_h M u_h^{\theta,k},$$

thus $u_h^{\theta,k} \in V_h$, then, the problem (25) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities

$$b\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right) \geq \left(f^{\,\theta,k} + \mu u_h^{k-1}, v_h - u_h^{\theta,k}\right)_\Omega$$

$$+ \left(\varphi^{\theta,k}, v_h - u_h^{\theta,k}\right)_{\Gamma_0},\ v_h,\ u_h^{\theta,k} \in V_h \tag{26}$$

such that

$$\begin{cases} b\left(u_h^{\theta,k}, v - u_h^{\theta,k}\right) = \mu\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right)_\Omega \\ +a\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right),\ v_h,\ u_h^{\theta,k} \in V_h, \\ \mu = \dfrac{1}{\theta \Delta t} = \dfrac{T}{\theta n} \end{cases}. \tag{27}$$

3.3. Stability analysis for the discrete PQVI

It is possible to analyze stability taking advantage of the structure of eigenvalues of the bilinear form $a(.,.)$, and we recall that W is compactly embedded in $L^2(\Omega)$ since Ω is bounded. Thus, there is a non decreasing sequence of eigenvalues $\delta \leq \lambda_1 \leq \lambda_2 \leq$ .....for the bilinear form $a(.,.)$, i.e.,

$$\begin{cases} \omega_j \in L^2,\ \omega_j \neq 0 : \\ a(\omega_j, v_h) = \lambda_j (\omega_j, v_h)_\Omega,\ \forall v_h \in V^h. \end{cases}$$

The corresponding eigenfunctions $\{\omega_j\}$ form a complete orthonormal basis in $L^2(\Omega)$. In analogous way, when considering the finite dimensional problem in W^h, we find a sequence of eigenvalues $\delta \leq \lambda_{1h} \leq \lambda_{2h} \leq \ldots \leq \lambda_{m(h)}$ and L^2-orthonormal basis of eigenvectorss $\omega_{ih} \in W^h$, $i = 1, 2, \ldots, m(h)$. Any function v_h in V^h can thus be expanded with respect to the system ω_{ih} as

$$v_h = {}_{i=1}^{m(h)} (v_h, \omega_{ih})_\Omega\, \omega_{ih},$$

in particular, we have

$$u_h^k = {}_{i=1}^{m(h)} u_i^k \omega_{ih} \text{ and } u_i^k = \left(u_h^k, \omega_{ih}\right)_\Omega.$$

Moreover, let $f_h{}^k$ be the L^2-orthogonal projection of $\theta f^{\,k} + (1-\theta)\, f^{\,k-1}$ into W^h i.e., $f_h{}^k \in W^h$ and

$$\begin{cases} \left(f_h{}^{\theta,k}, v_h\right)_\Omega = \left(\theta f^{\,k} + (1-\theta)\, f^{\,k-1}, v_h\right)_\Omega \\ \left(\varphi_h{}^{\theta,k}, v_h\right)_{\Gamma_0} = \left(\theta \varphi^{\,k} + (1-\theta)\, \varphi^{\,k-1}, v_h\right)_{\Gamma_0} \end{cases}, \tag{28}$$

and set

$$\begin{cases} f_h{}^k = {}_{i=1}^{m(h)} f_i{}^k \omega_{ih}\ ;\ f_i{}^k = \left(f_h{}^k, \omega_{ih}\right), \\ \varphi_h{}^k = {}_{i=1}^{m(h)} f_i{}^k \omega_{ih}\ ;\ \varphi_i{}^k = \left(\varphi_h{}^k, \omega_{ih}\right)_{\Gamma_0} \end{cases}.$$

We are now in a position to prove the stability for $\theta \in \left[0, \frac{1}{2}\right[$

Choosing in (22) $v_h = 0$ and by using the trace theorem, thus we have for $u_h^{\theta,k} \in V^h$

$$\frac{1}{\Delta t}\left(u_h^k - u_h^{k-1}, u_h^{\theta,k}\right) + a\left(u_h^{\theta,k}, u_h^{\theta,k}\right)$$
$$\leq \left(f_h^{\theta,k}, u_h^{\theta,k}\right)_\Omega + \left(\varphi_h^{\theta,k}, u_h^{\theta,k}\right)_{\Gamma_0}$$
$$\leq \left(f_h^{\theta,k}, u_h^{\theta,k}\right)_\Omega + \varepsilon\left(\varphi_h^{\theta,k}, u_h^{\theta,k}\right)_\Omega, \ \varepsilon \geq 0 .$$

Thus the inequality (22) is equivalent to

$$\begin{cases} \frac{1}{\Delta t}\left(u_i^k - u_i^{k-1}\right) + \lambda_{ih}\left(u_i^{\theta,k}, u_i^{\theta,k}\right) \leq \\ \leq {f_i}^k + \varepsilon{\varphi_i}^k, \ \varepsilon \geq 0. \end{cases} \tag{29}$$

Since ω_{ih} are the eigenfunctions means
$a(\omega_{ih}, \omega_{ih}) = \lambda_{ih}(\omega_{ih}, \omega_{ih})$

$$= \lambda_{ih}.\delta_{ii} = \lambda_{ih},$$

for each $k = 1, ..., p$, we can rewrite (29) as

$$\begin{cases} u_i^k \leq \frac{1-(1-\theta).\Delta t.\lambda_{ih}}{1+\theta\Delta t.\lambda_{ih}} u_i^{k-1} + \\ + \frac{\Delta t}{1+\theta\Delta t.\lambda_{ih}}({f_i}^k + \varepsilon{\varphi_i}^k), \end{cases} \tag{30}$$

the inequality (30) stable if and only if

$$\left|\frac{1-(1-\theta).\Delta t.\lambda_{ih}}{1+\theta\Delta t.\lambda_{ih}}\right| < 1,$$

that is to say

$$2\theta - 1 > -\frac{2}{\lambda_{ih}.\Delta t},$$

means

$$\Delta t < \frac{2}{(1-2\theta)\lambda_{ih}}.$$

So that this relation satisfied for all the eigenvalues λ_{ih} of bilinear form $a(.,.)$, we have to choose their highest value, we take it for $\lambda_{mh} = \rho(A)$ (spectral radius of A).

We deduce that if $\theta \geq \frac{1}{2}$ the θ-scheme way is stable unconditionally (i.e., stable $\forall \Delta t$). However, if $0 \leq \theta < \frac{1}{2}$ the θ-scheme is stable unless

$$\Delta t < \frac{2}{(1-2\theta)\rho(A)}. \tag{31}$$

Notice that this condition is always satisfied if $0 \leq \theta < \frac{1}{2}$ Hence, taking the absolute value of (30) we have

$$|u_i^m| < |u_i^0| + \left|\frac{\Delta t}{1+\theta\Delta t.\lambda_{ih}}\right| \sum_{i=1}^{m-1} ({f_i}^k + \varepsilon{\varphi_i}^k),$$

also we deduce that

$$\|u_i^m\|_\infty < \|u_i^0\|_\infty + \left\|\frac{\Delta t}{1+\theta\Delta t.\lambda_{ih}}\right\|_\infty \sum_{i=1}^{m-1} \|{f_i}^k + \varepsilon{\varphi_i}^k\|_\infty . \tag{32}$$

Proposition 2. *We have for each $k = 1, ..., p$*

$$\|u_h^p\|_2^2 + \Delta t \sum_{k=1}^{p} a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) \leq$$
$$\leq C(n)\left(\|u_{0h}\|_2^2 + \sum_{k=1}^{p} \Delta t \left\|f^{\ \theta,k} + \varepsilon\varphi^{\ \theta,k}\right\|_2^2\right) \tag{33}$$

Proof. We take $v_h = 0$ in (22), for the left-hand side we can easily show

$$\left(u_h^k - u_h^{k-1},\ u_h^{\theta,k}\right)_\Omega + \Delta t.a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) =$$
$$= \frac{1}{2}\left(\|u_h^k\|_2^2 - \|u_h^{k-1}\|_2^2\right) +$$
$$+ \left(\theta - \frac{1}{2}\right)\|u_h^k - u_h^{k-1}\|_2 + \Delta t.a\left(u_h^{\theta,k}, u_h^{\theta,k}\right),$$

then we get for $\theta \geq \frac{1}{2}$

$$\left(u_h^k - u_h^{k-1},\ u_h^{\theta,k}\right)_\Omega + \Delta t.a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) \geq$$
$$\geq \frac{1}{2}\left(\|u_h^k\|_2^2 - \|u_h^{k-1}\|_2^2\right) + \Delta t.a\left(u_h^{\theta,k}, u_h^{\theta,k}\right)$$
$$\geq \frac{1}{2}\left(\|u_h^k\|_2^2 - \|u_h^{k-1}\|_2^2\right) + (\Delta t)\left\|u_h^{\theta,k}\right\|_1^2.$$

For the right-hand side we make use of the following algebraic inequality

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right), \quad \forall a,\ b \in \mathbb{R},$$

Also, by using the Trace theorem and the coerciveness assumption of $a(.,.)$

$$(\Delta t)\left[\left(f^{\ \theta,k}, u_h^{\theta,k}\right)_\Omega + \left(\varphi^{\ \theta,k}, u_h^{\theta,k}\right)_{\Gamma_0}\right] \leq$$
$$\leq \Delta t\left(\tfrac{1}{2}\left\|u_h^{\theta,k}\right\|_2^2 + \tfrac{1}{2}\left\|f^{\ \theta,k} + \varepsilon\varphi^{\ \theta,k}\right\|_2^2\right)$$
$$\leq \Delta t\left(\tfrac{1}{2}a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) + \tfrac{1}{2}\left\|f^{\ \theta,k} + \varepsilon\varphi^{\ \theta,k}\right\|_2^2\right),$$

thus

$$\begin{cases} \|u_h^k\|_2^2 - \|u_h^{k-1}\|_2^2 + \Delta t.a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) \leq \\ \leq \Delta t\left\|f^{\ \theta,k} + \varepsilon\varphi^{\ \theta,k}\right\|_2^2. \end{cases}$$

We deduce that $\forall k = 1, ..., p$

$$\sum_{k=1}^{p-1} \left\|u_h^k\right\|_2^2 + \Delta t \sum_{k=1}^{p} a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) \leq$$

$$\leq \sum_{k=1}^{p} \left\|u_h^{k-1}\right\|_2^2 + \Delta t \sum_{k=1}^{p} \left\|f^{\ \theta,k} + \varepsilon\varphi^{\ \theta,k}\right\|_2^2$$

The sum becomes

$$\left\|u_h^p\right\|_2^2 + \Delta t \sum_{k=1}^{p} a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) \leq$$

$$\leq C(p)\left(\left\|u_{0h}\right\|_2^2 + \sum_{k=1}^{p} \Delta t \left\|f^{\ \theta,k} + \varepsilon\varphi^{\ \theta,k}\right\|_2^2\right)$$

where $C(p)$ is a constant independent of h and Δt, thus we have proved that the scheme is unconditionally stable for $\theta \geq \frac{1}{2}$.

On the other hand, since in a finite dimensional space all norms are equivalent, we can infer that

$$\left\|u_h^p\right\|_2^2 + \Delta t \sum_{k=1}^{p} a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) \leq$$

$$\leq C(p)\left(\left\|u_{0h}\right\|_2^2 + \sum_{k=1}^{p} \Delta t \left\|f^{\ \theta,k} + \varepsilon\varphi^{\ \theta,k}\right\|_2^2\right)$$

3.4. The discrete Schwarz sequences.

As we have defined before Ω be a bounded open domain in $\mathbb{R}^2$ and we assume that Ω is a smooth and connected.

Then we decompose Ω in two sub-domains Ω_1, Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2 \tag{34}$$

and u satisfies the local regularity condition

$$u\mid_{\Omega_i} \in L^2(0, T, W^{2,p}(\Omega_i)) \tag{35}$$

and we denote by $\Gamma = \partial\Omega, \Gamma_1 = \partial\Omega_1, \Gamma_2 = \partial\Omega_2, \gamma_1 = \partial\Omega_1 \cap \Omega_2, \gamma_2 = \partial\Omega_2 \cap \Omega_1, \Omega_{1,2} = \Omega_1 \cap \Omega_2$.

For $i = 1, 2$, let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i; $h_i\,(h_1 = h_2 = h)$, being the meshsize. We assume that the two triangulations are mutually independent on $\Omega_{1,2}$ in the sense that a triangle belonging to one triangulation does not necessarily belong to the other.

Let V^{h_i} be the space of continuous piecewise linear functions on τ^{h_i} which vanish on $\Omega_i \cap \partial\Omega_j$, $i \neq j$, $i, j = 1, 2$. For $w \in C\left(\partial\bar{\Omega}_i\right)$ we define

$$V_w^{h_i} = \begin{cases} v_h \in V^{h_i} : \ v_h = \pi_{h_i}(w) \ \text{ on } \Omega_i \cap \partial\Omega_j; \\ v_h(.,0) = v_{h0} \text{ in } \Omega, \\ \dfrac{\partial v_h}{\partial \eta} = \varphi \text{ in } \Gamma_0\,, \\ v_h = 0 \text{ in } \Gamma/\Gamma_0; \ \ i \neq j, \ i, \ j = 1, \ 2, \end{cases} \tag{36}$$

where π_{h_i} denotes the interpolation operator on $\partial\Omega_i$ and V^{h_i} defined in (19) .

We consider the model obstacle problem: Find $u_h^{\theta,k} \in V_h$such that

$$b\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right) \geq \left(f^{\ \theta,k} + \mu u_h^{k-1}, v_h - u_h^{\theta,k}\right)_{\Omega} +$$
$$+\left(\varphi^{\theta,k}, \left(v_h - u_h^{\theta,k}\right)\right)_{\Gamma_0}, \ v_h, \ u_h^{\theta,k} \in V_h \tag{37}$$

We define the discrete counterparts of the discrete Schwarz sequences, respectively by $u_h^{\theta,k,2n+1}, \ v_h \ \in \ V^h_{\left(u_h^{\theta,k,2n}\right)}$, such that

$$\begin{cases} b\left(u_h^{\theta,k,2n+1}, v_h - u_h^{2n+1}\right) - \\ -\left(f^{\ \theta,k} + \mu u_h^{\theta,k-1,2n-1}, v_h - u_h^{\theta,k,2n+1}\right)_{\Omega_1} - \\ -\left(\varphi^{\theta,k}, v - u_h^{\theta,k,2n+1}\right)_{\Gamma_0} \geq 0, \\ u_h^{\theta,k,2n+1} = u_h^{\theta,k,2n} \text{ on } \partial\Omega_1, \ v_h = u_h^{\theta,k,2n} \text{ on } \partial\Omega_1, \\ u_h^{\theta,k,2n+1} \leq r_h M u_h^{\theta,k,2n-1} \end{cases} \tag{38}$$

and $u_h^{\theta,k,2n}, v^h \in V^h_{\left(u_h^{\theta,k,2n-1}\right)}$ such that

$$\begin{cases} b\left(u_h^{\theta,k,2n}, v_h - u_h^{2n}\right) - \\ -\left(f^{\ \theta,k} + \mu u_h^{\theta,k-1,2n-2}, v_h - u_h^{\theta,k,2n}\right)_{\Omega_2} - \\ -\left(\varphi^{\theta,k}, v - u_h^{\theta,k,2n}\right)_{\Gamma_0} \geq 0, \\ u_h^{\theta,k,2n} = u_h^{\theta,k,2n-1} \text{ on } \partial\Omega_2, \ v_h = u_h^{\theta,k,2n-1} \text{ on } \partial\Omega_2, \\ u_h^{\theta,k,2n} \leq r_h M u_h^{\theta,k,2n-2}. \end{cases} \tag{39}$$

4. Existence and uniqueness for discrete PQVI.

Next using the preceding assumptions, we shall prove the existence of a unique solution for problem (3.20) by means of the Banach's fixed point theorem.

4.1. A fixed point mapping associated with discrete problem

We consider the following mapping

$$T_h : L^{\infty}_{+}(\Omega) \longrightarrow V^h$$
$$w \longrightarrow T_h(w) = \xi_h, \quad (40)$$

where ξ_h is the unique solution of the following PQVI: find $\xi_h \in V^h$

$$b(\xi_h, v_h - \xi_h) \geq \left(f^{\theta,k} + \mu w, v_h - \xi_h\right)_{\Omega} +$$
$$+ \left(\varphi^{\theta,k}, (v_h - \xi_h)\right)_{\Gamma_0}, \quad v_h \in V^h.$$

Proposition 3. *Under the previous hypotheses and notations, if we set $\theta \geq \frac{1}{2}$, the mapping T_h is a contraction in $L^{\infty}(\Omega \cup \Gamma)$ with a rate of contraction $\left(\frac{1}{1+\theta.\Delta t}/\right)$. Therefore, T_h admits a unique fixed point which coincides with the solution of PQVI (38).*

Proof. For w, $\tilde{w}$ in $L^{\infty}(\Omega \cup \Gamma)$, we consider

$$\xi_h = T_h(w) = \partial\left(f^{\theta,k} + \varphi^{\theta,k} + \mu w,\ r_h M \xi_h\right)$$

and

$$\tilde{\xi}_h = T_h(\tilde{w}) = \partial\left(f^{\theta,k} + \varphi^{\theta,k} + \mu\,\tilde{w},\ r_h M \tilde{\xi}_h\right)$$

solution to quasi-variational inequalities (38) with right-hand side

$$F^{\theta,k} = f^{\theta,k} + \varphi^{\theta,k} + \mu w_h,\ \tilde{F}^{\theta,k} = f^{\theta,k} + \varphi^{\theta,k} + \mu\,\tilde{w}_h.$$

Now setting

$$\phi = \frac{1}{1+\mu}\left\|F^{\theta,k} - \tilde{F}^{\theta,k}\right\|_{\infty},$$

then for $\xi_h + \phi$ is solution of

$$\begin{cases} b(\xi_h + \phi, (v_h + \phi) - (\xi_h + \phi)) \geq \\ \geq \left(f^{\theta,k} + \mu w + \phi, (v_h + \phi) - (\xi_h + \phi)\right)_{\Omega} + \\ + \left(\varphi^{\theta,k} + \phi, (v_h + \phi) - (\xi_h + \phi)\right)_{\Gamma_0} \\ \xi_h + \phi \leq r_h M \xi_h + \phi, \\ v_h + \phi \leq r_h M \xi_h + \phi \ , \forall v_h \in V^h. \end{cases}$$

We have

$$F^{\theta,k} \leq \tilde{F}^{\theta,k} + \left\|F^{\theta,k} - \tilde{F}^{\theta,k}\right\|_{\infty}$$
$$\leq \tilde{F}^{\theta,k} + \frac{1}{\beta+\mu}\left\|F^{\theta,k} - \tilde{F}^{\theta,k}\right\|_{\infty}$$
$$\leq \tilde{F}^{\theta,k} + \phi,$$

so, due to lemma 2 and assumption (2.5), it follows that

$$\partial_h\left(f^{\theta,k} + \varphi^{\theta,k} + \mu w,\ r_h M \xi_h\right) \leq$$
$$\leq \partial_h\left(f^{\theta,k} + \varphi^{\theta,k} + \mu\,\tilde{w} + \phi,\ r_h M\left(\tilde{\xi}_h + \phi\right)\right) \leq$$
$$\leq \partial_h\left(f^{\theta,k} + \varphi^{\theta,k} + \mu\,\tilde{w},\ r_h M \tilde{\xi}_h\right) + \phi,$$

hence

$$\xi_h \leq \tilde{\xi}_h + \phi.$$

Similarly, interchanging the roles of w and $\tilde{w}$ we also get

$$\tilde{\xi}_h \leq \xi_h + \phi.$$

It finally yields

$$\begin{cases} \left\|\partial_h\left(F^{\theta,k}, r_h M \xi_h\right) - \partial_h\left(\tilde{F}^{\theta,k}, r_h M \tilde{\xi}_h\right)\right\|_{\infty} \leq \\ \leq \frac{1}{1+\mu}\left\|F^{\theta,k} - \tilde{F}^{\theta,k}\right\|_{\infty} \\ \leq \frac{1}{1+\mu}\left\|f^{\theta,k} + \varphi^{\theta,k} + \mu w - f^{\theta,k} - \varphi^{\theta,k} - \mu\,\tilde{w}\right\|_{\infty} \\ \leq \frac{1}{1+\theta\Delta t}\left\|w - \tilde{w}\right\|_{\infty}, \end{cases}$$

which completes the proof.

Proposition 4. *If we set $0 \leq \theta < \frac{1}{2}$ the mapping T_h is a contraction in $L^{\infty}(\Omega \cup \Gamma)$ with the rate of contraction $\frac{2}{2+\theta(1-2\theta)\rho(A)}$, where $\rho(A)$ spectral radius of the operator A.*

Proof. Under condition of stability we have shown the θ-scheme is stable if and only if $\Delta t < \frac{2}{(1-2\theta)\rho(A)}$,

thus it can easily shown

$$\left\|\partial_h\left(F^{\theta,k}, r_h M \xi_h\right) - \partial_h\left(\tilde{F}^{\theta,k}, r_h M \tilde{\xi}_h\right)\right\|_{\infty} \leq$$
$$\leq \frac{1}{1+\theta\Delta t}\left\|w - \tilde{w}\right\|_{\infty} \leq$$
$$\leq \frac{2}{2+\theta(1-2\theta)\rho(A)}\left\|w - \tilde{w}\right\|_{\infty}$$

thus the mapping T_h is a contraction in $L^\infty(\Omega \cup \Gamma)$ with rate of contraction $\frac{2}{2+\theta(1-2\theta)\rho(A)}$. Therefore, T_h admits a unique fixed point which coincides with the solution of PQVI (37).

4.2. Iterative discrete algorithm

We choose u_h^0 as the solution of the following discrete equation

$$b(u_h^0, v_h) = \left(g^0, v_h\right), \; v_h \in V^h, \tag{41}$$

where g^0 is a regular function give.

Now we give our following discrete algorithm

$$u_h^{\theta,k,2n+1} = T_h u_h^{k-1,2n+1}, k = 1, ..., p, \; u_h^{\theta,k,2n+1} \in V^h_{\left(u_h^{\theta,k,2n}\right)} \tag{42}$$

or

$$u_h^{\theta,k,2n} = T_h u_h^{k-1,2n}, k = 1, .., p, \; u_h^{\theta,k,2n} \in V^h_{\left(u_h^{\theta,k,2n-1}\right)} \tag{43}$$

where $u_h^{\theta,k}$ is the solution of the problem (37).

Remark. If we choose $\theta = 1$ in (42) or (43) we get Bensoussan's algorithm. The idea of this choice has been studied by [8].

Proposition 5. *[cf .4] Under the previous hypotheses and notations, we have the following estimate of convergence if $\theta \geq \frac{1}{2}$*

$$\left\| u_h^{\theta,k,2n+1} - u_h^\infty \right\|_\infty \leq \left(\tfrac{1}{1+\theta\Delta t}\right)^k \|u_h^\infty - u_{h0}\|_\infty, \tag{44}$$

if $0 \leq \theta < \frac{1}{2}$, we have

$$\left\| u_h^{\theta,k,2n+1} - u_h^\infty \right\|_\infty \leq \left(\tfrac{2}{2+\theta(1-2\theta)\rho(A)}\right)^k \|u_h^\infty - u_{h0}\|_\infty. \tag{45}$$

5. L^∞-Asymptotic Behavior

Theorem 2. *[cf. 3] If $A = (a_{ij})_{i,j=\{1....N\}}$ is the M-matrix. Then there exists two constants $k_1, \; k_2$*

$$k_1 = \sup\{w_h(x), \; x \in \gamma_2\} \in (0, 1)$$

and

$$k_1 = \sup\{w_h(x), \; x \in \gamma_1\} \in (0, 1)$$

such that

$$\sup_{\gamma_1} \left| u_h^\infty - u_h^{\infty,2n+1} \right| \leq k_1 \sup_{\gamma_1} \left| u_h^\infty - u_h^{\infty,2n} \right| \tag{46}$$

and

$$\sup_{\gamma_2} \left| u_h^\infty - u_h^{\infty,2n+1} \right| \leq k_2 \sup_{\gamma_2} \left| u_h^\infty - u_h^{\infty,2n} \right|. \tag{47}$$

In [3] we proved the following main convergence result

Theorem 3. *[cf. 3] The sequences $\left(u_h^{\theta,k,2n+1}\right); \left(u_h^{\theta,k,2n}\right); n \geq 0$ produced by the Schwarz alternating method converge geometrically to the solution u of the obstacle problem (5). More precisely, there exist $k_1, \; k_2 \in (0,1)$ which depend only respectively of (Ω_1, γ_2) and (Ω_2, γ_1) such that all $n \geq 0$.*

$$\sup_{\bar{\Omega}_1} \left| u_h^\infty - u_h^{\infty,2n+1} \right| \leq k_1^n k_2^n \sup_{\gamma_1} \left| u_h^\infty - u_h^0 \right| \tag{48}$$

and

$$\sup_{\bar{\Omega}_2} \left| u_h^\infty - u_h^{\infty,2n} \right| \leq k_1^n k_2^{n-1} \sup_{\gamma_2} \left| u_h^\infty - u_h^0 \right|. \tag{49}$$

Proof. The continuous case has been proved in [19, 20] and for the discrete case has been proved in the previous work [cf. 3]. This theorem remains true for the problem introduces in this paper, because it is system of coercive elliptic quasi-variational inequality according the step of time discretization k.

Also in [3], we proved the following error estimate for the elliptic Q.V.I related to impulse control problem:

Theorem 4. *[3,17] Under the results of the theorem 2, and the theorem 4. Then there exists a constant C independent of both h and n such that*

$$\left\| u^\infty - u_h^{\infty,2n+1} \right\|_{L^\infty(\bar{\Omega}_1)} \leq Ch^2 \left|\log h\right|^3 \tag{50}$$

and

$$\left\| u^\infty - u_h^{\infty,2n} \right\|_{L^\infty(\bar{\Omega}_2)} \leq Ch^2 \left|\log h\right|^3. \tag{51}$$

where u^∞ is the continuous solution of elliptic quasi variational inequality

5.1. Asymptotic behavior

This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in L^∞-norm for parabolic variational inequalities, where we evaluate the variation in L^∞ between $u_h^\theta(T, x)$, the discrete solution calculated at the moment $T = p\Delta t$ and u^∞,the asymptotic continuous solution

Theorem 5.*(The main result). Under the results of the proposition 5 and the theorem 4, we have*

for the first case $\theta \geq \frac{1}{2}$

$$\left\| u_h^{\theta,p,2n+1} - u^{\infty} \right\|_{\infty} \leq C \left[h^2 \left|\log h\right|^3 + \left(\tfrac{1}{1+\theta \Delta t} \right)^p \right], \tag{52}$$

and

$$\left\| u_h^{\theta,p,2n} - u^{\infty} \right\|_{\infty} \leq C \left[h^2 \left|\log h\right|^3 + \left(\tfrac{1}{1+\theta \Delta t} \right)^p \right], \tag{53}$$

and for the second case $0 \leq \theta < \frac{1}{2}$

$$\left\| u_h^{\theta,p,2n+1} - u^{\infty} \right\|_{\infty} \leq C \left[h^2 \left|\log h\right|^3 + \left(\tfrac{2}{2+\theta(1-2\theta)\rho(A)} \right)^p \right], \tag{54}$$

and

$$\left\| u_h^{\theta,p,2n} - u^{\infty} \right\|_{\infty} \leq C \left[h^2 \left|\log h\right|^3 + \left(\tfrac{2}{2+\theta(1-2\theta)\rho(A)} \right)^p \right], \tag{55}$$

where C *is a constant independent of* h *and* k

Proof. We have

$$\left\| u_h^{\theta,p,2n+1} - u^{\infty} \right\|_{\infty} \leq \left\| u_h^{\theta,p,2n+1} - u_h^{\infty} \right\|_{\infty} + \left\| u_h^{\infty} - u^{\infty} \right\|_{\infty}.$$

Using the proposition 5 and the theorem 4, we have for $\theta \geq \frac{1}{2}$

$$\left\| u_h^{\theta,p,2n+1} - u^{\infty} \right\|_{\infty} \leq C \left[h^2 \left|\log h\right|^3 + \left(\tfrac{1}{1+\theta \Delta t} \right)^p \right],$$

and for $0 \leq \theta < \frac{1}{2}$ we have

$$\left\| u_h^{\theta,p,2n+1} - u^{\infty} \right\|_{\infty} \leq C \left[h^2 \left|\log h\right|^3 + \left(\tfrac{2}{2+\theta(1-2\theta)\rho(A)} \right)^p \right]$$

The proof for (54) and (55) case is similar.

Remark. It can be seen in the previous estimates (52, 53, 54, 55), $\left(\frac{1}{1+\beta\theta\Delta t} \right)^p$, $\left(\frac{2}{2+\theta(1-2\theta)\rho(A)} \right)^p$, tend to 0 when p approach to infinity. Therefore, (cf. [3]), the estimation order for both the coercive and noncoercive problems is

$$\left\| u^{\infty} - u_h^{\infty,2n+1} \right\|_{L^{\infty}(\bar{\Omega}_1)} \leq Ch^2 \left|\log h\right|^3$$

and

$$\left\| u^{\infty} - u_h^{\infty,2n} \right\|_{L^{\infty}(\bar{\Omega}_2)} \leq Ch^2 \left|\log h\right|^3.$$

6. Conclusion

In this paper, we have introduced a new approach for an overlapping Schwarz method on non-matching grids for parabolic quasi-variational inequalities related to impulse control problem with respect to the mixed boundary conditions, where we have established the asymptotic behavior in uniform norm similar to that in the previous published paper [3] regarding the overlapping Schwarz method for the stationary free boundary problems,. The type of estimation, which we have obtained here, is important for the calculus of quasi-stationary state for the simulation of petroleum or gaseous deposit.

Acknowledgement

The authors would like to thank the editor and the referee for her/ his careful reading and relevant remarks/suggestions which permit them to improve the paper.

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