

Constrained Controllability for Distributed Hyperbolic Systems

Ali Boutoulout*, Hamid Bourray, Fatima Zahrae El Alaoui and Layla Ezzahri

Faculty of sciences, Moulay Ismail University, Meknes, Morocco

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Abstract: The purpose of this paper is to characterize the minimum energy control that steers a hyperbolic system to a final state between two prescribed functions only on a subregion ω of the system evolution domain Ω . We give some definitions and properties of this new concept, and then we concentrate on the determination of the control which would realize a given final state with output constraints in ω with minimum energy. This problem is solved using the Lagrangian approach and leads to an algorithm for the computation of the optimal control. The obtained results are illustrated by numerical simulations which lead to some conjectures.

Keywords: Energy minimum, hyperbolic systems, optimal control, Lagrangian approach.

1 Introduction

Most mechanical, biological or economical problems are modelled using partial differential equations, the formulation of these phenomena in a distributed system has the advantage to describe them accurately and keeps for each parameter its true physical meaning. Applied mathematics and control theory aim to rigorously develop methods for solving problems related to real applications. In the field of analysis and control of these systems, several notions have been developed particularly controllability, stability and by duality observability and detectability, etc. These various concepts have been widely studied and leads to a vast and disparate literature [1], [2].

The concept of controllability is one of the most important concepts in the analysis of distributed systems. This notion can be done in an abstract way by considering various types of functional spaces and operators to introduce some definitions and establish various characterization and properties.

The term of regional controllability has been used to refer to control problems in which the target of our interest is not fully specified as a state, but refers only to a smaller region (which can be internal or boundary) of the system domain. This concept has been widely developed and interesting results have been obtained, in particular, the possibility to reach a state only on an internal subregion

[3] or on a part of the boundary [4].

The mathematical model of a real system is obtained from measurements or from the approximation techniques and is often affected by disturbances [5], and the solution of such a system is approximately known. For these reasons we are here interested in introducing the concept of controllability with constraints, which the aim is to steer a system from an initial state to a final one between two prescribed functions given only on a part of a subregion ω of the geometric area Ω where the system is considered.

This work is a contribution to the enlargement of the regional analysis of distributed systems, representing a new concept of controllability with constraints [6], limited mainly to systems described by hyperbolic partial differential equations. It aims to explore this notion and to give approach which leads to characterize the optimal control that satisfied the output constraints. The paper is organized as follows: In section 2, we introduce the notion of regional controllability of hyperbolic systems, we provide results on this type of controllability and we give definitions and properties related to this notion. In section 3, we solve the problem of minimum energy control using Lagrangian approach devoted to the computation of the optimal control problem for the hyperbolic equations excited by an internal zone actuator. The last section is devoted to compute the obtained algorithm with numerical example and simulations.

* Corresponding author e-mail: boutouloutali@yahoo.fr

2 Problem statement

Let Ω be an open bounded and regular subset of \mathbb{R}^n ($n=1,2,3$) with a boundary $\partial\Omega$.

For $T > 0$, let $Q = \Omega \times]0, T[$ and $\Sigma = \partial\Omega \times]0, T[$, we consider the following hyperbolic system

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x, t) - Ay(x, t) = Bu(t) & Q \\ y(x, 0) = y_0(x), \frac{\partial y}{\partial t}(x, 0) = y_1(x) & \Omega \\ y(\xi, t) = 0 & \Sigma \end{cases} \quad (1)$$

Where A is a second-order elliptic linear operator, $B \in \mathcal{L}(\mathbb{R}^p, L^2(\Omega))$, $u \in U = L^2(0, T; \mathbb{R}^p)$ (p depends on the number of the considered actuators) and $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$. We design by $Z_u(\cdot) = (y_u(\cdot), \frac{\partial y_u}{\partial t}(\cdot)) \in H_0^1(\Omega) \times L^2(\Omega)$ the solution of (1) when it is excited by a control u .

If we denote by $\bar{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$, $z = \begin{bmatrix} y \\ \frac{\partial y_u}{\partial t} \end{bmatrix}$ and $\bar{B}u = \begin{bmatrix} 0 \\ Bu \end{bmatrix}$ then, the system (1) can be written as follows:

$$\begin{cases} \frac{\partial z}{\partial t}(x, t) + \bar{A}z(x, t) = \bar{B}u(t) & Q \\ z(0) = (y_0, y_1)^\top & \Omega \end{cases} \quad (2)$$

The operator \bar{A} is closed and linear, with dense domain in $H_0^1(\Omega) \times L^2(\Omega)$. Hence the system (2) admits a unique solution which is expressed using a semigroup $(\bar{S}(t))_{t \geq 0}$ (for more details about semigroups, see [7]) generated by \bar{A} and given as follow:

$$z(T) = \bar{S}(T)z_0 + \int_0^T \bar{S}(T - \tau)\bar{B}u(\tau)d\tau$$

With the assumption that the operator \bar{A} admits a basis orthogonal eigenfunctions (w_{n_j}) associated with the eigenvalues γ_n of multiplicity r_n , the semigroup $(\bar{S}(t))_{t \geq 0}$ can be written as:

$$\bar{S}(t)z(\cdot) = \left(\begin{array}{l} \sum_{n=1}^{r_n} \left[\langle z_1, w_{n_j} \rangle \cos(\sqrt{-\gamma_n}t) + \frac{1}{\sqrt{-\gamma_n}} \langle z_2, w_{n_j} \rangle \sin(\sqrt{-\gamma_n}t) \right] w_{n_j}(\cdot) \\ \sum_{n=1}^{r_n} \left[(-\sqrt{-\gamma_n}) \langle z_1, w_{n_j} \rangle \sin(\sqrt{-\gamma_n}t) + \langle z_2, w_{n_j} \rangle \cos(\sqrt{-\gamma_n}t) \right] w_{n_j}(\cdot) \end{array} \right)$$

Then we have

$$\int_0^T \bar{S}(T-t)\bar{B}u(t)dt = \left(\begin{array}{l} \sum_{n=1}^{r_n} \sum_{j=1}^{r_n} \left[\int_0^T \frac{1}{\sqrt{-\gamma_n}} \langle Bu(t), w_{n_j} \rangle \sin(\sqrt{-\gamma_n}(T-t))dt \right] w_{n_j}(\cdot) \\ \sum_{n=1}^{r_n} \sum_{j=1}^{r_n} \left[\int_0^T \langle Bu(t), w_{n_j} \rangle \cos(\sqrt{-\gamma_n}(T-t))dt \right] w_{n_j}(\cdot) \end{array} \right)$$

Let ω be an open set of Ω with Lebesgue positive measure, and the restriction operator in ω defined as follows:

$$\chi_\omega : L^2(\Omega) \times L^2(\Omega) \rightarrow L^2(\omega) \times L^2(\omega) \\ (z_1, z_2) \mapsto (z_1, z_2)|_\omega$$

While χ_ω^* is its adjoint operator defined from $L^2(\omega) \times L^2(\omega)$ to $L^2(\Omega) \times L^2(\Omega)$ by

$$\chi_\omega^*(z_1, z_2)(x) = \begin{cases} (z_1, z_2)x, & x \in \omega \\ 0, & x \in \Omega \setminus \omega \end{cases}$$

And let's consider

$$\tilde{\chi}_\omega : L^2(\Omega) \rightarrow L^2(\omega) \\ z \mapsto z|_\omega$$

Let $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ ($i = 1, 2$) be functions in $L^2(\omega)$ such that $\alpha_i(\cdot) \leq \beta_i(\cdot)$ a.e in ω .

Throughout the paper we set

$$I := [\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)] = \{(y_1(\cdot), y_2(\cdot)) \in L^2(\omega) \times L^2(\omega) \mid \alpha_1(\cdot) \leq y_1(\cdot) \leq \beta_1(\cdot) \text{ and } \alpha_2(\cdot) \leq y_2(\cdot) \leq \beta_2(\cdot) \text{ a.e in } \omega\}$$

We recall that an actuator is conventionally defined by a couple (D, f) , where $D \subset \bar{\Omega}$ is the geometric support of the actuator and f is the spatial distribution of the action on the support D .

In the case of a pointwise actuator (internal or boundary) $D = \{b\}$ and $f = \delta(b - \cdot)$, where δ is the Dirac mass concentrated in b , and the actuator is then denoted by (b, δ_b) . For definitions and properties of strategic actuators we refer to [4, 8].

We also recall that the system (1) is said to be ω -exactly (resp. ω -approximately) controllable, if for all $(p^d, v^d) \in L^2(\omega) \times L^2(\omega)$ (resp. for all $\varepsilon > 0$) there exists a control $u \in U$ such that $\tilde{\chi}_\omega y_u(T) = p^d$ and $\tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) = v^d$ (resp. $\|\tilde{\chi}_\omega y_u(T) - p^d\|_{L^2(\omega)} + \|\tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) - v^d\|_{L^2(\omega)} < \varepsilon$) [9].

Let H be the operator from $U \rightarrow L^2(\Omega) \times L^2(\Omega)$, for $u \in U$, defined by:

$$Hu = \int_0^T \bar{S}(T - \tau)\bar{B}u(\tau)d\tau$$

Definition 1.

We say that the system (1) is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -Controllable in ω if

$$(Im\chi_\omega H) \cap ([\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]) \neq \emptyset$$

Remark.

The above definition is equivalent to say that:

The system (1) is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -Controllable in ω at the time T if there exists $u \in U$ such that :

$$\alpha_1(\cdot) \leq \tilde{\chi}_\omega y_u(T) \leq \beta_1(\cdot) \text{ and } \alpha_2(\cdot) \leq \tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) \leq \beta_2(\cdot)$$

Definition 2.

The actuator (D, f) is said to be $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -strategic in ω if the excited system is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -controllable in ω

Remark.

- 1.A system (1) which is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -Controllable in ω_1 is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -Controllable for any $\omega_2 \subseteq \omega_1$.
- 2.Let

$$\mathcal{J}(u) = \frac{1}{2} \int_0^T \|u(t)\|_{\mathbb{R}^p}^2 dt$$

be the transfer cost, $(p^d, v^d) \in I$, and consider the sets

$$\mathcal{W}_\omega = \left\{ u \in L^2(0, T, \mathbb{R}^p) / \chi_\omega(y_u(T), \frac{\partial y_u}{\partial t}(T)) = (p^d, v^d) \right\}$$

$$\mathcal{W}_I = \left\{ u \in L^2(0, T, \mathbb{R}^p) / \alpha_1(\cdot) \leq \tilde{\chi}_\omega y_u(T) \leq \beta_1(\cdot) \text{ and } \alpha_2(\cdot) \leq \tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) \leq \beta_2(\cdot) \text{ a.e in } \omega \right\}$$

We have $\mathcal{W}_\omega \subseteq \mathcal{W}_I$, then

$$\inf_{\mathcal{W}_I} \mathcal{J}(u) \leq \inf_{\mathcal{W}_\omega} \mathcal{J}(u)$$

This means that the cost of steering the system in I is less than steering it to a fixed desired state $(p^d, v^d) \in [\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$.

The $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -controllability in ω may be characterized by the following result:

Proposition 1.

The system (1) is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -Controllable in ω if and only if

$$(Ker\chi_\omega + ImH) \cap ([\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]) \neq \emptyset$$

Proof

We suppose that there exists $z \in ([\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)])$, and $u \in U$ such that $\chi_\omega Z_u(T) = \chi_\omega z$, let's consider $z_1 = z - Z_u(T)$ and $z_2 = Z_u(T)$, then $z = z_1 + z_2$ where $z_1 \in ker(\chi_\omega)$ and $z_2 \in Im(H)$ which prove that $z \in (Ker\chi_\omega + ImH)$.

Conversely,

if $(Ker\chi_\omega + ImH) \cap ([\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]) \neq \emptyset$ then there exists $z \in ([\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)])$ such that $z \in (Ker\chi_\omega + ImH)$,

so $z = z_1 + z_2$, where $\chi_\omega z_1 = 0$ and $\exists u \in U | z_2 = Hu$. It follows that there exists $z \in [\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$, and $u \in U$ such that $\chi_\omega Z_u(T) = z$.

3 Minimum energy control

The purpose of this section is to explore the Lagrangian multiplier approach devoted to the computation of the optimal control problem, for the hyperbolic equation excited by an internal zone actuator, which steers the system (1) from $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ to a final state $(p^d, v^d) \in L^2(\omega) \times L^2(\omega)$ such that $\alpha_1(\cdot) \leq p^d \leq \beta_1(\cdot)$ and $\alpha_2(\cdot) \leq v^d \leq \beta_2(\cdot)$ in a subregion ω .

More precisely we are interested to the following minimization problem

$$\begin{cases} \inf \mathcal{J}(u) \\ u \in U_{ad} \end{cases} \quad (3)$$

Where

$$U_{ad} = \{ u \in U | \alpha_1(\cdot) \leq \tilde{\chi}_\omega y_u(T) \leq \beta_1(\cdot) \text{ and } \alpha_2(\cdot) \leq \tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) \leq \beta_2(\cdot) \text{ a.e in } \omega \},$$

is the set of admissible controls.

The following result ensure the existence and the uniqueness of the solution of the problem (3).

Proposition 2.

If the system (1) is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -Controllable in ω then the problem (3) has a unique solution u^* .

Proof

If the system (1) is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -Controllable in ω then U_{ad} is a non-empty subset of the reflexive U , then the mapping $u \mapsto (y_u(T), \frac{\partial y_u}{\partial t}(T))$ is linear, so U_{ad} is convex, and to prove that U_{ad} is closed, we consider a sequence $(u_n)_n$ in U_{ad} such that $u_n \rightarrow u$ strongly in U . Since $\chi_\omega H$ is continuous, then $\chi_\omega H u_n$ converges strongly to $\chi_\omega H u$ in $L^2(\omega) \times L^2(\omega)$, and $\chi_\omega(y_{u_n}(T), \frac{\partial y_{u_n}}{\partial t}(T)) \in I$ which is closed so U_{ad} is closed. Furthermore $\lim_{\|u\| \rightarrow +\infty} \mathcal{J}(u) = +\infty$ and the mapping

$u \mapsto \frac{1}{2} \|u\|^2$ is continue and strictly convex then (3) has a unique solution.

Remark.

The solution u^* of (3) is characterized by $\langle u^*, u - u^* \rangle \geq 0, \forall u \in U_{ad}$, but this characterization is difficult to be implemented from a numerical point of view. In the following, we give the Lagrangian multiplier approach characterizing the optimal control solution of (3).

We consider the problem (3), when the system is excited by one zone actuator (D, f) . The following result gives a useful characterization of the problem (3):

Theorem 1.

If the actuator (D, f) is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -strategic in ω then the solution of (3) is given by :

$$u^* = -(\chi_\omega H)^*(\lambda_1^*, \lambda_2^*) \quad (4)$$

Where $(\lambda_1^*, \lambda_2^*)$ is the solution of:

$$\begin{cases} (p^{d^*}, v^{d^*}) = P_I[\rho(\lambda_1^*, \lambda_2^*) + (p^{d^*}, v^{d^*})] \\ (p^{d^*}, v^{d^*}) + R_\omega(\lambda_1^*, \lambda_2^*) = \chi_\omega \bar{S}(T)(y_0, y_1) \end{cases} \quad (5)$$

While $P_I : L^2(\omega) \times L^2(\omega) \rightarrow I$ denotes the projection operator, $\rho > 0$ and $R_\omega = (\chi_\omega H)(\chi_\omega H)^*$.

Proof

If the actuator (D, f) is $[\alpha_1(\cdot), \beta_1(\cdot)] \times [\alpha_2(\cdot), \beta_2(\cdot)]$ -strategic in ω then $U_{ad} \neq \emptyset$ and (3) has a unique solution. The problem (3) is equivalent to the following saddle point problem:

$$\begin{cases} \inf \mathcal{J}(u) \\ (u, p^d, v^d) \in V \end{cases} \quad (6)$$

Where

$$V = \{(u, p^d, v^d) \in U \times I \mid \tilde{\chi}_\omega y_u(T) = p^d, \tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) = v^d\}.$$

To study this constraints, we'll use a Lagrangian functional and steers the problem (6) to a saddle points problem.

We associate to the problem (6) the Lagrangian functional defined by:

$$\forall (u, p^d, v^d, \lambda_1, \lambda_2) \in U \times I \times L^2(\omega) \times L^2(\omega),$$

$$\begin{aligned} L(u, p^d, v^d, \lambda_1, \lambda_2) &= \frac{1}{2} \|u\|^2 + \langle \lambda_1, \tilde{\chi}_\omega y_u(T) - p^d \rangle_{L^2(\omega)} \\ &\quad + \langle \lambda_2, \tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) - v^d \rangle_{L^2(\omega)} \end{aligned}$$

Where $\langle \cdot, \cdot \rangle_{L^2(\omega)}$ is the scalar product in $L^2(\omega)$.

Let recall that $(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*)$ is a saddle point of the functional L if:

$$\begin{aligned} \max_{(\lambda_1, \lambda_2) \in L^2(\omega) \times L^2(\omega)} L(u^*, p^{d^*}, v^{d^*}, \lambda_1, \lambda_2) &= L(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*) \\ &= \min_{\substack{u \in U \\ (p^d, v^d) \in I}} L(u, p^d, v^d, \lambda_1^*, \lambda_2^*) \end{aligned}$$

The proof will be continued in three steps.

• Step 1

$U \times I$ are non-empty, closed and convex subset. The Functional L satisfies conditions

$(u, p^d, v^d) \mapsto L(u, p^d, v^d, \lambda_1, \lambda_2)$ is convex and lower semi-continuous for all $(\lambda_1, \lambda_2) \in L^2(\omega) \times L^2(\omega)$.

$(\lambda_1, \lambda_2) \mapsto L(u, p^d, v^d, \lambda_1, \lambda_2)$ is concave and upper semi-continuous for all $(u, p^d, v^d) \in U \times I$

Moreover there exists $(\lambda_1^0, \lambda_2^0) \in L^2(\omega) \times L^2(\omega)$ such that

$$\lim_{\|(u, p^d, v^d)\| \rightarrow +\infty} L(u, p^d, v^d, \lambda_1^0, \lambda_2^0) = +\infty \quad (7)$$

And there exists $(u_0, p_0^d, v_0^d) \in U \times I$ such that

$$\lim_{\|(\lambda_1, \lambda_2)\| \rightarrow +\infty} L(u_0, p_0^d, v_0^d, \lambda_1, \lambda_2) = -\infty \quad (8)$$

Then, the functional L admits a saddle point. For more details we refer to [10].

• Step 2

Let $(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*)$ be a saddle point of L and prove that u^* is the solution of (3). We have

$$L(u^*, p^{d^*}, v^{d^*}, \lambda_1, \lambda_2) \leq L(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*) \leq L(u, p^d, v^d, \lambda_1^*, \lambda_2^*)$$

For all $(u, p^d, v^d, \lambda_1, \lambda_2) \in U \times I \times L^2(\omega) \times L^2(\omega)$
From the first inequality

$$L(u^*, p^{d^*}, v^{d^*}, \lambda_1, \lambda_2) \leq L(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*)$$

it follows that :

$$\begin{aligned} \langle \lambda_1, \tilde{\chi}_\omega y_{u^*}(T) - p^{d^*} \rangle + \langle \lambda_2, \tilde{\chi}_\omega \frac{\partial y_{u^*}}{\partial t}(T) - v^{d^*} \rangle &\leq \\ \langle \lambda_1^*, \tilde{\chi}_\omega y_{u^*}(T) - p^{d^*} \rangle + \langle \lambda_2^*, \tilde{\chi}_\omega \frac{\partial y_{u^*}}{\partial t}(T) - v^{d^*} \rangle & \end{aligned}$$

which implies that $\tilde{\chi}_\omega y_{u^*}(T) = p^{d^*}$ and $\tilde{\chi}_\omega \frac{\partial y_{u^*}}{\partial t}(T) = v^{d^*}$, hence $\tilde{\chi}_\omega y_{u^*}(T) \in [\alpha_1(\cdot), \beta_1(\cdot)]$ and $\tilde{\chi}_\omega \frac{\partial y_{u^*}}{\partial t}(T) \in [\alpha_2(\cdot), \beta_2(\cdot)]$.

From the second inequality it follows that :

$$\begin{aligned} \frac{1}{2} \|u^*\|^2 + \langle \lambda_1^*, \tilde{\chi}_\omega y_{u^*}(T) - p^{d^*} \rangle + \langle \lambda_2^*, \tilde{\chi}_\omega \frac{\partial y_{u^*}}{\partial t}(T) - v^{d^*} \rangle &\leq \\ \frac{1}{2} \|u\|^2 + \langle \lambda_1^*, \tilde{\chi}_\omega y_u(T) - p^d \rangle + \langle \lambda_2^*, \tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) - v^d \rangle & \end{aligned}$$

and $(p^d, v^d) \in I$.

Since $\tilde{\chi}_\omega y_{u^*}(T) = p^{d^*}$ and $\tilde{\chi}_\omega \frac{\partial y_{u^*}}{\partial t}(T) = v^{d^*}$ we have,

$$\frac{1}{2} \|u^*\|^2 \leq \frac{1}{2} \|u\|^2 + \langle \lambda_1^*, \tilde{\chi}_\omega y_u(T) - p^d \rangle + \langle \lambda_2^*, \tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) - v^d \rangle$$

taking $p^d = \tilde{\chi}_\omega y_u(T) \in [\alpha_1(\cdot), \beta_1(\cdot)]$ and $v^d = \tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) \in [\alpha_2(\cdot), \beta_2(\cdot)]$, we obtain $\frac{1}{2} \|u^*\|^2 \leq \frac{1}{2} \|u\|^2$ which implies that u^* is the minimum energy.

• Step 3

$(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*)$ is a saddle point of L then the following assumptions hold:

$$\langle u^*, u - u^* \rangle + \langle (\lambda_1^*, \lambda_2^*), \chi_\omega H(u - u^*) \rangle = 0 \quad \forall u \in U \quad (9)$$

$$\langle (\lambda_1^*, \lambda_2^*), (p^d, v^d) - (p^{d^*}, v^{d^*}) \rangle \leq 0 \quad \forall (p^d, v^d) \in I \quad (10)$$

$$\langle (\lambda_1, \lambda_2) - (\lambda_1^*, \lambda_2^*), \chi_\omega (y_{u^*}(T), \frac{\partial y_{u^*}}{\partial t}(T)) - (p^{d^*}, v^{d^*}) \rangle = 0, \quad \forall (\lambda_1, \lambda_2) \in L^2(\omega) \times L^2(\omega) \quad (11)$$

Details on the saddle point theory and its applications can be found for instance in [11, 12, 13].

From (9) we deduce that (4) and (11) is equivalent to $(p^{d^*}, v^{d^*}) = \chi_\omega \bar{S}(T)(y_0, y_1) + \chi_\omega H(u^*)$, and with (4) the second part of (5) is obtained. From the inequality (10)

we obtain

$$\langle (\rho(\lambda_1^*, \lambda_2^*) + (p^{d^*}, v^{d^*})) - (p^{d^*}, v^{d^*}), (p^d, v^d) - (p^{d^*}, v^{d^*}) \rangle \leq 0, \text{ for all } (p^d, v^d) \in I, \text{ which is equivalent to the first part of (5).}$$

Corollary 1.

If the system (1) is exactly controllable in ω , and ρ conveniently chosen, then the system (5) has only one solution $(\lambda_1^*, \lambda_2^*, p^{d^*}, v^{d^*})$.

Proof

The regional controllability in ω implies that $(\chi_\omega H)^*$ and R_ω are bijective, so if $(u^*, p^{d^*}, v^{d^*}, \lambda_1^*, \lambda_2^*)$ is a saddle point of L then the system (5) is equivalent to

$$\begin{aligned} (\lambda_1^*, \lambda_2^*) &= R_\omega^{-1}(\chi_\omega \bar{S}(T)(y_0, y_1) + (p^{d^*}, v^{d^*})) \\ (p^{d^*}, v^{d^*}) &= P_I(-\rho R_\omega^{-1}(p^{d^*}, v^{d^*}) + \rho R_\omega^{-1} \chi_\omega \bar{S}(T)(y_0, y_1) + (p^{d^*}, v^{d^*})) \end{aligned} \quad (12)$$

It follows that $(p^{d^*}, v^{d^*}) \in I$ is a fixed point of the function

$$F_\rho : I \rightarrow I \\ (Z_1, Z_2) \mapsto (P_I(-\rho R_\omega^{-1}(Z_1, Z_2) + \rho R_\omega^{-1} \chi_\omega \bar{S}(T)(y_0, y_1) + (Z_1, Z_2))) \quad (13)$$

The operator R_ω^{-1} is coercive, then there exists $m > 0$ such that

$$\langle R_\omega^{-1}(Z_1, Z_2), (Z_1, Z_2) \rangle \geq m \| (Z_1, Z_2) \|^2 \\ \forall (Z_1, Z_2) \in L^2(\omega) \times L^2(\omega)$$

It follows that

$$\| F_\rho(Z_1, Z_2) - F_\rho(Y_1, Y_2) \|^2 \\ \leq (1 + \rho^2 \| R_\omega^{-1} \|^2 - 2\rho m) \| (Z_1, Z_2) - (Y_1, Y_2) \|^2$$

for all (Z_1, Z_2) and (Y_1, Y_2) in I , then we deduce that if

$$0 < \rho < \frac{2m}{\| R_\omega^{-1} \|^2}$$

Then F_ρ is contractant, which implies the uniqueness of $p^{d^*}, v^{d^*}, \lambda_1^*$ and λ_2^* .

Remark.

1.If $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$ we find the notion of exact regional controllability and the solution of (3) is given by

$$u^*(t) = (\chi_\omega H)^* R_\omega^{-1}((\alpha_1, \alpha_2) - \chi_\omega \bar{S}(T)(y_0, y_1))$$

2.Similar results can be obtained in pointwise actuator case.

4 Numerical approach

In this subsection we describe a numerical scheme which allows the calculation of the initial state (position and speed) between the constraints functions. So from theorem (1), the solution of the problem (3) arises to

compute the saddle points of L , which is equivalent to solve the following problem

$$\inf_{(u, p^d, v^d) \in U \times I} \left(\sup_{(\lambda_1, \lambda_2) \in L^2(\omega) \times L^2(\omega)} L(u, p^d, v^d, \lambda_1, \lambda_2) \right) \quad (14)$$

To attain this, the implementation can be based on the following algorithm of Uzawa type [13]

1.Choose:

.The inner region ω , the actuator (D, f) and a precision threshold ε small enough.

.Functions $p_0^d \in [\alpha_1(\cdot), \beta_1(\cdot)], v_0^d \in [\alpha_2(\cdot), \beta_2(\cdot)],$

$\lambda_1^1 \in L^2(\omega)$ and $\lambda_2^1 \in L^2(\omega)$

2. $(p_{n-1}^d, v_{n-1}^d, \lambda_1^n, \lambda_2^n)$ known, we determine u_n, p_n^d, v_n^d with the formula

. $u_n = -(\chi_\omega H)^*(\lambda_1^n, \lambda_2^n),$

. $p_n^d = P_{[\alpha_1(\cdot), \beta_1(\cdot)]}(\rho \lambda_1^n + p_{n-1}^d),$

. $v_n^d = P_{[\alpha_2(\cdot), \beta_2(\cdot)]}(\rho \lambda_2^n + v_{n-1}^d)$

3. $\lambda_1^{n+1} = \lambda_1^n + (\tilde{\chi}_\omega y_{u_n}(T) - p_n^d)$

and $\lambda_2^{n+1} = \lambda_2^n + (\tilde{\chi}_\omega \frac{\partial y_u}{\partial t}(T) - v_n^d)$

4.If $\| p_{n+1}^d - p_n^d \|_{L^2(\omega)} + \| v_{n+1}^d - v_n^d \|_{L^2(\omega)} \leq \varepsilon$ we stop, else we return to 2.

Example

Here we give a numerical example that leads to some results related to the choice of the subregion, the constraints functions and the actuator location. Let's consider the following one-dimensional system in $\Omega =]0, 1[$ excited by a pointwise actuator:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}(x, t) = \frac{\partial^2 y}{\partial x^2}(x, t) + \delta(x - b)u(t) & \Omega \times]0, T[\\ y(x, 0) = 0, \frac{\partial y}{\partial t}(x, 0) = 0 & \Omega \\ y(0, t) = y(1, t) = 0 &]0, T[\end{cases} \quad (15)$$

We take $T=2, b = 0.85$ (location of the pointwise actuator),

$$\alpha_1(x) = \frac{1}{5}x^2(x - 1)^2, \beta_1(x) = \frac{-1}{3}x(x - 1),$$

$$\alpha_2(x) = \frac{1}{2}x^2(x - 1)^2 \text{ and } \beta_2(x) = \frac{1}{5}x(x - 1).$$

Applying the previous algorithm we obtain the following results:

For $\omega = \Omega$

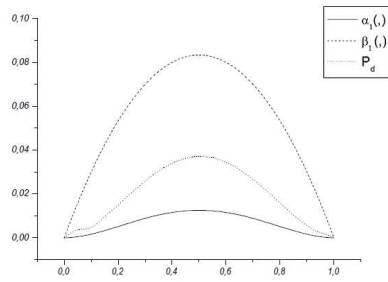


Fig. 1: Desired position between $\alpha_1(\cdot)$ and $\beta_1(\cdot)$

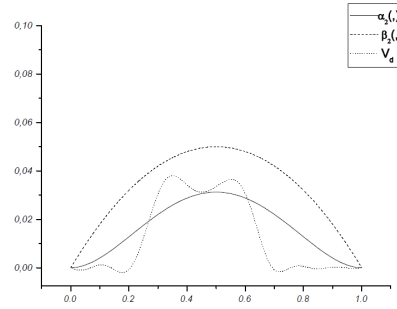


Fig. 4: Desired speed between $\alpha_2(\cdot)$ and $\beta_2(\cdot)$.

From figure 3 and 4, we note that the reached state (resp. speed) is between $\alpha_1(\cdot)$ and $\beta_1(\cdot)$ (resp. $\alpha_2(\cdot)$ and $\beta_2(\cdot)$) in the subregion ω , the location of the actuator is $[\alpha_i(\cdot), \beta_i(\cdot)]$ -strategic and the reached state and speed are obtained with reconstruction error $\varepsilon = 2.46 \times 10^{-4}$ and cost $\|u^*\|^2 = 3.07 \times 10^{-5}$

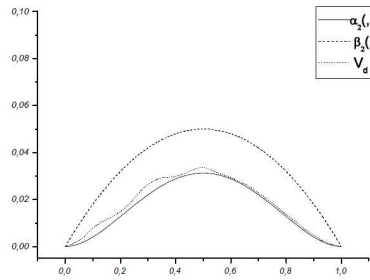


Fig. 2: Desired speed between $\alpha_2(\cdot)$ and $\beta_2(\cdot)$.

Figure 1 and 2 show that the reached state (resp. speed) is between $[\alpha_1(\cdot), \beta_1(\cdot)]$ (resp. $[\alpha_2(\cdot), \beta_2(\cdot)]$) in the whole domain so the sensor is $[\alpha_i(\cdot), \beta_i(\cdot)]$ -strategic in Ω .

For $\omega =]0.25, 0.65[$ we obtain the following figures.

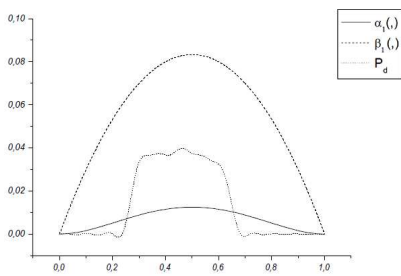


Fig. 3: Desired position between $\alpha_1(\cdot)$ and $\beta_1(\cdot)$

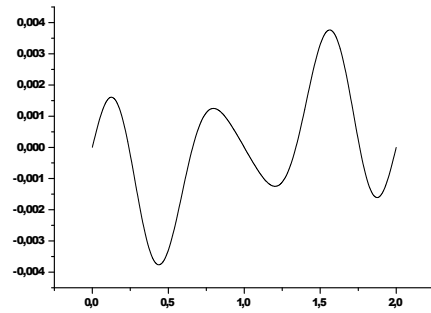


Fig. 5: Evolution of the control function

Figure 5 shows the evolution of the optimal control u^* which steers the system from the initial states to the desired ones between $\alpha_i(\cdot)$ and $\beta_i(\cdot)$.

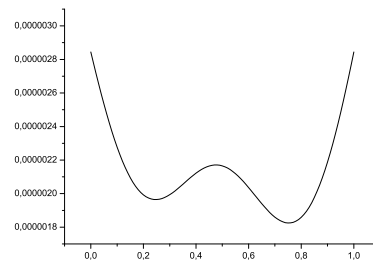


Fig. 6: Cost evolution with respect to actuator location.

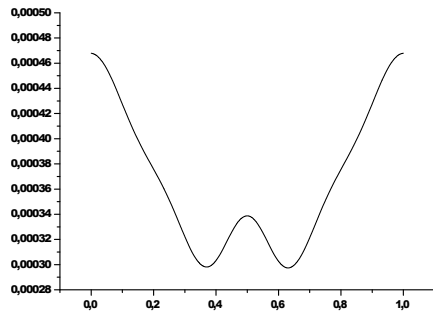


Fig. 7: Error evolution with respect to actuator location.

Table 1: Relation between the subregion and the cost

Subregion	The cost
]0.2,0.8[3.7×10^{-1}
]0.25,0.75[5.28×10^{-3}
]0.25,0.65[3.07×10^{-5}
]0.33,0.58[9.19×10^{-7}
]0.08,0.22[7.84×10^{-8}

Table 2: Relation between the subregion and the reconstruction error

Subregion	The reconstruction error
]0.2,0.8[6.47×10^{-1}
]0.25,0.75[3.53×10^{-3}
]0.25,0.65[2.46×10^{-4}
]0.33,0.58[7.76×10^{-5}
]0.08,0.22[1.89×10^{-5}

Table (1) and (2) show numerically how both the cost and the reconstruction error grow with respect to the subregion area. This shows that, the cost and the error increase with the width of the subregion.

5 Conclusion

We have developed an extension of the notion of controllability for hyperbolic systems with constraints, we characterized the optimal control using the Lagrangian approach, and interesting results are obtained and illustrated with numerical example and simulations. Future works aim to extend this notion of regional controllability with constrained to the case where ω is a part of the boundary of the evolution domain Ω .

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Ali Boutoulout is a professor at Moulay Ismail the University of Meknes in Morocco. He obtained his State Doctorate in Regional System Analysis (2000) at Moulay Ismail University. Professor Boutoulout has published many papers in the area of system analysis and

control. Now he is the head of the research at MACS Laboratory (Modelling, Analysis and Control Systems) and a director of System Theory and Informatics Master at the department of Mathematics and Informatics, Faculty of Sciences, Moulay Ismail University, Meknes, Morocco



Hamid Bourray is a professor at Moulay Ismail University of Meknes in Morocco. He is an assistant professor at the same University. He got his Doctorate in Systems Analysis (2002) at the Faculty of Sciences in Meknes. He had published many papers in

the area of system analysis and control. He is a researcher at STI team (System Theory and Informatics), MACS Laboratory at Moulay Ismail University, Meknes, Morocco.



Layla Ezzahri is a researcher, preparing her thesis in Applied Mathematics at Moulay Ismail University, of Meknes in Morocco. Her studies are focused on analysis and control of distributed systems and numerical analysis. At present she is a Ph.D student

at STI team (System Theory and Informatics) in MACS Laboratory.



Fatima El Alaoui is a professor at Moulay Ismail University of Meknes in Morocco. She is an assistant professor at the same University. She got his Doctorate in Systems Analysis (2011) at the Faculty of Sciences in Meknes. She had published many papers in the area of system analysis

and control. She is a researcher at STI team (System Theory and Informatics), MACS Laboratory at Moulay Ismail University, Meknes, Morocco.