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Generalized I-Proximity Spaces

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Abstract: An ideal on a set X is a nonempty collection of subsets of X with heredity property which is also closed finite unions. The purpose of this paper is to construct a new approach of generalized proximity based on the ideal notion. For $I = \{\phi\}$, we have the generalized proximity structure [15] and for the other types of I, we have many types of generalized proximity structures. In addition, if (X,τ) is an IR_2-topological space, then τ^* is a compatible with an I-Pervin proximity relation on $P(X)$. It is also shown that if (X,τ) is a $*-$normal space and (X,τ^*) is a R_o-space, then τ^* is a compatible with an I-Lodato proximity relation on $P(X)$.

Keywords: Generalized $I-$proximity space, compatibility, $*-$normal space, ideal.

1 Introduction

The notion of ideal topological spaces was first studied by Kuratowski [10] and Vaidyanathaswamy [21]. Compatibility of the topology with an ideal I was first defined by Njastad [16]. In 1990, Jankovic and Hamlett [6] investigated further properties of ideal topological spaces. The fundamental concept of Efremovič proximity space has been introduced by Efremovič [2]. In addition to, Leader [11,12] and Lodato [13,14] have worked with weaker axioms than those of Efremovič proximity space enabling them to introduce an arbitrary topology on the underlying set. Furthermore, proximity relations are useful in solving problems based on human perception [17] that arise in areas such as image analysis [5] and face recognition [4]. Cyclic contraction and best proximity point are among the popular topics in the fixed point theory and many results have been obtained, for instance, [1,3,9,19]. Recently, A. Kandil et.al. [7,8] introduced a new approach of proximity structures [15] based on the ideal and soft set notions. In this paper, we generalize the notion of generalized proximity by using the concepts of ideal in the ordinary topology. In addition, the notions of I-Leader, I-Pervin, and I-Lodato proximities have been introduced. The main theorems in our work is to exhibit the relation between the topology generated via these proximities and the topology τ^* which generated via ideal. Also, we show that our generalizations are good extension of the old proximity relations.

2 Preliminaries

Let (X,τ) be a topological space. For a subset A of a topological space (X,τ), $\overline{A}$ and A^o denote the closure and the interior of A in (X,τ), respectively. An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies the following properties:

1. $A\in I$ and $B\in I \Rightarrow A\cup B\in I$,
2. $A\in I$ and $B\subseteq A \Rightarrow B\in I$.

An ideal topological space is a topological space (X,τ) with an ideal I on X and is denoted by (X,τ,I) . For a subset $A\subseteq X$, $A^*(I,\tau) := \{x\in X : A\cap U \notin I$ for every open set U containing $x\}$ is called the local function of A with respect to I and τ (see [6,10,18]). We simply write A^* instead of $A^*(I,\tau)$ in case there is no chance for confusion.

Proposition 2.1.[6] Let (X,τ) be a topological space and I be an ideal on X. Then the operator

$$Cl^* : P(X) \to P(X)$$

defined by:

$$Cl^*(A) = A\cup A^* \tag{1}$$

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satisfies Kuratwski's axioms and induces a topology on X called τ^* given by

$$\tau^* = \{A \subseteq X : Cl^*(A^c) = A^c\}. \quad (2)$$

Where A^c denotes the complement of A.

Indeed, for every ideal topological space (X,τ,I), there exists a topology $\tau^*(I)$ finer than τ. When there is no ambiguity, $\tau^*(I)$ is denoted by τ^*. For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of A in (X,τ^*) respectively.

Example 2.2.[6] Let X be a nonempty set. Then the following families are ideals on X

1. $I = \{\phi\}$,
2. $I = P(X) = \{A : A \subseteq X\}$,
3. $I_f = \{A \subseteq X : A \text{ is } finite\}$, called ideal of finite subsets,
4. $I_c = \{A \subseteq X : A \text{ is } countable\}$, called ideal of countable subsets,
5. $I_A = \{B \subseteq X : B \subseteq A\}$,
6. If (X,τ) is a topological space, then the family of nowhere dense subsets, namely $I_n = \{A \subseteq X : \overline{A}^o = \phi\}$ forms an ideal on X.

Definition 2.3.[15] Let δ be a binary relation on $P(X)$. For any $A,B,C \in P(X)$, consider the following axioms:-

$(S_1) A\delta B \Rightarrow B\delta A$
$(S_2) A\delta(B\cup C) \Leftrightarrow A\delta B$ or $A\delta C$, and
$(B\cup C)\delta A \Leftrightarrow B\delta A$ or $C\delta A$.
$(S_3) A\delta B \Rightarrow A \neq \phi$ and $B \neq \phi$.
$(S_4) A\delta B$ and $b\delta C\ \forall b \in B \Rightarrow A\delta C$.
$(S_5) A\cap B \neq \phi \Rightarrow A\delta B$.
$(S_6) A\not\delta B \Rightarrow \exists C,D \subseteq X$ such that $A\not\delta C^c, D^c\not\delta B$ and $C\cap D = \phi$.

Then δ is said to be:-

1. A Leader proximity on X, if it satisfies (S_2), (S_3), (S_4) and (S_5).
2. A Lodato proximity on X, if it is Leader proximity and satisfies (S_1).
3. A Pervin proximity on X, if it satisfies (S_2), (S_3), (S_5) and (S_6).

If δ is a Leader (respectively Lodato and Pervin) proximity on X, then the pair (X,δ) is called a Leader (respectively Lodato and Pervin) proximity space.

Definition 2.4.[7] Let I be an ideal on a nonempty set X. A binary relation δ_I on $P(X)$ is called an I-proximity on X if δ_I satisfies the following conditions:-

$(IP_1) A\delta_I B \Rightarrow B\delta_I A$,
$(IP_2) A\delta_I(B\cup C) \Leftrightarrow A\delta_I B \text{ or } A\delta_I C$,
$(IP_3) A\not\delta_I B\ \forall A \in I,\ B \in P(X)$,
$(IP_4) A\cap B \notin I \Rightarrow A\delta_I B$,
$(IP_5) A\not\delta_I B \Rightarrow \exists\, C,D \subseteq X$ such that $A\not\delta_I C^c,\ D^c\not\delta_I B$ and $C\cap D \in I$.

An I-proximity space is a pair (X,δ_I) consisting of a set X and an $I-$proximity relation on X. We shall write $A\delta_I B$ if the sets $A,B \subseteq X$ are δ_I-related, otherwise we shall write $A\not\delta_I B$.

δ_I is said to be separated, if it satisfies:

$(IP_6) x\delta_I y \Rightarrow x = y$.

Definition 2.5.[15] A binary relation δ on $P(X)$ is called a generalized Proximity relation if it is Leader or Pervin or Lodato proximity. Furthermore, if δ is a generalized proximity relation on $P(X)$, then the pair (X,δ) is called a generalized Proximity Space.

3 New Approach of Generalized Proximity Spaces

Definition 3.1. Let I be an ideal on a nonempty set X and δ_I be a binary relation on $P(X)$. For any $A,B,C \in P(X)$, consider the following axioms:-

$(IL_1) A\delta_I B \Rightarrow B\delta_I A$,
$(IL_2) A\delta_I(B\cup C) \Leftrightarrow A\delta_I B$ or $A\delta_I C$, and
$(B\cup C)\delta_I A \Leftrightarrow B\delta_I A$ or $C\delta_I A$,
$(IL_3) A\not\delta_I B\ \forall A \in I,\ B \in P(X)$,
$(IL_4) A\delta_I B$ and $b\delta_I C\ \forall b \in B \Rightarrow A\delta_I C$,
$(IL_5) A\cap B \notin I \Rightarrow A\delta_I B$.
$(IL_6) A\not\delta_I B \Rightarrow \exists C,D \subseteq X$ such that $A\not\delta_I C^c, D^c\not\delta_I B$ and $C\cap D \in I$,

Then δ_I is said to be:-

(a) An $I-$Leader proximity on X, if it satisfies (IL_2), (IL_3), (IL_4) and (IL_5).
(b) An $I-$Lodato proximity on X, if it is $I-$Leader proximity and satisfies (IL_1).
(c) An $I-$Pervin proximity on X, if it satisfies (IL_2), (IL_3), (IL_5) and (IL_6).

If δ_I is an $I-$Leader (respectively $I-$Lodato and $I-$Pervin) proximity on X, then the pair (X,δ_I) is called an $I-$Leader (respectively $I-$Lodato and $I-$Pervin) proximity space.

By this generalized definition, we obtain all preceding definitions introduced by Leader [11], Lodato [13], and Pervin [15] as special cases of the current definition, as follows

Proposition 3.2. If $I = \{\phi\}$ in Definition 3.1, then we get the generalized proximity relations in Definition 2.3.

Proof. Straightforward.

Definition 3.3. A binary relation δ_I on $P(X)$ is called a generalized I-Proximity relation if it is an I-Leader or an I-Pervin or an I-Lodato proximity. Moreover, if δ_I is a generalized I-proximity relation on $P(X)$, then the pair (X,δ_I) is called a generalized I-Proximity Space.

Lemma 3.4. Let (X,δ_I) be a Generalized $I-$proximity space, $A\delta_I B$, $A \subseteq C$, and $B \subseteq D$, then $C\delta_I D$.

Proof. The result follows immediately from (IL_2).

Example 3.5. Let I be an ideal on a nonempty set X and δ_I be a binary relation on $P(X)$ defined as:

$$A\delta_I B \Leftrightarrow A,B \notin I. \tag{3}$$

Indeed, one easily sees that δ_I satisfies conditions $IL_1 - IL_6$. Therefore, it is a generalized I-proximity relation.

Theorem 3.6. Every $I-$Pervin proximity on X is also $I-$Leader proximity on X.

Let δ_I be an $I-$Pervin proximity on X. It is sufficient to show that δ_I satisfies (IL_4). Let $A\delta_I B$ and $\forall b \in B$, $b\delta_I H$. If $A \not\delta_I H$, then $\exists C,D \subseteq X$ such that $A \not\delta_I C^c, D^c \not\delta_I B$ and $C \cap D \in I$. This result, combined with $A\delta_I B$ and (IL_2), implies $B \not\subseteq C^c$, i.e. $B \cap C \neq \phi$. It follows that $\exists k \in X$ such that $k\delta_I H$ and $k \in C$. Two cases exist: either $k \in D$ or $k \in D^c$. If $k \in D$. Hence $\{k\} \in I$. (IL_3) implies that $k \not\delta_I H$, which is contradiction. If $k \in D^c$, then $k \not\delta_I B$. This result, combined with (IL_5) and (IL_3), implies that $k \not\delta_I H$, which is contradiction. So, $A\delta_I H$.

Theorem 3.7. Let (X, δ_I) be a generalized $I-$ proximity space. Then the δ_I- operator

$$\delta_I : P(X) \to P(X)$$

defined by:

$$A^{\delta_I} = \{x \in X : x\delta_I A\} \tag{4}$$

satisfies the following:-

1. $\phi^{\delta_I} = \phi$,
2. $A \subseteq B \Rightarrow A^{\delta_I} \subseteq B^{\delta_I}$,
3. $(A \cup B)^{\delta_I} = A^{\delta_I} \cup B^{\delta_I}$,
4. $(A \cap B)^{\delta_I} \subseteq A^{\delta_I} \cap B^{\delta_I}$,
5. $(A^{\delta_I})^{\delta_I} \subseteq A^{\delta_I}$,
6. $A \not\subseteq A^{\delta_I}$, in general.

Proof.

1. If $\exists\, x \in X$ such that $x\delta_I \phi$. Then (IL_3) implies $\phi \notin I$, which is contradiction. So, $\phi^{\delta_I} = \phi$.
2. Let $x \in A^{\delta_I}$. Then formula (4) implies that $x\delta_I A$ and Lemma 3.4 implies that $x\delta_I B$. Hence $x \in B^{\delta_I}$.
3. By part (2), we get $A^{\delta_I} \cup B^{\delta_I} \subseteq (A \cup B)^{\delta_I}$. To prove the other inclusion, let $x \in (A \cup B)^{\delta_I}$. Then $x\delta_I(A \cup B)$. Hence (IP_2) implies that $x\delta_I A$ or $x\delta_I B$, consequently $x \in (A^{\delta_I} \cup B^{\delta_I})$. Hence the result.
4. The result is a direct consequence of part (2).
5. Let $x \in (A^{\delta_I})^{\delta_I}$. Then $x\delta_I A^{\delta_I}$ and $\forall\, y \in A^{\delta_I}$, we have $y\delta_I A$. (IL_4) implies that $x\delta_I A$. Hence the result.
6. We give an example. Let X be a nonempty set, $I = I_f$, A be a nonempty subset of X and δ_I is any generalized $I-$proximity relation on X. Then $A^{\delta_I} = \phi$.

Theorem 3.8. Let (X, δ_I) be a generalized $I-$proximity space. Then

$$A\delta_I B^{\delta_I} \Rightarrow A\delta_I B. \tag{5}$$

Proof. Let $A\delta_I B^{\delta_I}$ and $\forall\, y \in B^{\delta_I}$, we have $y\delta_I B$. Hence (IL_4) implies $A\delta_I B$.

Remark 3.9. The converse of Theorem 3.8 is not true. Consider X an infinite set, $I = I_f$ and δ_I is a generalized I-proximity relation defined as Example 3.5. If A, B are infinite subsets of X, then $B^{\delta_I} = \phi$ and hence $A \not\delta_I B^{\delta_I}$ but $B\delta_I A$.

Theorem 3.10. Let (X, δ_I) be a generalized I-proximity space. Then the operator

$$Cl^{\delta_I} : P(X) \to P(X)$$

defined by

$$Cl^{\delta_I}(A) = A \cup A^{\delta_I} \tag{6}$$

satisfies Kuratwski's axioms and induces a topology on X called τ_{δ_I} given by:

$$\tau_{\delta_I} = \{A \subseteq X : Cl^{\delta_I}(A^c) = A^c\} \tag{7}$$

Proof.

1. By Theorem 3.7 (1), we have $Cl^{\delta_I}(\phi) = \phi$.
2. formula (6) implies that $A \subseteq Cl^{\delta_I}(A)$.
3. By Theorem 3.7 (3), we have $Cl^{\delta_I}(A \cup B) = Cl^{\delta_I}(A) \cup Cl^{\delta_I}(B)$.
4. By Theorem 3.7 (2), we have

$$Cl^{\delta_I}(A) \subseteq Cl^{\delta_I}(Cl^{\delta_I}(A)). \tag{8}$$

So, it suffices to show that $\forall A \subseteq X$, we have $Cl^{\delta_I}(Cl^{\delta_I}(A)) \subseteq Cl^{\delta_I}(A)$ or equivalently that

$$if\, x \notin Cl^{\delta_I}(A),\ then\, x \notin Cl^{\delta_I}(Cl^{\delta_I}(A)). \tag{9}$$

Let $x \notin Cl^{\delta_I}(A)$. Hence $x \notin A$ and $x \not\delta_I A$. Theorem 3.8 implies that $x \not\delta_I A^{\delta_I}$ and (IP_2) implies that $x \not\delta_I (A \cup A^{\delta_I})$, i.e. $x \not\delta_I Cl^{\delta_I}(A)$. This result, combined with $x \not\delta_I A$ and formula (8), completes the proof.

4 Compatibility of Generalized I-Proximity Spaces

Definition 4.1. A topological space (X, τ) is compatible with the generalized I-Proximity relation δ_I, denoted $\tau \sim \delta_I$, if $\tau = \tau_{\delta_I}$.

Example 4.2. Let I be an ideal on a nonempty set X, (X, τ) be a topological space, and δ_I be a binary relation on $P(X)$ defined as:

$$A\delta_I B \Leftrightarrow A \cap \overline{B} \notin I. \tag{10}$$

Then δ_I is an I-Pervin Proximity relation on $P(X)$. Indeed, one easily sees that δ_I satisfies conditions (IL_2), (IL_3) and (IL_5). So, to check that δ_I also satisfies condition (IL_6), let $A \not\delta_I B$. It follows that $A \cap \overline{B} \in I$ and by taking $C = (\overline{B})^c$ and $D = \overline{B}$ have the required properties.

The following theorem shows that the topology generated by the formula (10) is finer than the topology (X, τ).

Theorem 4.3. Let (X,τ) be a topological space and δ_I is the formula (10). Then $\tau \subseteq \tau_{\delta_I}$.

Proof. To prove the theorem, we want to show that $Cl^{\delta_I}(A) \subseteq \overline{A}\ \forall A \subseteq X$. Let $x \in Cl^{\delta_I}(A)$. Then $x \in A$ or $x \in A^{\delta_I}$. If $x \in A$, hence the result. Now, let $x \in A^{\delta_I}$, then $x\delta_I A$ and hence $\{x\} \cap \overline{A} \notin I$. Consequently $x \in \overline{A}$. Then the result.

Example 4.4. Let I be an ideal on a nonempty set X, (X,τ) be a topological space, and δ_I be a binary relation on $P(X)$ defined as:

$$A\delta_I B \Leftrightarrow A \cap Cl^*(B) \notin I. \tag{11}$$

Then δ_I is an I-Pervin Proximity relation on X. Indeed, one easily sees that δ_I satisfies conditions (IL_2), (IL_3) and (IL_5). So, to check that δ_I also satisfies condition (IL_6), let $A \not\!\delta_I B$. It follows that $A \cap Cl^*(B) \in I$ and by taking $C = (Cl^*(B))^c$ and $D = Cl^*(B)$ have the required properties.

Theorem 4.5. Let (X,τ) be a topological space and δ_I is the formula (11). Then $\tau^* \subseteq \tau_{\delta_I}$.

Proof. To prove the theorem, it suffices to show that $Cl^{\delta_I}(A) \subseteq Cl^*(A)\ \forall A \subseteq X$. Let $x \in Cl^{\delta_I}(A)$. Then $x \in A$ or $x \in A^{\delta_I}$. If $x \in A$, hence the result. Now, if $x \in A^{\delta_I}$, then $x\delta_I A$ and hence $\{x\} \cap Cl^*(A) \notin I$. Consequently $x \in Cl^*(A)$. Hence $Cl^{\delta_I}(A) \subseteq Cl^*(A)$.

The following example shows that the topological space (X,τ) is not compatible,in general, with δ_I defined as Example 4.2. Also, (X,τ^*) is not compatible, in general, with δ_I defined as Example 4.4.

Example 4.6. Let (X,τ) be a cofinite topological space, $I = I_f$, $A = \mathbb{R}^+$, δ_I^1 is the formula (10), and δ_I^2 is the formula (11). It is clear that $\tau \neq \tau_{\delta_I^1}$ and $\tau^* \neq \tau_{\delta_I^2}$ as $\overline{A} = \mathbb{R}$, $Cl^{\delta_I^1}(A) = \mathbb{R}^+$, and $Cl^*(A) = \mathbb{R}$.

As a matter of fact, We got the idea of the following definition from [20].

Definition 4.7. A topological space (X,τ) is said to be

1. $*$-normal space if $\forall\, F_1, F_2 \in \tau^{*c}$ such that $F_1 \cap F_2 \in I$ then $\exists\, H, G \in \tau$ such that $F_1 \subseteq H$, $F_2 \subseteq G$ and $H \cap G \in I$.
2. IR_2-space if $\forall\, x \in X$, $F \in \tau^{*c}$ such that $\{x\} \cap F \in I$ then $\exists\, H, G \in \tau$ such that $x \in H$, $F \subseteq G$ and $H \cap G \in I$.

Theorem 4.8. Let I be an ideal on a nonempty set X, (X,τ) be an IR_2-topological space and δ_I is the formula (11). Then $\tau^* \sim \delta_I$ and δ_I is the smallest compatible $I-$Leader or $I-$Pervin proximity relation on $P(X)$.

Proof. Let $x \notin Cl^{\delta_I}(A)$. It follows that $x \notin A$ and $x \not\!\delta_I A$. Hence $\{x\} \cap Cl^*(A) \in I$. Since (X,τ) is IR_2-space, then $\exists\, H, G \in \tau$ such that

$$x \in H,\ Cl^*(A) \subseteq G \text{ and } H \cap G \in I \tag{12}$$

. By the definition of ideal part (2) and formula (12), we get $H \cap A \in I$, i.e. $\exists\, H \in \tau, x \in H$ such that $H \cap A \in I$. Hence $x \notin A^*$ and we have $x \notin A$. So, $x \notin Cl^*(A)$. It follows that

$$Cl^*(A) \subseteq Cl^{\delta_I}(A).$$

This result, combined with Theorem 4.5, implies $\tau_{\delta_I} = \tau^*$. Hence $\tau^* \sim \delta_I$. Finally, to prove that δ_I is the smallest compatible $I-$Pervin Proximity. Let α_I be another compatible $I-$Pervin Proximity and $A \not\!\alpha_I B$. Hence Theorem 3.8 implies $A \not\!\alpha_I Cl^{\alpha_I}(B)$ and (IL_5) implies $A \cap Cl^*(A) \in I$. Hence $A \not\!\delta_I B$, Hence the result.

Example 4.9. Let I be an ideal on a nonempty set X, (X,τ^*) be a topological R_o-space, and δ_I be a binary relation on $P(X)$ defined as:

$$A\delta_I B \Leftrightarrow Cl^*(A) \cap Cl^*(B) \notin I. \tag{13}$$

Then δ_I is an I-Lodato Proximity relation on $P(X)$. It follows directly from formula (13) that δ_I satisfies conditions (IL_1)-(IL_3) and (IL_5). So, to check that δ_I also satisfies condition (IL_4), let $A\delta_I B$ and $b\delta_I C\ \forall\ b \in B$. It follows that $Cl^*(A) \cap Cl^*(B) \notin I$ and $Cl^*(\{b\}) \cap Cl^*(C) \notin I$. Hence there exists a $c \in Cl^*(C)$ such that $c \in Cl^*(\{b\})$. Since (X,τ^*) is R_o-space, then $b \in Cl^*(\{c\}) \subseteq Cl^*(C)$, showing that $B \subseteq Cl^*(C)$. As a consequence, $Cl^*(A) \cap Cl^*(C) \notin I$, i.e. $A\delta_I C$. Then the result.

Theorem 4.10. Let I be an ideal on a nonempty set X, (X,τ) be a $*-$normal space, (X,τ^*) be a R_o-space, and δ_I is the formula (13). Then $\tau^* \sim \delta_I$.

Proof. To prove the theorem, it suffices to show that the topology generated by the closure operator Cl^* coincide with the topology generated by Cl^{δ_I}. In other words, we show that $\forall A \subseteq X$,

$$Cl^*(A) = Cl^{\delta_I}(A). \tag{14}$$

Let $x \in Cl^{\delta_I}(A)$, then $x \in A$ or $x \in A^{\delta_I}$. If $x \in A$, hence the result. Now, let $x \in A^{\delta_I}$, then $x\delta_I A$, and hence $Cl^*(\{x\}) \cap Cl^*(A) \notin I$. It follows that $\exists\, y \in Cl^*(A)$ such that $y \in Cl^*(\{x\})$. Since (X,τ^*) is R_o-space, then $x \in Cl^*(\{y\}) \subseteq Cl^*(A)$. Consequently, $x \in Cl^*(A)$. Hence

$$Cl^{\delta_I}(A) \subseteq Cl^*(A). \tag{15}$$

Now, we want to prove that $Cl^*(A) \subseteq Cl^{\delta_I}(A)$ or equivalently, if $x \notin Cl^{\delta_I}(A)$, then $x \notin Cl^*(A)$. Let $x \notin Cl^{\delta_I}(A)$, then $x \notin A$ and $x \notin A^{\delta_I}$. It follows that $x \not\!\delta_I A$ and hence formula (13) implies that $Cl^*(\{x\}) \cap Cl^*(A) \in I$. Since (X,τ) is $*-$normal space, then $\exists H, G \in \tau$ such that

$$Cl^*(\{x\}) \subseteq H, Cl^*(A) \subseteq G \text{ and } H \cap G \in I. \tag{16}$$

By the definition of ideal part (2) and formula (16), we get $H \cap A \in I$,.i.e $\exists H \in \tau, x \in H$ such that $H \cap A \in I$. Hence $x \notin A^*$and we have $x \notin A$. So, $x \notin Cl^*(A)$. It follows that

$$Cl^*(A) \subseteq Cl^{\delta_I}(A).$$

This result, combined with formula (15) and Definition 4.1, completes the proof of the theorem.

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