

# On Some New Retarded Nonlinear Integral Inequalities and Their Applications

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**Abstract:** In this paper we present some new generalized retarded nonlinear integral inequalities of Gronwall-Bellman type. Using integral and differential skills, some new results which provide explicit bounds on unknown functions in integral inequalities are investigated. Some applications are also presented to illustrate the usefulness of some of our results.

**Keywords:** Integral inequality, retarded integral and differential equation, boundedness.

## 1 Introduction

Gronwall inequality is an important tool in the study of existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of differential equations and integral equations see for instance [1–3, 5–7, 9, 11, 13, 14, 16, 17]. Many results on its generalization can be found for example in [2, 5–15]. However, in certain situations the bounds provided by the above mentioned inequalities are not directly applicable, and it's desirable to find some new estimates which will be equally important in order to achieve of desired goals; see [3, 4, 13–22]. The main purpose of this paper is to establish explicit bounds on retarded Gronwall-Bellman, Bihari and Pachpatte-like inequalities which can be used to study the qualitative behavior of the solutions of certain classes of retarded integral and differential equations. Some applications of some of our results are also given. Pachpatte in [15] investigated the retarded inequality

$$u(t) \leq k + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds, \forall t \in J, \quad (1)$$

where  $k$  is a constant. Replacing  $k$  by a nondecreasing continuous function  $f(t)$  in (1), Rashid in [16] studied the following retarded inequality

$$u(t) \leq f(t) + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds, \forall t \in J. \quad (2)$$

However, sometimes we need to study such inequalities with differentiable function in place of nondecreasing continuous function term  $f(t)$ . In this paper, some of our results concern with integral inequalities with such a differentiable function  $f(t)$ .

Throughout this paper,  $\mathbb{R}$  denoted the set of real numbers;  $\mathbb{R}_+ = [0, \infty)$ ,  $\mathbb{R}_+^* = (0, \infty)$ ,  $J = [a, b]$  is the subset of  $\mathbb{R}$ ,  $\prime$  denotes the derivative.  $\mathcal{C}(J, \mathbb{R}_+)$  denotes the set of all continuous functions from  $J$  into  $\mathbb{R}_+$  and  $\mathcal{C}^1(J, J)$  denotes the set of all continuously differentiable functions from  $J$  into  $J$ .

## 2 Main results

In this section, several new retarded integral inequalities of Gronwall-Bellman type are introduced.

**Theorem 2.1.** Let  $u(t)$ ,  $g(t)$ ,  $h(t) \in \mathcal{C}(J, \mathbb{R}_+)$ ,  $f(t) \in \mathcal{C}(J, \mathbb{R}_+^*)$ , be nondecreasing functions and  $p < 1$  is a constant. Suppose that  $\alpha(t) \in \mathcal{C}^1(J, J)$  be nondecreasing function with  $\alpha(t) \leq t$  on  $J$ . If the inequality

$$u(t) \leq f(t) + \int_a^t g(s)u^p(s)ds + \int_a^{\alpha(t)} h(s)u^p(s)ds, \quad (3)$$

holds for all  $t \in J$ , then

$$u(t) \leq f(t) \left[ 1 + (1-p)[\xi(t) + \eta(t)] \right]^{\frac{1}{1-p}}, \forall t \in J, \quad (4)$$

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where

$$\xi(t) = \int_a^t f^{p-1}(s)g(s)ds, \forall t \in J, \quad (5)$$

and

$$\eta(t) = \int_a^{\alpha(t)} f^{p-1}(s)h(s)ds, \forall t \in J. \quad (6)$$

**Proof.** Since  $f(t)$  is a positive and nondecreasing function, we observe from (3) that

$$\begin{aligned} \left[ \frac{u(t)}{f(t)} \right] &\leq 1 + \int_a^t g(s)f^{p-1}(s) \left[ \frac{u(s)}{f(s)} \right]^p ds \\ &+ \int_a^{\alpha(t)} h(s)f^{p-1}(s) \left[ \frac{u(s)}{f(s)} \right]^p ds, \forall t \in J. \end{aligned}$$

Let

$$r(t) = \frac{u(t)}{f(t)}, \forall t \in J, \quad r(0) \leq 1, \quad (7)$$

then

$$\begin{aligned} r(t) &\leq 1 + \int_a^t g(s)f^{p-1}(s)r^p(s)ds \\ &+ \int_a^{\alpha(t)} h(s)f^{p-1}(s)r^p(s)ds, \forall t \in J. \end{aligned}$$

Define a function  $z(t)$  by the right hand side of the above inequality, then we have

$$r(t) \leq z(t), \quad r(\alpha(t)) \leq z(\alpha(t)) \leq z(t), \quad z(a) = 1. \quad (8)$$

Differentiating  $z(t)$  with respect to  $t$  and using (8), we have

$$z^{-p}(t)z'(t) \leq g(t)f^{[p-1]}(t) + h(\alpha(t))f^{[p-1]}(\alpha(t))\alpha'(t),$$

for all  $t \in J$ . By taking  $t = s$  in the above inequality and integrating both sides from  $a$  to  $t$ , and making the change of the variable we get

$$z(t) \leq \left[ 1 + (1-p)[\xi(t) + \eta(t)] \right]^{\frac{1}{1-p}}, \forall t \in J. \quad (9)$$

where  $\xi(t)$  and  $\eta(t)$  are defined by (5) and (6) respectively. Therefore from (7), (8) and (9), we get the required inequality in (4). The proof is complete.

**Theorem 2.2.** Let  $u(t), g(t), h(t) \in \mathcal{C}(J, \mathbb{R}_+)$ , and  $f(t) \in \mathcal{C}(J, \mathbb{R}_+^*)$ ,  $\alpha(t) \in \mathcal{C}^1(J, J)$  be nondecreasing functions with  $\alpha(a) = a$ , and  $\alpha(t) \leq t$  on  $J$ . If the inequality

$$u^p(t) \leq f^p(t) + \int_a^t g(s)u^p(s)ds + \int_a^{\alpha(t)} h(s)u^q(s)ds, \forall t \in J, \quad (10)$$

holds, where  $p > q \geq 0$ , are constants. Then

$$u(t) \leq f(t)[\Theta(t)]^{\frac{1}{p-q}}, \forall t \in J, \quad (11)$$

where

$$\begin{aligned} \Theta(t) &= \exp\left(P_1 \int_a^t g(s)ds\right) \\ &\times \left[ 1 + P_1 \int_a^{\alpha(t)} h(s)f^{-[p-q]}(s) \exp\left(-P_1 \int_a^{\alpha^{-1}(s)} g(\lambda)d\lambda\right) ds \right], \end{aligned} \quad (12)$$

for all  $t \in J$ , where  $P_1 = \left[\frac{p-q}{p}\right]$ .

**Proof.** Since  $f(t)$  is a positive, monotonic, nondecreasing function, we observe from (10) that

$$\begin{aligned} \left[ \frac{u(t)}{f(t)} \right]^p &\leq 1 + \int_a^t g(s) \left[ \frac{u(s)}{f(s)} \right]^p ds \\ &+ \int_a^{\alpha(t)} h(s)f^{-[p-q]}(s) \left[ \frac{u(s)}{f(s)} \right]^q ds, \forall t \in J. \end{aligned}$$

Let

$$r_1(t) = \frac{u(t)}{f(t)}, \quad \forall t \in J, \quad r_1(a) \leq 1, \quad (13)$$

hence

$$r_1^p(t) \leq 1 + \int_a^t g(s)r_1^p(s)ds + \int_a^{\alpha(t)} h(s)f^{-[p-q]}(s)r_1^q(s)ds, \quad (14)$$

for all  $t \in J$ . Define a function  $\kappa(t)$  such that

$$\kappa^p(t) = 1 + \int_a^t g(s)r_1^p(s)ds + \int_a^{\alpha(t)} h(s)f^{-[p-q]}(s)r_1^q(s)ds, \quad (15)$$

for all  $t \in J$ , we can easily obtain

$$r_1(t) \leq \kappa(t), \quad r_1(\alpha(t)) \leq \kappa(\alpha(t)) \leq \kappa(t), \quad \kappa(a) = 1, \forall t \in J. \quad (16)$$

Differentiating (15) with respect to  $t$  and using (16), we have

$$\begin{aligned} p\kappa^{[p-1]}\kappa'(t) &\leq g(t)\kappa^p(t) \\ &+ h(\alpha(t))f^{-[p-q]}(\alpha(t))\alpha'(t)\kappa^q(t), \forall t \in J, \end{aligned}$$

but  $\kappa > 0$ . Thus, we have

$$p\kappa^{[p-q-1]}\kappa'(t) - g(t)\kappa^{[p-q]}(t) \leq h(\alpha(t))f^{-[p-q]}(\alpha(t))\alpha'(t), \quad (17)$$

if we let

$$\kappa^{[p-q]}(t) = Y(t), \quad (18)$$

then we have  $Y(a) = 1$ , and  $p\kappa^{[p-q-1]}\kappa'(t) = \left[\frac{1}{P_1}\right]Y'(t)$ , thus from (17) we obtain

$$Y' - P_1g(t)Y(t) \leq P_1h(\alpha(t))f^{-[p-q]}(\alpha(t))\alpha'(t), \forall t \in J.$$

The above inequality implies the estimation for  $Y(t)$  such that

$$Y(t) \leq \Theta(t), \forall t \in J,$$

where  $\Theta(t)$  as defined in (12). Then from the above inequality and (2.16) in (2.14), we have

$$r_1(t) \leq [\Theta(t)]^{\frac{1}{p-q}}, \forall t \in J.$$

The desired bound in (11) follows from the above inequality and (13). The proof is complete.

**Remark 2.1.** Theorem 2.2 gives the explicit estimation in Theorem 2.3 in [16] when  $p = 1$ .

**Theorem 2.3.** Let  $u(t), g(t), h(t) \in \mathcal{C}(J, \mathbb{R}_+)$ , and  $f(t) \in \mathcal{C}(J, \mathbb{R}_+^*)$ ,  $\alpha(t) \in \mathcal{C}^1(J, J)$  be nondecreasing functions with  $\alpha(a) = a, \alpha(t) \leq t$  on  $J$ . If the inequality

$$u(t) \leq f(t) + \int_a^{\alpha(t)} g(s)u(s)ds + \int_a^{\alpha(t)} g(s)u(s)[u(s) + \int_a^{\alpha(s)} h(\lambda)u(\lambda)d\lambda]ds, \quad (19)$$

holds for all  $t \in J$ . Then

$$u(t) \leq f(t) \exp\left(\int_a^{\alpha(t)} g(s)(1 + f(s)\Theta_1(s))ds\right), \forall t \in J, \quad (20)$$

where

$$\Theta_1(t) = \frac{\exp(\int_a^{\alpha(t)} [g(s) + h(s)]ds)}{1 - \int_a^{\alpha(t)} g(s)f(s) \exp(\int_a^s [g(\tau) + h(\tau)]d\tau)ds}, \quad (21)$$

for all  $t \in J$ , such that

$$\int_a^{\alpha(t)} g(s)f(s) \exp(\int_a^s [g(\tau) + h(\tau)]d\tau)ds < 1, \forall t \in J.$$

**Proof.** Since  $f(t)$  is a positive, monotonic, nondecreasing function, we observe from (19) that

$$\frac{u(t)}{f(t)} \leq 1 + \int_a^{\alpha(t)} g(s) \frac{u(s)}{f(s)} ds + \int_a^{\alpha(t)} g(s)f(s) \frac{u(s)}{f(s)} \left[ \frac{u(s)}{f(s)} + \int_a^{\alpha(s)} h(\lambda) \frac{u(\lambda)}{f(\lambda)} d\lambda \right] ds,$$

for all  $t \in J$ . Let

$$r_2(t) = \frac{u(t)}{f(t)}, \forall t \in J \quad r_2(a) \leq 1, \quad (22)$$

hence

$$r_2(t) \leq 1 + \int_a^{\alpha(t)} g(s)r_2(s)ds + \int_a^{\alpha(t)} g(s)f(s)r_2(s)[r_2(s) + \int_a^{\alpha(s)} h(\lambda)r_2(\lambda)d\lambda]ds, \forall t \in J,$$

for all  $t \in J$ . Let  $V(t)$  equal the right hand side in the above inequality, we have

$$r_2(t) \leq V(t), r_2(\alpha(t)) \leq V(\alpha(t)) \leq V(t), V(a) = 1, \forall t \in J. \quad (23)$$

Differentiating  $V(t)$  with respect to  $t$ , and using (23) we obtain

$$V'(t) \leq g(\alpha(t))\alpha'(t)V(t)[1 + f(\alpha(t))\gamma(t)], \forall t \in J, \quad (24)$$

where  $\gamma(t) = V(t) + \int_a^{\alpha(t)} h(s)V(s)ds$ , hence  $\gamma(a) = 1$ , and  $V(t) \leq \gamma(t)$ .

Differentiating  $\gamma(t)$  with respect to  $t$ , and using (24) we get

$$\gamma'(t) \leq [g(\alpha(t)) + h(\alpha(t))]\alpha'(t)\gamma(t) + g(\alpha(t))\alpha'(t)f(\alpha(t))\gamma^2(t), \forall t \in J,$$

but  $\gamma(t) > 0$ , thus from the above inequality we get

$$\gamma^{-2}(t)\gamma'(t) - [g(\alpha(t)) + h(\alpha(t))]\alpha'(t)\gamma^{-1}(t) \leq g(\alpha(t))\alpha'(t)f(\alpha(t)), \forall t \in J. \quad (25)$$

If we let

$$l(t) = \gamma^{-1}(t), \forall t \in J, \quad (26)$$

then we get  $l(a) = 1$  and  $\gamma^{-2}\gamma'(t) = -l'(t)$ , thus from (25) we have

$$l'(t) + [g(\alpha(t)) + h(\alpha(t))]\alpha'(t)l(t) \geq -g(\alpha(t))\alpha'(t)f(\alpha(t)).$$

The above inequality implies the estimation for  $l(t)$  such that

$$l(t) \geq \frac{1 - \int_a^{\alpha(t)} g(s)f(s) \exp(\int_a^s [g(\tau) + h(\tau)]d\tau)}{\exp(\int_a^{\alpha(t)} [g(s) + h(s)]ds)}, \forall t \in J.$$

Then from the above inequality in (26), we have

$$\gamma(t) \leq \Theta_1(t), \forall t \in J,$$

where  $\Theta_1(t)$  as defined in (21), thus from (24) and the above inequality we have

$$V'(t) \leq g(\alpha(t))\alpha'(t)V(t)[1 + f(t)\Theta_1(t)], \forall t \in J.$$

Integrating the above inequality from  $a$  to  $t$ , and making the change of variable yield

$$V(t) \leq \exp\left(\int_a^{\alpha(t)} g(s)(1 + f(s)\Theta_1(s))ds\right), \forall t \in J.$$

Using the above inequality and (23) in (22), we get the required inequality in (20). The proof is complete.

### 3 Further Inequalities

In this section, we present a number of more retarded nonlinear integral inequalities of Gronwall-Bellman, Bihari and Pachpatte-like, which are further generalizations for some known results and can be used as ready and powerful tools in developing the theory of nonlinear retarded differential and integral equations.

**Theorem 3.1.** Let  $u(t), g(t), h(t) \in \mathcal{C}(J, \mathbb{R}_+)$ ,  $f(t), \alpha_1(t), \alpha_2(t) \in \mathcal{C}^1(J, J)$  be nondecreasing functions with  $\alpha_i(a) = a$ , and  $\alpha_i(t) \leq t$  on  $J$  for  $i = 1, 2$ , and  $p > 1$  is a constant. Suppose that

$$u^p(t) \leq f^p(t) + \int_a^{\alpha_1(t)} g(s)u(s)ds + \int_a^{\alpha_2(t)} h(s)u(s)ds, \quad (27)$$

for all  $t \in J$ . If  $f(t) \geq 1, \forall t \in J$ , then

$$u(t) \leq \left[ f^{[p-1]}(a) + \left( \frac{p-1}{p} \right) [f^p(t) - f^p(a) + G(t) + H(t)] \right]^{\frac{1}{p-1}}, \forall t \in J, \quad (28)$$

where

$$G(t) = \int_a^{\alpha_1(t)} g(s)ds, \forall t \in J, \quad (29)$$

and

$$H(t) = \int_a^{\alpha_2(t)} h(s)ds, \forall t \in J. \quad (30)$$

**Proof.** Let  $V_1^p(t)$  equal the right hand side in (27), we have

$$u(t) \leq V_1(t); u(\alpha(t)) \leq V_1(\alpha(t)) \leq V_1(t); V_1(a) = f(a), \quad (31)$$

for all  $t \in J$ . Differentiating  $V_1^p(t)$  with respect to  $t$  and using (31), we obtain

$$pV_1^{[p-1]}V_1'(t) \leq pf^{[p-1]}(t)f'(t) + g(\alpha_1(t))\alpha_1'(t)V_1(t) + h(\alpha_2(t))\alpha_2'(t)V_1(t), \forall t \in J,$$

since  $V_1(t) > 0$ , we get

$$pV_1^{[p-2]}V_1'(t) \leq pf^{[p-1]}(t)\frac{f'(t)}{V_1(t)} + g(\alpha_1(t))\alpha_1'(t) + h(\alpha_2(t))\alpha_2'(t), \forall t \in J,$$

but  $f(t) \geq 1 \Rightarrow V_1(t) \geq 1 \Rightarrow \frac{f'(t)}{V_1(t)} \leq f'(t)$ , thus from the above inequality we get

$$pV_1^{[p-2]}V_1'(t) \leq pf^{[p-1]}(t)f'(t) + g(\alpha_1(t))\alpha_1'(t) + h(\alpha_2(t))\alpha_2'(t), \forall t \in J.$$

Integrating the above inequality from  $a$  to  $t$ , and making the change of variable yield

$$V_1(t) \leq \left[ f^{[p-1]}(a) + \left( \frac{p-1}{p} \right) [f^p(t) - f^p(a) + G(t) + H(t)] \right]^{\frac{1}{p-1}}, \forall t \in J,$$

where  $G(t)$  and  $H(t)$  as defined in (29) and (30) respectively. Using the above inequality in (31), we get the required inequality in (28). The proof is complete.

**Remark 3.1.** Theorem 3.1 gives the explicit estimation in Theorem 2.2 in [16] when  $\alpha_1(t) = 1$  and  $\alpha_2(t) = \alpha(t)$ .

**Theorem 3.2.** Let  $u(t), g(t), h(t) \in \mathcal{C}(J, \mathbb{R}_+)$ , and  $f(t), \alpha(t) \in \mathcal{C}^1(J, J)$  be nondecreasing functions with  $a \leq \alpha(t) \leq t$  on  $J$ .

(i) Suppose that

$$u(t) \leq f(t) + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds, \quad (32)$$

holds for all  $t \in J$ , then

$$u(t) \leq \Theta_2(t), \forall t \in J, \quad (33)$$

where

$$\Theta_2(t) = \exp \left( G_1(t) + H_1(t) \right) \times \left[ f(a) + \int_a^t f'(s) \exp \left( -[G_1(s) + H_1(s)] \right) ds \right], \quad (34)$$

for all  $t \in J$ , where  $G_1(t) = \int_a^t g(s)ds$ , and  $H_1(t) = \int_a^{\alpha(t)} g(s)ds, \forall t \in J$ .

(ii) Suppose that

$$u(t) \leq f(t) + \int_a^t g(s)u(s) \ln u(s)ds + \int_a^{\alpha(t)} h(s)u(s) \ln u(s)ds, \quad (35)$$

holds for all  $t \in J$ , then

$$u(t) \leq \exp \left( \Theta_2(t) \right), \forall t \in J, \quad (36)$$

where  $\Theta_2(t)$  as defined in (34).

**Proof. (i)** Let  $V_2(t)$  equal the right hand side in (32) we have

$$u(t) \leq V_2(t), u(\alpha(t)) \leq V_2(\alpha(t)) \leq V_2(t), V_2(a) = f(a), \quad (37)$$

for all  $t \in J$ . Differentiating  $V_2(t)$  with respect to  $t$  and using (37), we have

$$V_2'(t) - [g(t) + h(\alpha(t))\alpha'(t)]V_2(t) \leq f'(t), \forall t \in J.$$

The above inequality implies the estimation for  $V_2(t)$  such that

$$V_2(t) \leq \Theta_2(t), \forall t \in J,$$

where  $\Theta_2(t)$  as defined in (34), then from the above inequality in (37) we obtain the required inequality in (3). The proof is complete.

**(ii)** Let a function  $V_3(t)$  equal the right hand side of (35), then  $V_3(a) = f(a)$ , and  $u(t) \leq V_3(t)$ , and as in the proof of (i) we obtain

$$\frac{V_3'(t)}{V_3(t)} \leq \frac{f'(t)}{f(t)} + g(t) \ln V_3(t) + h(\alpha(t)) \ln V_3(t) \alpha'(t), \forall t \in J,$$

but  $f(t) \geq 1, \Rightarrow V_3(t) \geq 1, \Rightarrow \frac{f'(t)}{V_3(t)} \leq f'(t)$ , thus from the above inequality we get

$$\frac{V_3'(t)}{V_3(t)} \leq f'(t) + g(t) \ln V_3(t) + h(\alpha(t)) \ln V_3(t) \alpha'(t), \forall t \in J.$$

Integrating the above inequality from  $a$  to  $t$ , and making the change of variable yield

$$\ln V_3(t) \leq \left( \ln f(a) + f(t) - f(a) \right) + \int_a^t g(s) \ln V_3(s) ds + \int_a^{\alpha(t)} h(s) \ln V_3(s) ds, \forall t \in J. \tag{38}$$

Now by a suitable application of the inequality given in (i) to (38), we have

$$\ln V_3(t) \leq \Theta_2(t), \forall t \in J,$$

where  $\Theta_2(t)$  as defined in (34), then from the above inequality we get

$$V_3(t) \leq \exp\left(\Theta_2(t)\right), \forall t \in J.$$

Using the above inequality in  $u(t) \leq V_3(t)$ , we get the required inequality in (36). The proof is complete.

**Remark 3.2.** Theorem 3.2 (i) and (ii) gives the explicit estimations in  $(a_1)$  and  $(a_2)$  in Theorem 2.3 in [15] respectively when  $f(t)$  is a constant function.

**Theorem 3.3.** Let  $u(t), g(t), h(t) \in \mathcal{C}(J, \mathbb{R}_+)$  and  $f(t), \alpha_1(t)$  and  $\alpha_2(t) \in \mathcal{C}^1(J, J)$  be nondecreasing functions with  $\alpha_i(a) = a, i = 1, 2$ , and  $a \leq \alpha_i(t) \leq t$  on  $J$ , let  $w_i \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$  be nondecreasing functions with  $w_i(u) \geq 1$  for  $u \geq 1, i = 1, 2$ .

(i) Suppose that  $f(t) \geq 1, \forall t \in J$  and

$$u(t) \leq f(t) + \int_a^{\alpha_1(t)} g(s) w_1(u(s)) ds + \int_a^{\alpha_2(t)} h(s) w_2(u(s)) ds, \forall t \in J, \tag{39}$$

holds then for  $a \leq t \leq t_1$

$$u(t) \leq \begin{cases} W_2^{-1}[W_2(f(a)) + G(t) + H(t) + f(t) - f(a)] & \text{if } w_1(u) \leq w_2(u) \\ W_1^{-1}[W_1(f(a)) + G(t) + H(t) + f(t) - f(a)] & \text{if } w_2(u) \leq w_1(u). \end{cases} \tag{40}$$

where  $G(t)$  and  $H(t)$  as defined in (29) and (30) respectively. and for  $i = 1, 2, W_i^{-1}$  are the inverse functions of

$$W_i(\varepsilon) = \int_{\varepsilon_0}^{\varepsilon} \frac{ds}{w_i(s)}, \varepsilon_0 > 0, \varepsilon > 0, \tag{41}$$

and  $t_1 \in J$  is chosen so that  $W_i(f(a)) + G(t) + H(t) + f(t) - f(a) \in \text{Dom}(W_i^{-1})$ , for  $i = 1, 2$  respectively, for all  $t \in [a, t_1]$ .

(ii) Suppose that

$$u(t) \leq f(t) + \int_a^{\alpha_1(t)} g(s) u(s) w_1(\ln u(s)) ds + \int_a^{\alpha_2(t)} h(s) u(s) w_2(\ln u(s)) ds, \tag{42}$$

holds for all  $t \in J$ , then for  $a \leq t \leq t_2$

$$u(t) \leq \begin{cases} \exp(W_2^{-1}[W_2(\ln f(a)) + G(t) + H(t) + f(t) - f(a)]) & \text{if } w_1(u) \leq w_2(u) \\ \exp(W_1^{-1}[W_1(\ln f(a)) + G(t) + H(t) + f(t) - f(a)]) & \text{if } w_2(u) \leq w_1(u). \end{cases} \tag{43}$$

where  $W_i, W_i^{-1}, G(t), H(t)$  are as in (i) and  $t_2$  chosen so that

$$W_i(\ln f(a) + G(t) + H(t) + f(t) - f(a)) \in \text{Dom}(W_i^{-1}),$$

for  $i = 1, 2$ , respectively, for all  $t \in [a, t_2]$ .

**Proof. (i)** Let  $V_4(t)$  equal the right hand side in (39) we have

$$u(t) \leq V_4(t), u(\alpha(t)) \leq V_4(\alpha(t)) \leq V_4(t), V_4(a) = f(a); \tag{44}$$

for all  $t \in J$ . Differentiating  $V_4(t)$  with respect to  $t$  and using (44), leads to

$$V_4'(t) \leq f'(t) + g(\alpha_1(t)) w_2(V_4(t)) \alpha_1'(t) + h(\alpha_2(t)) w_1(V_4(t)) \alpha_2'(t), \forall t \in J.$$

In case  $w_1(u) \leq w_2(u)$ , then from the above inequality we have

$$V_4'(t) \leq f'(t) + w_2(V_4(t)) [g(\alpha_1(t)) \alpha_1'(t) + h(\alpha_2(t)) \alpha_2'(t)],$$

for all  $t \in J$ , but  $w_2(V_4(t)) > 0$ , thus we get

$$\frac{V_4'(t)}{w_2(V_4(t))} \leq \frac{f'(t)}{w_2(V_4(t))} + g(\alpha_1(t)) \alpha_1'(t) + h(\alpha_2(t)) \alpha_2'(t),$$

for all  $t \in J$ , but  $f(t) \geq 1 \Rightarrow V_4(t) \geq 1 \Rightarrow w_2(V_4(t)) \geq 1 \Rightarrow \frac{f'(t)}{w_2(V_4(t))} \leq f'(t)$ , thus from (41) we have

$$\frac{d}{dt} W_2(V_4(t)) = \frac{V_4'(t)}{w_2(V_4(t))} \leq f'(t) + g(\alpha_1(t)) \alpha_1'(t) + h(\alpha_2(t)) \alpha_2'(t).$$

Integrating the above inequality from  $a$  to  $t$  and making the change of variable we have

$$W_2[V_4(t)] \leq W_2(f(a)) + G(t) + H(t) + f(t) - f(a).$$

Using the above inequality in (44) gives the required inequality (40).

The proof of the case  $w_2(t) \leq w_1(t)$  can be completed similarly.

(ii) The proof of the inequality in this case can be completed by following the proof of the inequality in (i) in this theorem and the case (ii) in the Theorem 3.2. The proof is complete.

**Remark 3.3.** Theorem 3.3 (i) and (ii) gives the explicit estimations in  $(b_1)$  and  $(b_2)$  in Theorem 2 in [15] respectively when  $f(t)$  is a constant function,  $\alpha_1(t) = t$  and  $\alpha_2(t) = \alpha(t)$ .

**Remark 3.4.** Theorem 3.3 (i) gives the explicit estimation in Theorem 2.4 in [16] when  $\alpha_1(t) = t$  and  $\alpha_2(t) = \alpha(t)$ .

**Theorem 3.4.** Let  $u(t), h(t) \in \mathcal{C}(J, \mathbb{R}_+)$ ,  $\alpha(t), f(t) \in \mathcal{C}^1(J, J)$  be nondecreasing functions, with  $\alpha(t) \leq t$ ,  $\alpha(a) = a$ ,  $\alpha'(t) \geq 0$ , let  $w \in (\mathbb{R}_+, \mathbb{R}_+^*)$  nondecreasing function with  $w(u) \geq 1$  for  $u \geq 1$ , and  $k(t, s) \in \mathcal{C}(J \times J, \mathbb{R}_+)$  with  $\frac{\partial k}{\partial t}(t, s) \in \mathcal{C}(J \times J, \mathbb{R}_+)$ . Suppose that  $f(t) \geq 1$  and

$$u(t) \leq f(t) + \int_a^{\alpha(t)} h(s)w(u(s))ds + \int_a^{\alpha(t)} k(t, s)w(u(s))ds, \forall t \in J, \quad (45)$$

then

$$u(t) \leq W^{-1}[W(f(a)) + H_2(t) + f(t) - f(a) + \int_a^t F(s)ds], \quad (46)$$

for all  $t \in J$ , holds for all values of  $t_3$  for which

$$[W(f(a)) + H_2(t) + f(t) - f(a) + \int_a^t F(s)ds] \in \text{Dom}(W^{-1}),$$

for all  $t \in [a, t_3]$ , where

$$H_2(t) = \int_a^{\alpha(t)} [h(s) + k(t, s)]ds, \forall t \in J; \quad (47)$$

$$F(t) = \int_a^{\alpha(t)} \frac{\partial k}{\partial t}(t, s)ds, \forall t \in J; \quad (48)$$

$$W(r) = \int_{r_0}^r \frac{ds}{w(s)}, r_0 > 0, r > 0. \quad (49)$$

**Proof.** Let  $V_5(t)$  equal the right hand side in (45) we have

$$u(t) \leq V_5(t), u(\alpha(t)) \leq V_5(\alpha(t)) \leq V_5(t), V_5(a) = f(a), \quad (50)$$

for all  $t \in J$ . Differentiating  $V_5(t)$  with respect to  $t$  and using (50), we have

$$V_5'(t) \leq f'(t) + h(\alpha(t))\alpha'(t)w(V_5(t)) + \int_a^{\alpha(t)} \frac{\partial k}{\partial t}(t, s)w(V_5(t))ds + k(t, \alpha(t))w(V_5(t))\alpha'(t), \forall t \in J, \quad (51)$$

but  $w(V_5(t)) > 0$ , thus (51) written as

$$\frac{V_5'(t)}{w(V_5(t))} \leq \frac{f'(t)}{w(V_5(t))} + h(\alpha(t))\alpha'(t) + \int_a^{\alpha(t)} \frac{\partial k}{\partial t}(t, s)ds + k(t, \alpha(t))\alpha'(t), \forall t \in J,$$

but  $f(t) \geq 1 \Rightarrow V_5(t) \geq 1 \Rightarrow w(V_5(t)) \geq 1 \Rightarrow \frac{f'(t)}{w(V_5(t))} \leq f'(t)$ , thus from (49) we get

$$\begin{aligned} \frac{d}{dt}[W(V_5(t))] &= \frac{V_5'(t)}{w(V_5(t))} \\ &\leq f'(t) + h(\alpha(t))\alpha'(t) + \int_a^{\alpha(t)} \frac{\partial k}{\partial t}(t, s)ds \\ &\quad + k(t, \alpha(t))\alpha'(t), \forall t \in J. \end{aligned}$$

Integrating the above inequality from  $a$  to  $t$  and making the change of variable yield

$$V_5(t) \leq W^{-1}\left(W(f(a)) + H_2(t) + f(t) - f(a) + \int_a^t F(s)ds\right), \forall t \in J,$$

where  $H_2(t)$  and  $F(t)$  as defined in (47) and (48) respectively. Using the above inequality in (50) we get the result (46). The proof is complete.

## 4 Applications

In this section we apply our Theorems 3.4 and 2.3 to the following integral equation in the Corollary 4.1 and retarded integral equation in the Example 4.1 respectively, as follows: consider the integral equation

$$u(t) = y(t) + \int_a^t \Phi(s, u(\alpha(s)), A(t, s))ds, \forall t \in J, \quad (52)$$

where  $\Phi \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ , satisfy the following conditions:

$$|y(t)| \leq f(t), \forall t \in J; \quad (53)$$

$$|\Phi(t, u(\alpha(t)), k(t, s))| \leq h(\alpha(t))w(|u(\alpha(t))|) + k(t, \alpha(s))w(|u(\alpha(t))|), \quad (54)$$

$$W^{-1}(W(f(a)) + MH_2(t) + f(t) - f(a) + M \int_a^t F(s)ds) < \infty, \quad (55)$$

where  $H_2(t), F(t)$  as defined in Theorem 3.4,  $h(t), f(t), w(t) \in \mathcal{C}(J, \mathbb{R}_+)$  and

$$M = \max \frac{1}{\alpha'(t)}, \forall t \in J, \quad (56)$$

**Corollary 4.1.** Consider the nonlinear integral equation (52) and suppose that  $y, \Phi$  satisfy the conditions (53) and (54), and  $\alpha(t) \in \mathcal{C}^1(J, J)$  with  $\alpha(t) \leq t$ ,  $\alpha(a) = a$ ,  $w(u) \geq 1$ , for  $u \geq 1$  and  $k(t, s) \in \mathcal{C}(J \times J, \mathbb{R}_+)$  with  $\frac{\partial k}{\partial t}(t, s) \in \mathcal{C}(J \times J, \mathbb{R}_+)$ , for all  $t \in J$ . If  $f(t) \geq 1$  then all solution  $u(t)$  of the equation (52) exist on  $J$ , bounded and satisfy the following estimation:

$$|u(t)| \leq W^{-1}(W(f(a)) + MH_2(t) + f(t) - f(a) + M \int_a^t F(s)ds), \forall t \in J, \quad (57)$$

where  $M$  and  $W$  as defined in (56) and (49) respectively.

**proof.** Suppose that the hypotheses (53), (54) are satisfied, and let  $u(t)$ , be a solution of (52). Then from (52), (53) and (54), we get

$$|u(t)| \leq f(t) + \int_a^t h(\alpha(s))g(|u(\alpha(s))|)ds + \int_a^t k(t, \alpha(s))g(|u(\alpha(s))|)ds, \forall t \in J.$$

by making the change of variable for the above inequality we get

$$|u(t)| \leq f(t) + M \int_a^{\alpha(t)} h(s)w(|u(s)|)ds + M \int_a^{\alpha(t)} k(t,s)w(|u(s)|)ds, \forall t \in J.$$

Applying Theorem 3.4 to the above inequality, we get the estimation (57). Thus from the hypothesis (55) and the estimation in (57) implies the boundedness of the solution of (52). The proof is complete.

**Example 4.1.** Consider the retarded integral equation:

$$u(t) = y(t) + \int_a^{\alpha(t)} B(s, u(s))ds + \int_a^{\alpha(t)} A(s, u(s), B(s), \int_a^s D(\tau, u(\tau))d\tau)ds, \quad (58)$$

for all  $t \in J$ . Assume that

$$\begin{cases} |y(t)| \leq f(t); \\ |B(t, u(t))| \leq g(t)|u(t)|; \\ |D(t, u(t))| \leq h(t)|u(t)|; \end{cases} \quad (59)$$

$$|A(s, u(t), B(t), \int_a^t D(s, u(s))ds)| \leq |B(t, u(t))||u(t)| + \int_a^t |D(s, u(s))|ds; \quad (60)$$

$$f(t) \exp(\int_a^{\alpha(t)} g(s)(1 + f(s)\Theta_1(s))ds) < \infty, \quad (61)$$

for all  $t \in J$  where  $f(t), h(t), g(t), \alpha(t)$  and  $u(t)$  are as defined in Theorem 2.3, from (58), (59) and (60) we obtain

$$|u(t)| \leq f(t) + \int_a^{\alpha(t)} g(s)|u(s)|ds + \int_a^{\alpha(t)} g(s)|u(s)||u(s)| + \int_a^{\alpha(s)} h(\tau)|u(\tau)|d\tau ds, \forall t \in J, \quad (62)$$

for all  $t \in J$ . By Theorem 2.3 we get an explicit bound on an unknown function  $|u(t)|$  such that

$$|u(t)| \leq f(t) \exp(\int_a^{\alpha(t)} g(s)(1 + f(s)\Theta_1(s))ds), \forall t \in J, \quad (63)$$

where  $\Theta_1(t)$  as defined in (21). Thus from the hypotheses (61) and the estimation in (63) implies the boundedness of the solution of (58). The proof is complete.

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