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Some Spectral Properties of Direct Integral of Operators

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Abstract: In this work, a connection between some spectral properties of direct integral of operators in the direct integral of Hilbert spaces and their coordinate operators has been investigated.

Keywords: Direct integral of Hilbert spaces and operators; spectrum and resolvent sets; compact operators; Schatten-von Neumann operator classes; power and polynomially bounded operators.

1 Introduction

It is known that the general theory of linear closed operators in Hilbert spaces and its applications to physical problems have been investigated by many researchers (for example, see [1,2]). But many physical problems require studying the theory of linear operators in direct sums or general direct integrals of Hilbert spaces. The concepts of direct integral of Hilbert spaces and direct integral of operators as a generalization of the concept of direct sum of Hilbert spaces and direct sum of operators were introduced to mathematics and developed in 1949 by John von Neumann [3]. These subjects were incorporated in several works (see [4,5,6,7]). A spectral theory of some operators on a finite sum of Hilbert spaces was investigated by N. Dunford [8,9]. Note that, in terms of application, there are some results in papers [10,11,12] for the finite sum cases. Also, for the infinite direct sum cases the spectral and compactness properties are surveyed in [13]. Furthermore, some spectral investigations of the direct integral of operators in the direct integral of Hilbert spaces have been provided by T.R. Chow [14], T.R. Chow, F. Gilfeather [15], E.A. Azoff [16,17], and L.A. Fialkow [18]. It must be noted that the theory of direct integral of Hilbert spaces and operators on the these spaces has important role in the representation theory of locally compact groups, the theory of decomposition rings of operators to factors, invariant measures, reduction theory, on Neumann algebras and etc. On the other hand, many physical problems of today arising in the modelling of processes of multiparticle quantum mechanics, quantum field theory and the physics of rigid bodies require to study a theory of direct integral operators in the direct integral of Hilbert spaces (see [19] and references in it).

Numerical scientific investigations have been done for explain of the quantum measurements. Dealing with these subjects, S. Machida and M. Namiki [20,21,22] also [23] and [24]) have offered many-Hilbert-space theory (or continuous superselection-rule space method) lately. Note that a direct integral space of continuously many Hilbert spaces often arises in the quantum version of Lax-Phillips theory [25].

In second section of this paper connections between spectrum, resolvent set of direct integral of operators in the direct integral of Hilbert spaces and its coordinate operators are established. Note that the another approach to analogous problem has been used in the work [14]. In this paper sharp formulas for the connections are given.

In third section these connections are researched for compactness properties. Finally, in special case the analogous questions for the power and polynomially bounded operators are researched. Note that, these questions for the direct sum case of Hilbert spaces have been investigated in [13] and [28].

Along this paper the triplet (Λ,Σ,μ) is a measure space and all Hilbert spaces are infinite dimensional. In addition, the space of compact operators and Schatten-von Neumann classes in any Hilbert space will be denoted by $C_\infty(\cdot)$ and $C_p(\cdot), 1\leq p<\infty$ respectively. On the other hand $\sigma_p(\cdot),\sigma_c(\cdot),\sigma_r(\cdot),\sigma(\cdot),\rho(\cdot)$ and

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$R_\tau(\cdot), \tau \in \rho(\cdot)$ will be called point spectrum, continuous spectrum, residual spectrum, spectrum, resolvent set of an operator and resolvent operator respectively.

2 On the Spectrum of Direct Integral of Operators

In this section, the relationship between the spectrum and resolvent sets of the direct integral of operators and its coordinate operators will be investigated.

Before of all prove the following result.

Theorem 2.1. Let H_λ be a Hilbert space, $A_\lambda \in B(H_\lambda)$ for any $\lambda \in \Lambda$, $H = \int\limits_{\Lambda}^{\oplus} H_\lambda d\mu(\lambda)$ and $A = \int\limits_{\Lambda}^{\oplus} A_\lambda d\mu(\lambda)$. In this case the following relations are true

$$\sigma_p(A) \subset \bigcup_{\lambda\in\Lambda} \sigma_p(A_\lambda)$$

$$\{\tau \in \bigcap_{\lambda\in\Lambda} \sigma_p(A_\lambda) : A_\lambda x_\lambda^\tau = \tau x_\lambda^\tau, \|x_\lambda^\tau\| \in L^2(\Lambda)\} \subset \sigma_p(A)$$

Proof. For any $\tau \in \sigma_p(A)$ there exist element $x = (x_\lambda) \in D(A), \lambda \in \Lambda$ such that $x \neq 0$ and $Ax = \tau x$. Then almost everywhere $\lambda \in \Lambda$ with respect to measure μ it is true that $A_\lambda x_\lambda = \tau x_\lambda$. Since $x \neq 0$, then there exist $\lambda_\star \in \Lambda$ which satisfy the above equality and $x_{\lambda_\star} \in D(A_{\lambda_\star}), x_{\lambda_\star} \neq 0$. This means that $\tau \in \sigma_p(A_{\lambda_\star})$. Hence

$$\tau \in \bigcup_{\lambda\in\Lambda} \sigma_p(A_\lambda)$$

From this it is obtained that

$$\sigma_p(A) \subset \bigcup_{\lambda\in\Lambda} \sigma_p(A_\lambda)$$

The proof of the second proposition is clear.

Actually, in one special case the following stronger assertions are true.

Theorem 2.2. Assume that every one-point set is measurable and its measure is positive. Let H_λ be a Hilbert space, $A_\lambda \in B(H_\lambda)$ for any $\lambda \in \Lambda$, $H = \int\limits_{\Lambda}^{\oplus} H_\lambda d\mu(\lambda)$ and $A = \int\limits_{\Lambda}^{\oplus} A_\lambda d\mu(\lambda)$. In this case for the parts of spectrum and resolvent sets of the operator A the following claims are true

$$\sigma_p(A) = \bigcup_{\lambda\in\Lambda} \sigma_p(A_\lambda)$$

$$\sigma_c(A) = \left\{ \left(\bigcap_{\lambda\in\Lambda} (\sigma_c(A_\lambda) \cup \rho(A_\lambda)) \right) \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_c(A_\lambda) \right) \right\} \cup \left\{ \tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) : \sup \|R_\tau(A_\lambda)\| = \infty \right\}$$

$$\sigma_r(A) = \left(\bigcap_{\lambda\in\Lambda} (\sigma_c(A_\lambda) \cup \sigma_r(A_\lambda) \cup \rho(A_\lambda)) \right) \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_r(A_\lambda) \right)$$

$$\rho(A) = \left\{ \tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) : \sup \|R_\tau(A_\lambda)\| < \infty \right\}$$

Proof. Firstly let us prove the first relation of the theorem. Assumed that $\tau \in \sigma_p(A)$. Then there exist $x = (x_\lambda^\tau) \neq 0, (x_\lambda^\tau) \in D(A)$ such that $Ax = \tau x$. So for every $\lambda \in \Lambda$

$$A_\lambda x_\lambda^\tau = \tau x_\lambda^\tau, \quad x_\lambda^\tau \in D(A_\lambda)$$

and $x_{\lambda_\star}^\tau \neq 0$ for some $\lambda_\star \in \Lambda$. Hence $\tau \in \sigma_p(A_{\lambda_\star})$ and from this

$$\tau \in \bigcup_{\lambda\in\Lambda} \sigma_p(A_\lambda)$$

On the contrary, assumed that $\tau \in \bigcup\limits_{\lambda\in\Lambda} \sigma_p(A_\lambda)$. Then for at least one index $\lambda_\star \in \Lambda$ it is hold that $\tau \in \sigma_p(A_{\lambda_\star})$, i.e. for some $x_{\lambda_\star}^\tau \neq 0$, $x_{\lambda_\star}^\tau \in D(A_{\lambda_\star})$ it is true that $A_{\lambda_\star} x_{\lambda_\star}^\tau = \tau x_{\lambda_\star}^\tau$. In this case we have $Ax = \tau x$ for the element $x = (x_\lambda) \neq 0, (x_\lambda) \in D(A), \lambda \neq \lambda_\star, x_\lambda = 0$ and $x_{\lambda_\star} = x_{\lambda_\star}^\tau$.

Now we prove the second relation on the continuous spectrum. Let $\tau \in \sigma_c(A)$. In this case by the definition of continuous spectrum $A - \tau E$ is a one-to-one operator, $R(A - \tau E) \neq H$ and $R(A - \tau E)$ is dense in H. From this and definition of direct integral it implies that for every $\lambda \in \Lambda$ operator $A_\lambda - \tau E_\lambda$ is a one-to-one operator in H_λ , $R(A_\lambda - \tau E_\lambda)$ is dense in H_λ and $R(A_{\lambda_\star} - \tau E_{\lambda_\star}) \neq H_{\lambda_\star}$ for at least one $\lambda_\star \in \Lambda$ or $\tau \in \rho(A_\lambda)$ for every $\lambda \in \Lambda$, but $\sup \|R_\tau(A_\lambda)\| = \infty$. Hence

$$\tau \in \left(\bigcap_{\lambda\in\Lambda} (\sigma_c(A_\lambda) \cup \rho(A_\lambda)) \right) \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_c(A_\lambda) \right)$$

or

$$\tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) \quad \text{and} \quad \sup \|R_\tau(A_\lambda)\| = \infty$$

This means that

$$\sigma_c(A) \subset \left\{ \left(\bigcap_{\lambda\in\Lambda} (\sigma_c(A_\lambda) \cup \rho(A_\lambda)) \right) \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_c(A_\lambda) \right) \right\} \cup \left\{ \tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) : \sup \|R_\tau(A_\lambda)\| = \infty \right\}$$

On the contrary, suppose that

$$\tau \in \left\{ \left(\bigcap_{\lambda\in\Lambda} (\sigma_c(A_\lambda) \cup \rho(A_\lambda)) \right) \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_c(A_\lambda) \right) \right\} \cup \left\{ \tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) : \sup \|R_\tau(A_\lambda)\| = \infty \right\}$$

In this case $\tau \in \sigma_c(A_\lambda) \cup \rho(A_\lambda)$ for every $\lambda \in \Lambda$ and $\tau \in \sigma_c(A_{\lambda_\star})$ for at least one $\lambda_\star \in \Lambda$ or $\tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) : \sup \|R_\tau(A_\lambda)\| = \infty$ for every $\lambda \in \Lambda$. This means that for every $\lambda \in \Lambda$ operator $A_\lambda - \tau E_\lambda$ is a one-to-one operator in H_λ ,$R(A_\lambda - \tau E_\lambda)$ is dense in H_λ and $R(A_{\lambda_\star} - \tau E_{\lambda_\star}) \neq H_{\lambda_\star}$. From this $A - \tau E = \int_\Lambda^\oplus (A_\lambda - \tau E_\lambda) d\mu(\lambda)$ is a one-to-one operator, $R(A - \tau E) \neq H$ and $R(A - \tau E)$ is dense in H. Hence, $\tau \in \sigma_c(A)$. Moreover, when $\tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda)$ such that $\sup \|R_\tau(A_\lambda)\| = \infty$ for every $\lambda \in \Lambda$, it is clear that $\tau \in \sigma_c(A)$. This completes proof of second relation.

Now third relation of theorem will be proved. Let $\tau \in \sigma_r(A)$. In this case by the definition of residual spectrum $A - \tau E$ is a one-to-one operator and $\overline{R(A - \tau E)} \neq H$. From this for every $\lambda \in \Lambda$ an operator $A_\lambda - \tau E_\lambda$ is a one-to-one operator in H_λ and there exist at least one $\lambda_\star \in \Lambda$ such that $\overline{R(A_{\lambda_\star} - \tau E_{\lambda_\star}} \neq H_{\lambda_\star}$. Hence

$$\tau \in \left(\bigcap_{\lambda\in\Lambda} (\sigma_c(A_\lambda) \cup \sigma_r(A_\lambda) \cup \rho(A_\lambda)) \right) \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_r(A_\lambda) \right)$$

This means that

$$\sigma_r(A) \subset \left(\bigcap_{\lambda\in\Lambda} (\sigma_c(A_\lambda) \cup \sigma_r(A_\lambda) \cup \rho(A_\lambda)) \right) \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_r(A_\lambda) \right)$$

It is easy to prove the inverse implication.

Finally, let us prove the fourth claim of the theorem. Let $\tau \in \rho(A)$. In this case, $A - \tau E$ is a one-to-one operator, $R(A - \tau E) = H$ and $(A - \tau E)^{-1} \in B(H)$. From this for every $\lambda \in \Lambda$ operator $A_\lambda - \tau E_\lambda$ is a one-to-one operator in H_λ ,$R(A_\lambda - \tau E_\lambda) = H_\lambda$ and $(A_\lambda - \tau E_\lambda)^{-1} \in B(H_\lambda)$. This means that $\tau \in \rho(A_\lambda)$ for every $\lambda \in \Lambda$. Then $\tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda)$. Moreover, since $(A - \tau E)^{-1} = \int_\Lambda^\oplus (A_\lambda - \tau E_\lambda)^{-1} d\mu(\lambda) : \int_\Lambda^\oplus H_\lambda d\mu(\lambda) \longrightarrow \int_\Lambda^\oplus H_\lambda d\mu(\lambda)$ and $(A - \tau E)^{-1} \in B(H)$, then

$$\left\|(A - \tau E)^{-1}\right\| = \sup \left\|(A_\lambda - \tau E_\lambda)^{-1}\right\| < \infty$$

This means that

$$\|R_\tau(A)\| = \sup \|R_\tau(A_\lambda)\| < \infty$$

From this

$$\rho(A) \subset \left\{ \tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) : \sup \|R_\tau(A_\lambda)\| < \infty \right\}$$

. It is easy to prove the inverse of this relation. Consequently it is obtained that

$$\rho(A) = \left\{ \tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) : \sup \|R_\tau(A_\lambda)\| < \infty \right\}$$

On the other hand the simple calculations show that the following relations are true.

Corollary 2.3. Under the assumptions of last theorem we have

$$\sigma_c(A) = \left\{ \left[\left(\bigcup_{\lambda\in\Lambda} \sigma_p(A_\lambda) \right)^c \cup \left(\bigcup_{\lambda\in\Lambda} \sigma_r(A_\lambda) \right)^c \right] \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_c(A_\lambda) \right) \right\} \cup \left\{ \tau \in \bigcap_{\lambda\in\Lambda} \rho(A_\lambda) : \sup \|R_\tau(A_\lambda)\| = \infty \right\}$$

$$\sigma_r(A) = \left(\bigcup_{\lambda\in\Lambda} \sigma_p(A_\lambda \right)^c \cap \left(\bigcup_{\lambda\in\Lambda} \sigma_r(A_\lambda) \right)$$

Corollary 2.4. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}, n \leq \infty$ be any countable set, $\Sigma = P(\Lambda)$ and μ be any measure with property $\mu(\{\lambda\}) > 0$ for every point $\lambda \in \Lambda$. In this case the formulas

$$\sigma_p(A) = \bigcup_{m=1}^n \sigma_p(A_{\lambda_m})$$

$$\sigma_c(A) = \left\{ \left(\bigcap_{m=1}^n (\sigma_c(A_{\lambda_m}) \cup \rho(A_{\lambda_m})) \right) \cap \left(\bigcup_{m=1}^n \sigma_c(A_{\lambda_m}) \right) \right\} \cup \left(\bigcap_{m=1}^n \left\{ \tau \in \bigcap_{m=1}^n \rho(A_{\lambda_m}) : \sup \left\|R_\tau(A_{\lambda_m})\right\| = \infty \right\} \right)$$

$$\sigma_r(A) = \left(\bigcap_{m=1}^n (\sigma_c(A_{\lambda_m}) \cup \sigma_r(A_{\lambda_m}) \cup \rho(A_{\lambda_m})) \right) \cap \left(\bigcup_{m=1}^n \sigma_r(A_{\lambda_m}) \right)$$

$$\rho(A) = \left\{ \tau \in \bigcap_{m=1}^n \rho(A_{\lambda_m}) : \sup \left\|R_\tau(A_{\lambda_m})\right\| < \infty \right\}$$

are true.

Note that when $\Lambda = \mathbb{N}, \Sigma = P(\mathbb{N})$ is counting measure the analogous results have been established in work [13] and [28].

3 Some Compactness Properties of Direct Integral of Operators

In this section the compactness and spectral properties between direct integral of operators and their coordinate operators have been established. In general, there is not any relation between mentioned operators in compactness means.

Example 3.1. Let be $\Lambda = \mathbb{N}, \Sigma = P(\mathbb{N})$ μ-counting measure, $H_n = \mathbb{C}, A_n : \mathbb{C} \longrightarrow \mathbb{C}, A_n = E_n, n \geq 1, H = \overset{\infty}{\underset{n=1}{\oplus}} H_n, A = \overset{\infty}{\underset{n=1}{\oplus}} A_n.$. In this case $A_n \in C_\infty(H_n)$ for every $n \geq 1$,but $A \notin C_\infty(H)$.

Example 3.2. In some cases from the relations $A \notin C_\infty(H)$ no implies that $A_n \in C_\infty(H_n)$ for every $n \geq 1$.

Indeed, from the definition of direct integral of operators on the set having null μ-measure the coordinate operators may be defined by arbitrary way. But in certain situations there are concrete results.

Theorem 3.3. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, n \leq \infty$,Λ be any countable set, $\Sigma = P(\Lambda)$ and μ be any measure with property $\mu(\{\lambda\}) > 0$ for every point $\lambda \in \Lambda$.Then

(1) If $A = \overset{n}{\underset{m=1}{\oplus}} A_{\lambda_m} \in C_\infty(H), H = \overset{n}{\underset{m=1}{\oplus}} H_{\lambda_m}$,then $A_{\lambda_m} \in C_\infty(H_{\lambda_m})$ for every $1 \leq m \leq n$.

(2) Let Λ infinite countable set and $A_{\lambda_n} \in C_\infty(H_{\lambda_n})$ for every $n \geq 1$. In this case $A = \overset{\infty}{\underset{n=1}{\oplus}} A_{\lambda_n} \in C_\infty(H)$ if and only if $\lim_{n\to\infty} \|A_{\lambda_n}\| = 0$.

This theorem is proved by analogous scheme of the proof in theorem 4.6 in [13].

Now give one characterizating theorem on the point spectrum of compact direct integral of operators which can be easily proved.

Theorem 3.4. Let H_λ be a Hilbert space, $A_\lambda \in C_\infty(H_\lambda)$ for any $\lambda \in \Lambda$, $H = \overset{\oplus}{\int_\Lambda} H_\lambda d\mu(\lambda)$, $A = \overset{\oplus}{\int_\Lambda} A_\lambda d\mu(\lambda)$ and $A \in C_\infty(H)$.

In this case there exist countable subset $\Lambda_\star = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \Lambda, n \leq \infty$ such that the set $\Lambda_\star$ is minimal and

$$\sigma_p(A) = \bigcup_{m=1}^{n} \sigma_p(A_{\lambda_m})$$

From the definition of singular number $s(\cdot)$ (or characteristic numbers) of any compact operator in any Hilbert space [1] and Theorems 2.1 and 3.4 it is easy to prove the validity of the following result.

Theorem 3.5. Let H_λ be a Hilbert space, $A_\lambda \in C_\infty(H_\lambda)$ for any $\lambda \in \Lambda$, $H = \overset{\oplus}{\int_\Lambda} H_\lambda d\mu(\lambda)$, $A = \overset{\oplus}{\int_\Lambda} A_\lambda d\mu(\lambda)$ and $A \in C_\infty(H)$.

In this case there exist countable subset $\Lambda_\star = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \Lambda, n \leq \infty$ such that

(1) $\{s_k(A) : k \geq 1\} = \bigcup_{m=1}^{n} \{s_q(A_{\lambda_m}) : q \geq 1\}$;

(2) If $A \in C_p(H), 1 \leq p < \infty$, then $A_{\lambda_m} \in C_p(H_{\lambda_m})$ for every $m \in \mathbb{N} : 1 \leq m \leq n$;

(3) Let $A_{\lambda_m} \in C_{p(\lambda_m)}(H_{\lambda_m}), 1 \leq m \leq n, \quad 1 \leq p(\lambda_m) < \infty, \quad 1 \leq p = \sup\{p(\lambda_m) : 1 \leq m \leq n\} < \infty$ .Then $A \in C_p(H)$ if and only if the series $\sum_{m=1}^{n} \sum_{q=1}^{\infty} s_q^p(A_{\lambda_m})$ is convergent ;

(4) If $A_{\lambda_m} \in C_{p(\lambda_m)}(H_{\lambda_m}), 1 \leq m \leq n, \quad 1 \leq p(\lambda_m) < \infty, \quad p = \sup\{p(\lambda_m) : 1 \leq m \leq n\} < \infty$ and the series $\sum_{m=1}^{n} \sum_{q=1}^{\infty} s_q^{p(\lambda_m)}(A_{\lambda_m})$ is convergent, then $A \in C_p(H)$;

(5) If $A_{\lambda_m} \in C_{p(\lambda_m)}(H_{\lambda_m}), 1 \leq m \leq n, 1 \leq p(\lambda_m) < \infty, \quad p(\lambda_m) = \inf\{\alpha \in [1, \infty) : A_{\lambda_m} \in C_\alpha(H_{\lambda_m})\}$ and $\sup\{p(\lambda_m) : 1 \leq m \leq n\} = \infty$, then $A \notin C_p(H)$ for every $1 \leq p < \infty$;

(6) If $A_{\lambda_m} \in C_{p(\lambda_m)}(H_{\lambda_m}), 1 \leq m \leq n, \quad 1 \leq p(\lambda_m) \leq \infty, \quad p(\lambda_m) = \inf\{\alpha \in [1, \infty] : A_{\lambda_m} \in C_\alpha(H_{\lambda_m})\}$ and $\sup\{p(\lambda_m) : 1 \leq m \leq n\} = \infty$ and for some $k \in \mathbb{N}, \quad A_{\lambda_k} \in C_\infty(H_{\lambda_k})$, then $A \notin C_p(H)$ for every $1 \leq p < \infty$.

Proof. The validity of the claims (1) and (2) is clear. Prove third assertion of theorem. If the operator $A \in C_p(H)$, then the series $\sum_{k=1}^{\infty} s_k^p(A)$ is convergent. In this case by the first proposition of this theorem and important theorem on the convergence of the rearrangement series it is obtained that the series $\sum_{m=1}^{n} \sum_{q=1}^{\infty} s_q^p(A_{\lambda_m})$ is convergent.

On the contrary, if the series $\sum_{m=1}^{n} \sum_{q=1}^{\infty} s_q^p(A_{\lambda_m})$ is convergent, then the series $\sum_{k=1}^{\infty} s_k^p(A)$ which is a rearrangement of the above series is also convergent. So $A \in C_p(H)$.

Now prove (4). If $\|A_{\lambda_m}\| \leq 1$ for every $m, 1 \leq m \leq n$ then from the inequality

$$\sum_{m=1}^{n} \sum_{q=1}^{\infty} s_q^p(A_{\lambda_m}) \leq \sum_{m=1}^{n} \sum_{q=1}^{\infty} s_q^{p(\lambda_m)}(A_{\lambda_m}) < \infty$$

and first claim the validity of this assertion is clear. Now consider the general case. In this case the operator A can be written in form $A = CB$,where

$$C = \overset{n}{\underset{m=1}{\oplus}} \left(1 + \|A_{\lambda_m}\|\right) E_m \quad , \quad B = \overset{n}{\underset{m=1}{\oplus}} \left(\frac{A_{\lambda_m}}{\left(1 + \|A_{\lambda_m}\|\right)} \right)$$

Then $C \in B(H)$.

On the other hand, since $\|B_m\| \leq 1, 1 \leq m \leq n$ and

$$\sum_{m=1}^{n} \sum_{q=1}^{\infty} s_q^{p(\lambda_m)}(B_m) = \sum_{m=1}^{n} \sum_{q=1}^{\infty} \frac{s_q^{p(\lambda_m)}(A_{\lambda_m})}{\left(1 + \|A_{\lambda_m}\|\right)^{p(\lambda_m)}}$$

$$\leq \sum_{m=1}^{n} \sum_{q=1}^{\infty} s_q^{p(\lambda_m)}(A_{\lambda_m}) < \infty$$

then from the section (3) of this theorem it implies that $B \in C_p(H)$ with $p = \sup\{p(\lambda_m) : 1 \leq m \leq n\}$
Therefore, $A = CB \in C_p(H)$ [1].

Furthermore, by using proposition (2) of this theorem it is easy to prove the claim (5). On the other hand, the claim (6) is one of the corollary of (5).

Remark 3.6. Note that for the some $\lambda_\star \in \Lambda_\star$ in representation $\sigma_p(A) = \bigcup_{m=1}^{n} \sigma_p(A_{\lambda_m})$ in Theorem 3.4. it may be hold that

$$card\left[\sigma_p(A_{\lambda_\star}) \cap \sigma_p(A)\right] < \infty$$

In these situations corresponding conditions for such index in the Theorem 3.5(3-6) may be omitted, for example, as in the following assertion.

Theorem 3.7. Let H_λ be a Hilbert space, $A_\lambda \in C_\infty(H_\lambda)$ for any $\lambda \in \Lambda$, $H = \int_\Lambda^\oplus H_\lambda d\mu(\lambda)$, $A = \int_\Lambda^\oplus A_\lambda d\mu(\lambda)$ and $A \in C_\infty(H)$.

In this case there exist countable subset $\Lambda_\star = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \Lambda, n \leq \infty$ such that

$$\{s_k(A) : k \geq 1\} = \bigcup_{m=1}^{n} \{s_q(A_{\lambda_m}) : q \geq 1\}$$

If

$$card\left\{\lambda_\star \in \Lambda_\star : card\left[\sigma_p(A_{\lambda_\star}) \cap \sigma_p(A)\right] < \infty\right\} < \infty,$$

$$A_{\lambda_m} \in C_{p(\lambda_m)}(H_{\lambda_m}), \quad 1 \leq p(\lambda_m) < \infty, \quad \lambda_m \in \Lambda_\star,$$

$$\Lambda_{\star\star} = \left\{\lambda_\star \in \Lambda_\star : card\left[\sigma_p(A_{\lambda_\star}) \cap \sigma_p(A)\right] < \infty\right\},$$

$$\lambda_m \notin \Lambda_{\star\star}, 1 \leq p = \sup\{p(\lambda_m) : \lambda_m \in \Lambda_\star \setminus \Lambda_{\star\star}\} < \infty$$

and

$$\sum_{\lambda_m \in \Lambda_\star \setminus \Lambda_{\star\star}} \left(\sum_{k=1}^{\infty} s_q^{p(\lambda_m)}(A_{\lambda_m})\right) < \infty$$

Then $A = (A_\lambda) \in C_p(H)$.

4 Power and Polynomially Boundednessity of the Direct Sum Operators

In this section let us $\Lambda = \mathbb{N}, \Sigma = P(\mathbb{N})$ and μ is the counting measure. Here a connection of power (and polynomially) boundedness property of the direct sum operators in the direct sum Hilbert spaces and its coordinate operators were established. In advance, give some necessary definitions for the later.

Definition 4.1.[26,27] Let H be any Hilbert space.

(1) An operator $T \in B(H)$ is called power bounded $(T \in PW(H))$ if there exist a constant $M(\geq 1)$ such that for any $n \in \mathbb{N}$ it is satisfied that $\|T^n\| \leq M$ (3.1);

(2) Operator $T \in B(H)$ is called polynomially bounded $(T \in PB(H))$,if there exist a constant $M(\geq 1)$ such that for any polynomial $p(\cdot)$ it is satisfied that $\|p(T)\| \leq M\|p\|_\infty$ (3.2) ,
where $\|p\|_\infty = \sup\{|p(z)| : z \in \mathbb{C}, |z| \leq 1\}$.

(3) The smallest number M satisfying (3.1) (resp.(3.2)) is called the power bound (resp.polynomial bound) of the operator T and will be denoted by $M_w(T)$ (resp. $M_p(T)$).

Before of all note that the following theorem is true.

Theorem 4.2. If $H = \bigoplus_{n=1}^{\infty} H_n, A = \bigoplus_{n=1}^{\infty} A_n \in PW(H)$, then $A_n \in PW(H_n)$ for every $n \geq 1$.

The proof of this theorem is a result of the following equation

$$\sup_{m\geq 1}\left(\sup_{n\geq 1}\|A_n^m\|\right) = \sup_{n\geq 1}\left(\sup_{m\geq 1}\|A_n^m\|\right) < \infty$$

In general, the inverse of last assertion may be not true.

Example 4.3. Let us $H = \bigoplus_{n=1}^{\infty} H_n$, $H_n = L^2(-1,1)$, $A = \bigoplus_{n=1}^{\infty} A_n : H \longrightarrow H$, $A_n : L^2(-1,1) \longrightarrow L^2(-1,1)$, $A_n f(x) = \alpha_n \int_{-x}^{x} f(t)dt$, $\alpha_n \in \mathbb{R}$, $n \geq 1$, $\sup_{n\geq 1}|\alpha_n| = \infty$.

In this case it is easy to see that

$$\|A_n\| = \frac{4|\alpha_n|}{\pi} \quad \text{and} \quad A_n^2 = 0, n \geq 1$$

Consequently,

$$\sup_{m\geq 1}\|A^m\| = \sup_{m\geq 1}\left(\sup_{n\geq 1}\|A_n^m\|\right) = \sup_{n\geq 1}\|A_n\| = \sup_{n\geq 1}\frac{4|\alpha_n|}{\pi} = \infty$$

Hence $A_n \in PW(H_n)$ for any $n \geq 1$, but $A = \bigoplus_{n=1}^{\infty} A_n \notin PW(H)$.

Example 4.4. Let us $H = \bigoplus_{n=1}^{\infty} H_n$ $, H_n = \mathbb{C}^2, A = \bigoplus_{n=1}^{\infty} A_n$ $A : H \longrightarrow H$, $A_n : \mathbb{C}^2 \longrightarrow \mathbb{C}^2, A_n = \begin{pmatrix} 0 & 0 \\ \alpha_n & 0 \end{pmatrix}, \alpha_n \in \mathbb{C}$, $n \geq 1, \sup_{n\geq 1}|\alpha_n| = \infty$.

In this case $\|A_n\| = |\alpha_n|$ and $A_n^2 = 0, n \geq 1$,i.e. $A_n \in PW(\mathbb{C}^2)$ for any $n \geq 1$,but

$$\sup_{m\geq 1}\|A^m\| = \sup_{m\geq 1}\left(\sup_{n\geq 1}\|A_n^m\|\right) = \sup_{n\geq 1}\|A_n\| = \sup_{n\geq 1}|\alpha_n| = \infty$$

Therefore, $A = \overset{\infty}{\underset{n=1}{\oplus}} A_n \notin PW(H)$.

Actually, it is true the following result.

Theorem 4.5. Let $H = \overset{\infty}{\underset{n=1}{\oplus}} H_n, A = \overset{\infty}{\underset{n=1}{\oplus}} A_n$ and $A \in B(H)$. In this case $A \in PW(H)$ if and only if $A_n \in PW(H_n)$ for every $n \geq 1$ and $\sup\limits_{n\geq 1} M_w(A_n) < \infty$.

Proof. If $A \in PW(H)$, then from the following relation

$$\sup_{m\geq 1} \|A^m\| = \sup_{m\geq 1} \left(\sup_{n\geq 1} \|A_n^m\| \right) = \sup_{n\geq 1} \left(\sup_{m\geq 1} \|A_n^m\| \right) < \infty$$

it is implied that

$$\sup_{m\geq 1} \|A_n^m\| < \infty \quad \text{for each} \quad n \geq 1$$

From this it is determined that $A_n \in PW(H_n)$ for any $n \geq 1$. On the other hand, it is clear that for each $n \geq 1$

$$\begin{aligned} \sup_{m\geq 1} \|A_n^m\| &\leq \sup_{n\geq 1} \left(\sup_{m\geq 1} \|A_n^m\| \right) = \sup_{m\geq 1} \left(\sup_{n\geq 1} \|A_n^m\| \right) \\ &= \sup_{m\geq 1} \|A^m\| \leq M_w(A) < \infty \end{aligned}$$

Therefore

$$\sup_{n\geq 1} M_w(A_n) \leq M_w(A) < \infty$$

On the contrary, if for any $n \geq 1$

$$A_n \in PW(H_n), \quad \sup_{m\geq 1} \|A_n^m\| \leq M_w(A_n), \quad \sup_{n\geq 1} M_w(A_n) < \infty$$

Then from the equality

$$\begin{aligned} \sup_{m\geq 1} \|A^m\| &= \sup_{m\geq 1} \left(\sup_{n\geq 1} \|A_n^m\| \right) = \sup_{n\geq 1} \left(\sup_{m\geq 1} \|A_n^m\| \right) \\ &\leq \sup_{n\geq 1} M_w(A_n) < \infty \end{aligned}$$

it is obtained that $A \in PW(H)$

Now polynomially boundedness property of the direct sum operators will be investigated. In advance, note that the following proposition is true.

Theorem 4.6. If $H = \overset{\infty}{\underset{n=1}{\oplus}} H_n$ $, A = \overset{\infty}{\underset{n=1}{\oplus}} A_n \in PB(H)$, then $A_n \in PB(H_n)$ for every $n \geq 1$.

Unfortunately, the inverse of last theorem may be not true in general.

Example 4.7. Let us $H = \overset{\infty}{\underset{n=1}{\oplus}} H_n$, $H_n = L^2(-1,1)$,

$A = \overset{\infty}{\underset{n=1}{\oplus}} A_n : H \longrightarrow H, \quad A_n : L^2(-1,1) \longrightarrow L^2(-1,1)$,

$A_n f(x) = \alpha_n \int\limits_{-x}^{x} f(t)dt, \quad \alpha_n \in \mathbb{R}, \quad \alpha_n \geq \frac{\pi}{4}$, $n \geq 1$, $\sup\limits_{n\geq 1} |\alpha_n| = \infty$.

In this case it is known that $A_n \in C_\infty(H_n)$ and A_n is a nilpotent operator with power of nilpotency 2 for any $n \geq 1$. Then for any polynomial function

$$p(z) = \sum_{k=0}^{q} a_k z^k, z \in \mathbb{C}, q = 0, 1, 2, \ldots$$

we have for each $n \geq 1$

$$\begin{aligned} \|p(A_n)\| &\leq |a_0| + |a_1| \|A_n\| \leq \frac{4|\alpha_n|}{\pi}(|a_0| + |a_1|) \\ &\leq \|A_n\| \|p\|_\infty \end{aligned}$$

In other words, $A_n \in PB(H_n)$ for every $n \geq 1$.

Unfortunately, for the polynomial $p_\star(z) = z, z \in \mathbb{C}$ we have

$$\|p_\star(A)\| = \sup_{n\geq 1} \|p_\star(A_n)\| = \sup_{n\geq 1} \|A_n\| = \sup_{n\geq 1} \frac{4|\alpha_n|}{\pi} = \infty$$

i.e. $A \notin PB(H)$.

But in general case the following result is true.

Theorem 4.8. Let $H = \overset{\infty}{\underset{n=1}{\oplus}} H_n, A = \overset{\infty}{\underset{n=1}{\oplus}} A_n$ and $A \in B(H)$. In this case $A \in PB(H)$ if and only if $A_n \in PB(H_n)$ for every $n \geq 1$ and $\sup\limits_{n\geq 1} M_p(A_n) < \infty$.

Proof. Assumed that for every polynomial $p(\cdot)$

$$\|p(A_n)\| \leq M_p(A_n) \|p\|_\infty \quad \text{and} \quad \sup_{n\geq 1} M_p(A_n) < \infty$$

In this case since

$$p(A) = \overset{\infty}{\underset{n=1}{\oplus}} p(A_n)$$

for every polynomial function $p(\cdot)$, then

$$\|p(A)\| = \sup_{n\geq 1} \|p(A_n)\|$$

From last relation it is obtained that

$$\|p(A)\| \leq \sup_{n\geq 1} M_p(A_n) \|p\|_\infty$$

Hence $A \in PB(H)$.

Now let us $A \in PB(H)$, i.e. for any $n \geq 1$ and polynomial $p(\cdot)$ it is valid that

$$\|p(A_n)\| \leq \|p(A)\| = \sup_{n\geq 1} \|p(A_n)\| \leq \sup_{n\geq 1} M_p(A_n) \|p\|_\infty$$

Then it is clear that $A_n \in PB(H_n), n \geq 1$.
On the other hand, from last equality it is implied that $M_p(A_n) \leq M_p(A) < \infty$ for every $n \geq 1$. Hence, $\sup\limits_{n\geq 1} M_p(A_n) < \infty$. This completes the proof of the theorem.

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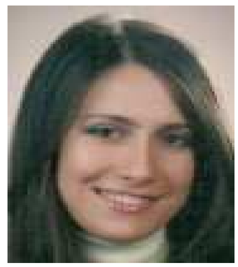

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