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Certain Types of the Orbits of Real Quadratic Fields by Hecke Groups

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Abstract: Erich Hecke (1936) introduced the groups $H(\lambda_q) = \langle S, T : S^2 = T^q = 1\rangle$ generated by two linear-fractional transformations $S(z) = \frac{-1}{z}$ and $T(z) = \frac{-1}{z+\lambda}$. In this paper, we discuss the action of hecke groups $H(\lambda_q)$ on real quadratic fields. In particular, we explore the orbits of $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$ where $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$ is the disjoint union of $\mathbb{Q}^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \neq 0, b = \frac{a^2-n}{c} \in \mathbb{Z} \mid (a,b,c) = 1\}$ for $n = k^2m$.

Keywords: Hecke groups, Quadratic Fields, Orbits

1 Introduction

In 1936 Erich Hecke [2] introduced the groups $H(\lambda)$ generated by two linear-fractional transformations $S(z) = \frac{-1}{z}$ and $T(z) = \frac{-1}{z+\lambda}$. Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2cos(\frac{\pi}{q})$, $q \in \mathbb{N}$, $q \geq 3$ or $\lambda \geq 2$. Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic group of order 2 and q, and it has a presentation

$$H(\lambda_q) = \langle S, T : S^2 = T^q = 1\rangle \cong C_2 * C_q$$

The first few of these groups are $H(\lambda_3) = PSL(2,\mathbb{Z})$, the modular group, $H(\lambda_4) = \langle S, T : S^2 = T^4 = 1\rangle$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$ and $H(\lambda_6) = H(\sqrt{3}) = \langle S, T : S^2 = T^6 = 1\rangle$. It was proved that the action of $H = \langle x, y : x^2 = y^4 = 1\rangle$, where $x(z) = \frac{-1}{2z}$ and $y(z) = \frac{-1}{2(z+1)}$, on the rational projective line $\mathbb{Q} \cup \{\infty\}$ is transitive [7,12]. The action of the modular group $G = \langle x', y' : x'^2 = y'^3 = 1\rangle$, where $x'(z) = \frac{-1}{z}$ and $y'(z) = \frac{-1}{z+1}$, on the real quadratic fields has been discussed in [3,9,11] and [10].
Let $n = k^2m$, $k \in \mathbb{N}$ and m is a square free positive integer. Then $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$ is a disjoint union of

$$\mathbb{Q}^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \neq 0, b = \frac{a^2-n}{c} \in \mathbb{Z} \mid (a,b,c) = 1\}.$$

If $\alpha = \frac{a+\sqrt{n}}{c} \in \mathbb{Q}^*(\sqrt{n})$ and its conjugate $\overline{\alpha}$ have opposite signs then α is called an ambiguous number [3]. The set of ambiguous numbers in $\mathbb{Q}^*(\sqrt{n})$ is denoted by $\mathbb{Q}_1^*(\sqrt{n})$ and $|\mathbb{Q}_1^*(\sqrt{n})|$ has been determined in [1] as a function of n. Since $\mathbb{Q}''(\sqrt{n}) = \mathbb{Q}^*(\sqrt{n}) \cup \frac{1}{2}\mathbb{Q}^*(\sqrt{n})$ and for $n \not\equiv 0(mod\ 4)$ $\mathbb{Q}^{**}(\sqrt{n}) = \{\alpha(a,b,c) \in \mathbb{Q}^*(\sqrt{n}) \mid c \equiv 0(mod\ 2)\}$ are two H-subsets of $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$.
The results of [12] are extended in [13] to all non-square $n \equiv 0(mod\ 4)$ and was proved that $\mathbb{Q}''(\sqrt{n}) = \mathbb{Q}^{*\sim}(\sqrt{n}) \cup \mathbb{Q}^{*\sim}(\sqrt{4n})$, where $\mathbb{Q}^{*\sim}(\sqrt{n}) = (\mathbb{Q}^*(\sqrt{\frac{n}{4}}) \setminus \mathbb{Q}^{**}(\sqrt{\frac{n}{4}})) \cup \mathbb{Q}^{**}(\sqrt{n})$. Moreover the proper H-subsets of $\mathbb{Q}^{**}(\sqrt{n})$ or $\mathbb{Q}''(\sqrt{n}) = \mathbb{Q}^{**}(\sqrt{n}) \cup \mathbb{Q}^{*\sim}(\sqrt{4n})$ according as $n \not\equiv 0(mod\ 4)$ or $n \equiv 0(mod\ 4)$ have been discovered. As we denote the number of H-orbits of $Q^{*\sim}(\sqrt{4p})$ by $o_H^{*\sim}(4p)$ and the number of H-orbits of $\mathbb{Q}''(\sqrt{p})$ by $o_H(p)$. In a recent paper [15], H-orbits of $\mathbb{Q}^{*\sim}(\sqrt{4p})$, $p \equiv 1(mod\ 4)$, have been found for the case $|(\frac{\sqrt{p}}{1})^H|_{amb} + |(\frac{\sqrt{p}}{-1})^H|_{amb} = |\mathbb{Q}_1^{*\sim}(\sqrt{4p})|$. In this paper we discuss the case whenever $|(\frac{\sqrt{p}}{1})^H|_{amb} + |(\frac{\sqrt{p}}{-1})^H|_{amb} < |\mathbb{Q}_1^{*\sim}(\sqrt{4p})|$.
We tabulate the actions on, $\alpha = \frac{a+\sqrt{n}}{c}$ with $b = \frac{a^2-n}{c}$, of x, y and their combinations in Table 1 and we cite the following results for later reference.
Lemma 1.1 [11] Let m be a square-free positive integer.

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Table 1: The action of elements of H on $\alpha=\frac{a+\sqrt{n}}{c}\in\mathbb{Q}^{''}(\sqrt{n})$

α	a	b	c
$x(\alpha)=\frac{-1}{2\alpha}$	$-a$	$\frac{c}{2}$	$2b$
$y(\alpha)=\frac{-1}{2(\alpha+1)}$	$-a-c$	$\frac{c}{2}$	$2(2a+b+c)$
$y^2(\alpha)=\frac{-(\alpha+1)}{2\alpha+1}$	$-3a-2b-c$	$2a+b+c$	$4a+4b+c$
$(xy)^k(\alpha)=\alpha+k$	$a+kc$	$2ka+b+k^2c$	c
$yx(\alpha)\frac{\alpha}{1-2\alpha}$	$a-2b$	b	$-4a+4b+c$
$(y^2x)(\alpha)=\frac{1-2\alpha}{2(-1+\alpha)}$	$3a-2b-c$	$\frac{-4a+4b+c}{2}$	$2(-2a+b+c)$
$(yx)^k(\alpha)=\frac{\alpha}{1-2k\alpha}$	$a-2kb$	b	$-4ka+4k^2b+c$
$(y^3x)^k(\alpha)=\alpha-k$	$a-kc$	$2ka+b+k^2c$	c

Then
$|\mathbb{Q}_1^*(\sqrt{m}\ )|=2\tau(m)+4\sum_{a=1}^{\lfloor\sqrt{m}\rfloor}\tau(m-a^2)$ where $\tau(m)$ stands for the number of positive divisors of m and $\lfloor\sqrt{m}\rfloor$ is the largest integer less than $\sqrt{m}$.

Lemma 1.2 [9] Let $p\equiv 1(mod\ 4)$. Then $\mathbb{Q}^*(\sqrt{p})$ splits into at least two G-orbits, namely, $(\sqrt{p})^G$ and $(\frac{1+\sqrt{p}}{2})^G$ under the action of G.

Lemma 1.3 [11] Let n be square free positive integer. Then $|\mathbb{Q}_1^{**}(\sqrt{n})|=2\tau''(n)+4\sum_{a=1}^{\lfloor\sqrt{n}\rfloor}\tau''(n-a^2)$ where $\tau''(u)$ denotes those divisors of u, which are divisible by 2. □

Lemma 1.4 [12] Let $\alpha\in Q''(\sqrt{n})$. Then $\alpha^H=(\overline{\alpha})^H$ if and only if there exists an element β in α^H such that $x(\beta)=\overline{\beta}$.

2 Types of G-orbits of $\mathbb{Q}^*(\sqrt{p})$ and H-orbits of $\mathbb{Q}^{''}(\sqrt{p})$

We start this section by describing the closed paths (circuits) for the action of group $H(\lambda_4)$ (see [6] and figure 1).

Definition 2.1. If $n_1,n_2,\ n_3,n_4,...,n_k$ is a sequence of positive integers and

$$i_j=0,1,2,i_l\neq i_{l+1}\ (l=1,2,...,k-1),i_1\neq i_k$$

Then by a circuit of the type

$$(n_{1i_1},n_{2i_2},n_{3i_3},n_{4i_4},...,n_{ki_k})$$

we shall mean the circuit (counter clockwise) in which n_j, $j=1,2,3,...,k$ squares have i_j vertices outside the circuit.

Remark 2.2. 1. Since it is immaterial with which ambiguous number of α^H the circuit begins, we can express type of the orbit in Definition 2.1. by any of the following k-equivalent forms

$$(n_{1i_1},n_{2i_2},...,n_{ki_k})=(n_{2i_2},n_{3i_3},...,n_{ki_k},n_{1i_1})$$
$$=...(n_{ki_k},n_{1i_1},...,n_{k-1i_{k-1}}) \quad (1)$$

2. This circuit induces an element

$$g=(xy^{i_k+1})^{n_k}...(xy^{i_2+1})^{n_2}(xy^{i_1+1})^{n_1}$$

of H and fixes a particular vertex of a square lying on the circuit and hence the ambiguous length of this circuit is given by $2(n_1+n_2+n_3+...+n_k)$

The following example and figure 2, both are the best description of the above definition and remark.

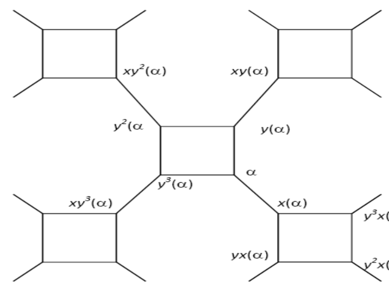

Fig. 1: The coset diagram for the action of H on $\alpha\in\mathbb{Q}^{''}(\sqrt{n})$

Example 2.3. By the circuit of the type $(2_0,1_1,1_2,2_0,1_2,1_1,2_0)$ we mean the circuit (see figure 2) induces an element $h=(xy)^2(xy^2)(xy^3)(xy)^2(xy^3)(xy^2)(xy)^2$ of H which fixes vertex $\frac{\sqrt{7}}{1}$. Let $l_1=\frac{\sqrt{7}}{1}$. $(xy)^2(l_1)=\frac{2+\sqrt{7}}{1}=l_2$, $(xy^2)(l_2)=\frac{1+\sqrt{7}}{4}=l_3$, $(xy^3)(l_3)=\frac{-2+\sqrt{7}}{2}=l_4$, $(xy)^2(l_4)=\frac{2+\sqrt{17}}{2}=l_5$, $(xy^3)(l_5)=\frac{-1+\sqrt{7}}{4}=l_6$, $(xy^2)(l_6)==\frac{-2+\sqrt{7}}{1}=l_7$ $(xy)^2(l_7)=l_1$,and the ambiguous length of this circuit is $2(2+1+1+2+1+1+2)$.

The following four results have been taken from [15] for our convenience in section 3.

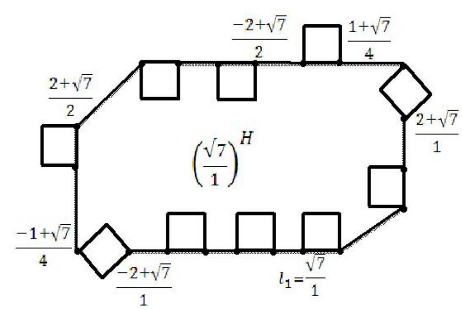

Fig. 2: Orbit of $l_1=\frac{\sqrt{7}}{1}$ and $h(l_1)=l_1$

Theorem 2.4. Let $n\equiv 1(mod\ 8)$. Then $\mathbb{Q}^{''}(\sqrt{n})$ splits into four H-subsets. In particular $(\frac{\sqrt{n}}{1})^H$, $(\frac{\sqrt{n}}{-1})^H$, $(\frac{1+\sqrt{n}}{2})^H$ and

$(\frac{1+\sqrt{n}}{4})^H$ are at least four H-orbits of $\mathbb{Q}''(\sqrt{n})$.

Remark 2.5. It can be easily seen that

1. $o_H^{*\sim}(p) = o_G(p)$ when $p \equiv 1(mod\ 4)$.
2. $o_G(4p) = 2\,o_G(p)$ if $p \equiv 1$, or $5(mod\ 8)$ such that $p-1$ is not a perfect square
3. $o_G(4p) = 2o_G(p)+2$ if $p \equiv 5(mod\ 8)$ such that $p-1$ is a perfect square

Theorem 2.6. Let $p \equiv 1(mod\ 4$. Then

1. $o_H(p) = 2\,o_G(p)$ if $p \equiv 1(mod\ 8)$.
2. $o_H(p) = o_G(p)+1$ if $p \equiv 5(mod\ 8)$ such that $p-1$ is not a perfect square.
3. $o_H(p) = 2o_G(p)+1$ if $p \equiv 5(mod\ 8)$ such that $p-1$ is a perfect square.

Remark 2.7. Let $p \equiv 1$ or $5(mod\ 8)$ such that $p-1$ is a perfect square. Then the numbers $\frac{\pm\lfloor\sqrt{p}\rfloor+\sqrt{p}}{1}$ and $\frac{\pm\lfloor\sqrt{p}\rfloor+\sqrt{p}}{-1}$ are contained in $(\frac{\sqrt{p}}{1})^H$ and $(\frac{\sqrt{p}}{-1})^H$ respectively. Also the numbers $\frac{\pm1+\sqrt{p}}{\pm(p-1)}$ are contained in $(\frac{1+\sqrt{p}}{2})^H$. Similarly the numbers $\frac{1+\sqrt{p}}{\pm\sqrt{p-1}}$ are contained in $(\frac{1+\sqrt{p}}{4})^H$ and $\frac{-1+\sqrt{p}}{\pm\sqrt{p-1}}$ are contained in $(\frac{-1+\sqrt{p}}{4})^H$ respectively.

Lemma 2.8. Let $n \equiv 0(mod\ 4)$. Then $|\mathbb{Q}_1^{*\sim}(\sqrt{n})| = 2(|\mathbb{Q}_1^{**}(\sqrt{n})|)$. Whereas if $n \not\equiv 0(mod\ 4)$ $|\mathbb{Q}_1^{*\sim}(\sqrt{4n})| = 2(|\mathbb{Q}_1^{*}(\sqrt{n})| - |\mathbb{Q}_1^{**}(\sqrt{n})|)$.

3 H-orbits of $\mathbb{Q}^{*\sim}(\sqrt{4p})$ with $o_H^{*\sim}(4p) > 4$

Let $p \equiv 1(mod\ 4)$. If $|(\sqrt{p})^H|_{amb} + |(\frac{\sqrt{p}}{-1})^H|_{amb} = |\mathbb{Q}_1^{*\sim}(\sqrt{4p})|$, then we have $o_H^{*\sim}(p) = 2$. If $|(\sqrt{p})^H|_{amb} + |(\frac{\sqrt{p}}{-1})^H|_{amb} < |\mathbb{Q}_1^{*\sim}(\sqrt{4p})|$, then we have the following results.

Lemma 3.1. Let $p \equiv 1(mod\ 4)$. Then

1. $(\alpha)^H \cap (\overline{\alpha})^H = \emptyset$ for all $\alpha \in \mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash ((\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H)$.
2. $(\alpha)^H \cap (-\alpha)^H = \emptyset$ for all $\alpha \in \mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash (\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H)$.

Proof. By [9] we know that $\frac{a+\sqrt{p}}{\pm c}, \frac{-a+\sqrt{p}}{\pm c}$ are contained in $(\sqrt{p})^H$ or $(\frac{\sqrt{p}}{-1})^H$ where $c \not\equiv 0(mod\ 2)$ and $\frac{c+\sqrt{p}}{\pm a}, \frac{-c+\sqrt{p}}{\pm a}$ are contained in $(\frac{1+\sqrt{p}}{2})^H$ or $(\frac{1+\sqrt{p}}{4})^H$ where $a \not\equiv 0(mod\ 2)$. Hence by Lemma 1.4. we have $(\alpha)^H \cap (\overline{\alpha})^H = \emptyset$ for all $\alpha \in Q^{*\sim}(\sqrt{4p}) \backslash ((\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H \cup (\frac{1+\sqrt{p}}{2})^H \cup (\frac{1+\sqrt{p}}{4})^H)$ The 2nd part directly follows from Theorem 3.3 [13]. □

In the following lemma we use $\mathbb{Q}''''(\sqrt{p}) = \mathbb{Q}'(\sqrt{p}) \cup \frac{1}{2}\mathbb{Q}'(\sqrt{p})$.

Lemma 3.2. Let $p \equiv 1(mod\ 4)$. Then $\frac{1+\sqrt{p}}{4} \in \mathbb{Q}''''(\sqrt{p})$ or $\mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash \mathbb{Q}''''(\sqrt{p})$ according as $p \equiv 1(mod\ 8)$ or $n \equiv 5(mod\ 8)$ for $p > 13$.

Proof. The proof is straightforward. □

Lemma 3.3. Let $p \equiv 5(mod\ 8)$ such that $p-1$ is a perfect square. If $(\mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash \mathbb{Q}''''(\sqrt{p})) \backslash ((\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H) \neq \emptyset$, then either $\frac{1+\sqrt{p}}{q_1}$ or $\frac{2+\sqrt{p}}{t_1} \in (\mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash Q''''(\sqrt{p})) \backslash ((\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H)$.

Proof. Using Remark 2.7, $(\sqrt{p})_a mb^H \cup (\frac{\sqrt{p}}{-1})_a mb^H = \{\frac{\pm a+\sqrt{p}}{\pm 1}, \frac{\pm a+\sqrt{p}}{\pm(p-a^2)}, 0 \leq a \leq \lfloor\sqrt{p}\rfloor\}$. If $(\mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash \mathbb{Q}''''(\sqrt{p})) \backslash ((\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H) \neq \emptyset$, then either $p-1$ is not power of two or is power of 2. In first case $p-1$ is not power of two then there exists $\frac{1+\sqrt{p}}{q_1} \in (\mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash \mathbb{Q}''''(\sqrt{p})) \backslash ((\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H)$. If $p-1$ is power of 2 then $p-4$ is not power of 2. Thus there exists $\frac{2+\sqrt{p}}{t_1} \in (\mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash \mathbb{Q}''''(\sqrt{p})) \backslash ((\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H)$. □

Corollary 3.4. Let $p \equiv 5(mod\ 8)$ such that $p-1$ is a perfect square. If $(\mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash \mathbb{Q}''''(\sqrt{p})) \backslash ((\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H) \neq \emptyset$, then $(\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H \cup (\frac{1+\sqrt{p}}{q_1})^H \cup (\frac{-1+\sqrt{p}}{q_1})^H \cup (\frac{1+\sqrt{p}}{-q_1})^H \cup (\frac{-1+\sqrt{p}}{-q_1})^H \subseteq \mathbb{Q}^{*\sim}(\sqrt{4p})$ or $(\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H \cup (\frac{2+\sqrt{p}}{t_1})^H \cup (\frac{-2+\sqrt{p}}{t_1})^H \cup (\frac{2+\sqrt{p}}{-t_1})^H \cup (\frac{-2+\sqrt{p}}{-t_1})^H \subseteq \mathbb{Q}^{*\sim}(\sqrt{4p})$.

Proof. The proof is straightforward and follows by Lemma 3.3. □

Lemma 3.5. Let $p \equiv 1(mod\ 8)$ such that $p-1$ is a perfect square. Then $\mathbb{Q}^{*\sim}(\sqrt{4p})$ splits into at least six H-orbits for $p > 17$.

Proof. Using Remark 2.7, $(\frac{\sqrt{p}}{1})^H_{amb} \cup (\frac{\sqrt{p}}{-1})^H_{amb} = \{\frac{\pm a+\sqrt{p}}{\pm 1}, \frac{\pm a+\sqrt{p}}{\pm(p-a^2)}, 0 \leq a < \lfloor\sqrt{p}\rfloor\}$ and $(\frac{1+\sqrt{p}}{2})^H_{amb} \cup (\frac{1+\sqrt{p}}{4})^H_{amb} = \{\frac{\pm a+\sqrt{p}}{\pm 2}, \frac{\pm a+\sqrt{p}}{\frac{p-a^2}{\pm 2}}, \frac{\pm 1+\sqrt{p}}{\pm\lfloor\sqrt{p}\rfloor} :$ $a = 1, 3, ..., \lfloor\sqrt{p}\rfloor - 1\}$. Also $(\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H \subseteq \mathbb{Q}^{*\sim}(\sqrt{4p}) \backslash \mathbb{Q}''''(\sqrt{p})$ and $(\frac{1+\sqrt{p}}{2})^H \cup (\frac{1+\sqrt{p}}{4})^H \subseteq \mathbb{Q}''''(\sqrt{p})$.

For $p > 17$ we have atleast four more H-orbits namely $(\frac{-1+\sqrt{p}}{4})^H$, $(\frac{1+\sqrt{p}}{8})^H$, $(\frac{3+\sqrt{p}}{8})^H$ and $(\frac{-3+\sqrt{p}}{8})^H$ contained in $\mathbb{Q}''''(\sqrt{p})$ since otherwise $\lfloor\sqrt{p}\rfloor = 4$ and hence $\frac{-1+\sqrt{p}}{4}, \frac{1+\sqrt{p}}{8}, \frac{3+\sqrt{p}}{8}$ and $\frac{3+\sqrt{p}}{8} \in (\frac{1+\sqrt{p}}{2})^H \cup (\frac{1+\sqrt{p}}{4})^H$.

Hence For $p > 17$, $\frac{-1+\sqrt{p}}{4}, \frac{1+\sqrt{p}}{8}, \frac{3+\sqrt{p}}{8}$ and $\frac{3+\sqrt{p}}{8} \notin (\frac{1+\sqrt{p}}{2})^H \cup (\frac{1+\sqrt{p}}{4})^H$. This shows $\mathbb{Q}''''(\sqrt{p})$ contains at least six H-orbits.

By Corollary 3.4 we have six H-orbits either $(\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H \cup (\frac{1+\sqrt{p}}{q_1})^H \cup (\frac{-1+\sqrt{p}}{q_1})^H \cup (\frac{1+\sqrt{p}}{-q_1})^H \cup (\frac{-1+\sqrt{p}}{-q_1})^H$ or $(\sqrt{p})^H \cup (\frac{\sqrt{p}}{-1})^H \cup (\frac{2+\sqrt{p}}{t_1})^H \cup (\frac{-2+\sqrt{p}}{t_1})^H \cup (\frac{2+\sqrt{p}}{-t_1})^H \cup$

$(\frac{-2+\sqrt{p}}{-t_1})^H$ contained in $\mathbb{Q}^{*\sim}(\sqrt{4p})\backslash\mathbb{Q}''''(\sqrt{p})$. Thus we have at least twelve H-orbits. □

Lemma 3.6. Let $p \equiv 5(mod\ 8)$ such that $p-1$ is a perfect square. Then $\mathbb{Q}^{*\sim}(\sqrt{4p})$ splits into at least six H-orbits for $p > 13$

Proof Using Lemma 3.2., $\frac{1+\sqrt{p}}{4} \in \mathbb{Q}^{*\sim}(\sqrt{4p})\backslash\mathbb{Q}''''(\sqrt{n})$. Also $(\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H \subseteq \mathbb{Q}^{*\sim}(\sqrt{4p})\backslash\mathbb{Q}''''(\sqrt{p})$. For $p > 13$, $\frac{\pm1+\sqrt{p}}{4} \notin (\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H$ otherwise for $p = 13$, $\frac{\pm1+\sqrt{13}}{\pm4} \in (\frac{\sqrt{13}}{1})^H \cup (\frac{\sqrt{13}}{-1})^H$ hence $(\frac{\pm1+\sqrt{p}}{\pm4})^H$ exists and contained in $\mathbb{Q}^{*\sim}(\sqrt{4p})\backslash\mathbb{Q}''''(\sqrt{p})$. Thus $(\frac{\sqrt{p}}{1})^H \cup (\frac{\sqrt{p}}{-1})^H \cup (\frac{1+\sqrt{p}}{\pm4})^H \cup (\frac{-1+\sqrt{p}}{\pm4})^H \subseteq \mathbb{Q}^{*\sim}(\sqrt{4p}) \setminus \mathbb{Q}''''(\sqrt{p})$ and $(\frac{1+\sqrt{p}}{2})^H \subseteq \mathbb{Q}''''(\sqrt{p})$. Hence we have eight H-orbits. □

Example 3.7. Let $p = 37$. By Theorem 2.4, $\mathbb{Q}^{*\sim}(\sqrt{4p})$ splits in at least six H-orbits, namely, $(\frac{\sqrt{37}}{1})^H, (\frac{\sqrt{37}}{-1})^H, (\frac{1+\sqrt{37}}{3})^H, (\frac{1+\sqrt{p}}{-3})^H, (\frac{-1+\sqrt{37}}{3})^H$ and $(\frac{-1+\sqrt{37}}{-3})^H$. By Theorem 2.8, $|(\frac{\sqrt{37}}{\pm1})^H|_{amb} = 36$ and $|(\frac{1+\sqrt{37}}{2})^H|_{amb} = 24$. By Lemma 1.1, $|\mathbb{Q}_1^*(\sqrt{37})| = 124$ and by Lemma 1.3, $|\mathbb{Q}_1^{**}(\sqrt{37})| = 56$. Using Theorem 2.8, $|\mathbb{Q}_1^{*\sim}(\sqrt{148})| = 2(124-56) = 136$. Since $|(\frac{\sqrt{p}}{1})^H|_{amb} + |(\frac{\sqrt{p}}{-1})^H|_{amb} = 72 < 136$. Therefore by Lemmas 3.2 and 3.3 at least four more H-orbits exists which are $(\frac{\pm1+\sqrt{37}}{\pm3})^H$. Also $|(\frac{1+\sqrt{37}}{3})^H|_{amb} = 16$. Here the sum of cardinalities of all six orbits is 144. Therefore we conclude that $\mathbb{Q}^{*\sim}(\sqrt{148})$ splits into exactly six H-orbits.

Example 3.8. Let $p = 577$. Then $\mathbb{Q}^{*\sim}(\sqrt{4p})$ splits into fourteen H-orbits, namely, $(\frac{577}{1})^H, (\frac{577}{-1})^H$, $(\frac{1+\sqrt{577}}{3})^H$, $(\frac{1+\sqrt{577}}{-3})^H$, $(\frac{-1+\sqrt{577}}{3})^H$, $(\frac{-1+\sqrt{577}}{-3})^H$, $(\frac{1+\sqrt{577}}{9})^H$, $(\frac{1+\sqrt{577}}{-9})^H$, $(\frac{-1+\sqrt{577}}{9})^H$, $(\frac{-1+\sqrt{577}}{-9})^H$, $(\frac{3+\sqrt{577}}{71})^H$, $(\frac{3+\sqrt{577}}{-71})^H$, $(\frac{-3+\sqrt{577}}{71})^H$ and $(\frac{-3+\sqrt{577}}{-71})^H$.

We conclude this paper with the following remarks.

Remark 3.9. Let $p \equiv 5(mod\ 8)$ such that $p-1$ is a perfect square. Then

1. $p \equiv 1$ or $5(mod\ 16)$ according as $\lfloor\sqrt{p}\rfloor \equiv 0$ or $2(mod\ 4)$.
2. Let $Y = \{\frac{\pm1+\sqrt{p}}{\pm c} \in \mathbb{Q}^*(\sqrt{p}) : c = 1, \lfloor\sqrt{p}\rfloor^2\}$ and $Z = \{\frac{\pm1+\sqrt{p}}{\pm c} \in \mathbb{Q}^*(\sqrt{p}) : c = 1, \frac{\lfloor\sqrt{p}\rfloor^2}{2}, \lfloor\sqrt{p}\rfloor\}$. Then $Y \cup x(Y) \subseteq (\frac{1+\sqrt{p}}{2})^H$ and $Z \cup x(Z) \subseteq (\frac{1+\sqrt{p}}{4})^H \cup (\frac{-1+\sqrt{p}}{4})^H$. □

Remark 3.10. Let $p \equiv 5(mod\ 8)$ such that $p-1 = \lfloor\sqrt{p}\rfloor^2 = (2q_1)^2$. Then

1. $|(\frac{1+\sqrt{p}}{4})^H|_{amb} = |(\frac{-1+\sqrt{p}}{4})^H|_{amb} = 2\sqrt{p-1}+4$.
2. $(\frac{1+\sqrt{p}}{4})^H = (\frac{1+\sqrt{p}}{q_1})^H$. □

Remark 3.11. It can be easily seen by Theorem 2.6; and Remark 2.5 that

1. 257 and 761 are the only primes $p \equiv 1(mod\ 8)$ and $p < 2011$ such that $o_H(p) = 12$.
2. 401 and 1601 are the only primes $p \equiv 1(mod\ 8)$ and $p < 2011$ such that $o_H(p) > 12$: For $p = 401$, $\mathbb{Q}''(\sqrt{p})$ splits into twenty H-orbits, namely, $(\frac{\sqrt{p}}{1})^H$, $(\frac{\sqrt{p}}{-1})^H$, $(\frac{1+\sqrt{p}}{2})^H$, $(\frac{-1+\sqrt{p}}{4})^H$, $(\frac{1+\sqrt{p}}{5})^H$, $(\frac{-1+\sqrt{p}}{5})^H$, $(\frac{1+\sqrt{p}}{-5})^H$, $(\frac{-1+\sqrt{p}}{-5})^H$, $(\frac{1+\sqrt{p}}{8})^H$, $(\frac{-1+\sqrt{p}}{8})^H$, $(\frac{1+\sqrt{p}}{10})^H$, $(\frac{-1+\sqrt{p}}{10})^H$, $(\frac{1+\sqrt{p}}{16})^H$, $(\frac{-1+\sqrt{p}}{16})^H$, $(\frac{1+\sqrt{p}}{25})^H$, $(\frac{1+\sqrt{p}}{-25})^H$, $(\frac{-1+\sqrt{p}}{25})^H$, $(\frac{-1+\sqrt{p}}{-25})^H$ and $(\frac{3+\sqrt{p}}{28})^H$.
   For $p = 1601$, $\mathbb{Q}''(\sqrt{p})$ splits into twenty eight H-orbits, namely, $(\frac{\sqrt{p}}{1})^H$, $(\frac{\sqrt{p}}{-1})^H$, $(\frac{1+\sqrt{p}}{2})^H$, $(\frac{1+\sqrt{p}}{4})^H$, $(\frac{-1+\sqrt{p}}{4})^H$, $(\frac{1+\sqrt{p}}{5})^H$, $(\frac{1+\sqrt{p}}{-5})^H$, $(\frac{-1+\sqrt{p}}{5})^H$, $(\frac{-1+\sqrt{p}}{-5})^H$, $(\frac{1+\sqrt{p}}{8})^H$, $(\frac{-1+\sqrt{p}}{8})^H$, $(\frac{1+\sqrt{p}}{10})^H$, $(\frac{-1+\sqrt{p}}{10})^H$, $(\frac{1+\sqrt{p}}{16})^H$, $(\frac{-1+\sqrt{p}}{16})^H$, $(\frac{1+\sqrt{p}}{25})^H$, $(\frac{1+\sqrt{p}}{-25})^H$, $(\frac{-1+\sqrt{p}}{25})^H$, $(\frac{-1+\sqrt{p}}{-25})^H$, $(\frac{1+\sqrt{p}}{32})^H$, $(\frac{1+\sqrt{p}}{50})^H$, $(\frac{-1+\sqrt{p}}{50})^H$, $(\frac{3+\sqrt{p}}{8})^H$, $(\frac{-3+\sqrt{p}}{8})^H$, $(\frac{3+\sqrt{p}}{199})^H$, $(\frac{3+\sqrt{p}}{-199})^H$, $(\frac{-3+\sqrt{p}}{199})^H$ and $(\frac{-3+\sqrt{p}}{-199})^H$.
3. The primes $p \equiv 5(mod\ 8)$ and $p < 2011$ such that $o_H(p) = 9$ are $101, 197, 269, 389, 557, 677, 701, 1301, 1613, 1949$ and 1973.
4. 1901 is the only prime $p \equiv 5(mod\ 8)$ and $p < 2011$ such that $o_H(p) > 12$.
5. $37, 349, 373, 709, 757, 829, 877, 997, 1213$ and 1861 are the primes $p \equiv 5(mod\ 8)$ and $p < 2011$ such that $o_H^{*\sim}(p) = 9$.

4 Conclusion

We have explored the action of hecke group $H(\lambda_4) = \langle S, T : S^2 = T^4 = 1\rangle$, on the subsets $\mathbb{Q}(\sqrt{m}) \setminus \mathbb{Q}$ of the real quadratic fields and different types of the orbits are introduced. The H-orbits of $\mathbb{Q}^{*\sim}(\sqrt{4p})$ with $o_H^{*\sim}(4p) > 4$ are investigated and the classification of H-orbits is given depending upon the nature of prime $p < 2011$, using modular arithmetic.

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