Applied Mathematics & Information Sciences
An International Journal

Unsteady flow of a viscous fluid between two oscillating cylinders

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Received: Jul 8, 2011; Revised Oct. 4, 2011; Accepted Oct. 16, 2011
Published online: 1 Sep. 2012

Abstract: The starting solutions corresponding to the motions of a Newtonian fluid, between two infinite circular cylinders, are determined by means of the Laplace and finite Hankel transforms. The general case, when both cylinders oscillate along their common axis and around the same axis, is considered. The solutions that have been obtained are presented as sum of the steady-state and transient solutions and satisfy all imposed initial and boundary conditions. Finally, the resulting shear stresses are also determined. The time required to attain the steady-state has been obtained by means of the graphical illustrations.

Keywords: Newtonian fluids, Velocity field, Exact solutions, Shear stress.

1. Introduction

The study on the flow of a viscous fluid in an annular region between two infinitely long coaxial circular cylinders is not only of fundamental theoretical interest but it also occurs in many applied problems. The flow between rotating cylinders, starting from rest, has applications in the food industry. The starting solutions for the motion of the second grade fluids due to longitudinal and torsional oscillations of a circular cylinder have been established in [1]. Other recent results regarding pulsatile or helical flows of non-Newtonian fluids have been obtained by Dapra and Scarpi [2] and Fetecau *et al* [3]. Vieru *et al* [4], by means of the Laplace transform and Cauchy's residue theorem, have determined the starting solutions for the oscillating motion of a Maxwell fluid.

The corresponding solutions for a Newtonian fluid, performing the same motion, are obtained from the general solutions as a particular case. Other interesting studies on the flow in pipe-like domains are in [5-10]. The stationary and nonstationary rotating Navier-Stokes equations with mixed boundary conditions are investigated in [11].

The aim of this paper is to study the flow of a Newtonian fluid in an annular pipe. More exactly, by means of the Laplace and finite Hankel transforms we establish the starting solutions corresponding to the motion of a Newtonian fluid between two infinite concentric circular cylinders. For completeness we consider the general case when both cylinders are oscillating. These solutions are presented as sum of the steady-state and transient solutions and, for high values of time, they tend to the steady-state solutions which are periodic in time. The solutions obtained in this paper can be used to make a comparison between flows of Newtonian and non-Newtonian fluids. The similar solutions obtained in [1] and [4] can be recovered as particular cases of the solutions that have been obtained here.

Finally, the profiles of velocities $w(r,t)$ and $u(r,t)$ are plotted as function on the radial coordinate for different values of time t and radii R_1 and R_2. The time required to attain the steady-state have been obtained by using the graphical illustrations of the transient solutions $w_t(r,t)$ and $u_t(r,t)$.

2. Statement of the problem

Consider that a Newtonian fluid at rest is situated in the annular region between two infinite straight circular cylinders of radii R_1 and $R_2(> R_1)$. At time zero, both cylinders begin to oscillate along their common axis ($r = 0$) and around the same axis.

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Due to the shear, the fluid is gradually moved, and its velocity is of the form

$$\mathbf{v} = \mathbf{v}(r,t) = \omega(r,t)\mathbf{e}_\theta + u(r,t)\mathbf{e}_z, \tag{1}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_z$ denote the unit vectors along the θ and z directions of the cylindrical coordinate system r, θ and z.

The governing equations, neglecting the body forces and the pressure gradients are [3, 4]

$$\frac{\partial \omega(r,t)}{\partial t} = \nu\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Big)\omega(r,t), \tag{2}$$

$$\frac{\partial u(r,t)}{\partial t} = \nu\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)u(r,t), \tag{3}$$

$$\tau_1(r,t) = \mu\Big(\frac{\partial}{\partial r} - \frac{1}{r}\Big)\omega(r,t), \tag{4}$$

$$\tau_2(r,t) = \mu\frac{\partial u(r,t)}{\partial r}, \tag{5}$$

where μ is the constant shear viscosity, $\nu = \mu/\rho$ is the kinematic viscosity of the fluid, ρ is the constant density of the fluid, $\tau_1(r,t) = S_{r\theta}(r,t)$ and $\tau_2(r,t) = S_{rz}(r,t)$ are the shear stresses which are different from zero.

The appropriate initial and boundary conditions are

$$\omega(r,0) = 0, \;\; u(r,0) = 0, \quad r \in [R_1, R_2], \tag{6}$$

$$\omega(R_1,t) = \Omega_1\sin(\alpha_1 t), \omega(R_2,t) = \Omega_2\sin(\alpha_2 t), t > 0, \tag{7}$$

$$u(R_1,t) = U_1\sin(\beta_1 t), u(R_2,t) = U_2\sin(\beta_2 t), t > 0, \tag{8}$$

where $\alpha_1, \; \alpha_2, \; \beta_1, \; \beta_2$ are the frequencies of the velocity of cylinders.

The uncoupled Eqs. (2), (3), with the initial and boundary conditions (6)–(8) can be solved in general by several methods. We shall use the Laplace and finite Hankel transforms [12, 13].

The boundary conditions can be considered in the more general forms

$$\omega(R_i,t) = A_i\sin(\alpha_i t) + B_i\cos(\beta_i t),$$

$$u(R_i,t) = C_i\sin(\gamma_i t) + D_i\cos(\delta_i t), \quad i = 1, 2. \tag{9}$$

In this paper we solve the problem (2)–(8). The problem (2)–(6) and (9) can be solved in the same manner which will be presented as follows.

3. Calculation of the velocity field

In order to determine the exact solutions of the problem (2), (3) with conditions (6)–(8), describing the motion of the fluid at low and high values of time after the start of the boundaries, the Laplace and finite Hankel transform method is used.

Applying the Laplace transform [12, 14] to Eqs. (2) and (3) and using the initial and boundary conditions (6)–(8) we obtain

$$q\bar{\omega}(r,q) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\bar{\omega}(r,q), \tag{10}$$

$$q\bar{u}(r,q) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{u}(r,q), \tag{11}$$

$$\bar{\omega}(R_1,q) = \frac{\alpha_1\Omega_1}{q^2+\alpha_1^2}, \;\; \bar{\omega}(R_2,q) = \frac{\alpha_2\Omega_2}{q^2+\alpha_2^2}, \tag{12}$$

$$\bar{u}(R_1,q) = \frac{\beta_1 U_1}{q^2+\beta_1^2}, \;\; \bar{u}(R_2,q) = \frac{\beta_2 U_2}{q^2+\beta_2^2}, \tag{13}$$

where $\bar{\omega}(r,q) = \int\limits_0^t \omega(r,t)e^{-qt}dt$ and $\bar{u}(r,q) = \int\limits_0^t u(r,t)e^{-qt}dt$, are the Laplace transforms of the functions $\omega(r,t)$ and $u(r,t)$, respectively.

In the following, let us denote by [12]

$$\bar{\omega}_H(r_{1n},q) = \int_{R_1}^{R_2} r\bar{\omega}(r,q)B_1(rr_{1n})dr, \tag{14}$$

the finite Hankel transform of $\bar{\omega}(r,q)$, where $r_{1n}, \;\; n = 1, 2, ...$, are the positive roots of the transcendental equation $B_1(R_2 r) = 0$ and

$$B_1(rr_{1n}) = J_1(rr_{1n})Y_1(R_1r_{1n}) - J_1(R_1r_{1n})Y_1(rr_{1n}), \tag{15}$$

respectively,

$$\bar{u}_H(r_{0n},q) = \int_{R_1}^{R_2} r\bar{u}(r,q)B_0(rr_{0n})dr, \tag{16}$$

the finite Hankel transform of $\bar{u}(r,q)$, where $r_{0n}, \;\; n = 1, 2, \ldots$, are the positive roots of the transcendental equation $B_0(R_2 r) = 0$ and

$$B_0(rr_{0n}) = J_0(rr_{0n})Y_0(R_1r_{0n}) - J_0(R_1r_{0n})Y_0(rr_{0n}). \tag{17}$$

In the above equations, $J_\nu(\cdot)$ and $Y_\nu(\cdot)$ are the Bessel functions of order ν of the first and second kind [15].

Integrating by parts and using the following formulae [15]

$$\frac{d}{dr}J_0[u(r)] = -J_1[u(r)]u'(r),$$

$$\frac{d}{dr}J_1[u(r)] = \Big\{J_0[u(r)] - \frac{1}{u(r)}J_1[u(r)]\Big\}u'(r), \tag{18}$$

$$J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z) = \frac{2}{\pi z},$$

we obtain

$$\int\limits_{R_1}^{R_2} r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\bar{\omega}(r,q)B_1(rr_{1n})dr$$
$$= -\frac{2}{\pi}\bar{\omega}(R_1,q) + \frac{2J_1(R_1r_{1n})}{\pi J_1(R_2r_{1n})}\bar{\omega}(R_2,q)$$
$$-r_{1n}^2\bar{\omega}_H(r_{1n},q), \tag{19}$$

$$\int\limits_{R_1}^{R_2} r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{u}(r,q)B_0(rr_{0n})dr$$
$$= -\frac{2}{\pi}\bar{u}(R_1,q) + \frac{2J_0(R_1r_{0n})}{\pi J_0(R_2r_{0n})}\bar{u}(R_2,q)$$
$$-r_{0n}^2\bar{u}_H(r_{0n},q). \tag{20}$$

Applying the finite Hankel transform to Eqs. (10) and (11) and using the boundary conditions (12) and (13) and Eqs. (19) and (20) we find

$$\bar{\omega}_H(r_{1n},q)=\Big(-\frac{2\nu}{\pi}\frac{\alpha_1\Omega_1}{q^2+\alpha_1^2}$$

$$+\frac{2\nu}{\pi}\frac{J_1(R_1r_{1n})}{J_1(R_2r_{1n})}\frac{\alpha_2\Omega_2}{q^2+\alpha_2^2}\Big)\frac{1}{q+\nu r_{1n}^2}, \quad (21)$$

and

$$\bar{u}_H(r_{0n},q)=\Big(-\frac{2\nu}{\pi}\frac{\beta_1U_1}{q^2+\beta_1^2}$$

$$+\frac{2\nu}{\pi}\frac{J_0(R_1r_{0n})}{J_0(R_2r_{0n})}\frac{\beta_2U_2}{q^2+\beta_2^2}\Big)\frac{1}{q+\nu r_{0n}^2}. \quad (22)$$

Now, for a more suitable presentation of the final results, we rewrite Eqs. (21) and (22) in the equivalent forms

$$\bar{\omega}_H(r_{1n},q)=-\frac{2\Omega_1}{\pi r_{1n}^2}\frac{\alpha_1}{q^2+\alpha_1^2}$$
$$+\frac{2\Omega_2J_1(R_1r_{1n})}{\pi r_{1n}^2J_1(R_2r_{1n})}\frac{\alpha_2}{q^2+\alpha_2^2}+\Big[\frac{2\alpha_1\Omega_1}{\pi r_{1n}^2}\frac{q}{q^2+\alpha_1^2}$$
$$-\frac{2\alpha_2\Omega_2J_1(R_1r_{1n})}{\pi r_{1n}^2J_1(R_2r_{1n})}\frac{q}{q^2+\alpha_2^2}\Big]\frac{1}{q+\nu r_{1n}^2}, \quad (23)$$

respectively,

$$\bar{u}_H(r_{0n},q)=-\frac{2U_1}{\pi r_{0n}^2}\frac{\beta_1}{q^2+\beta_1^2}$$
$$+\frac{2U_2J_0(R_1r_{0n})}{\pi r_{0n}^2J_0(R_2r_{0n})}\frac{\beta_2}{q^2+\beta_2^2}+\Big[\frac{2\beta_1U_1}{\pi r_{0n}^2}\frac{q}{q^2+\beta_1^2}$$
$$-\frac{2\beta_2U_2J_0(R_1r_{0n})}{\pi r_{0n}^2J_0(R_2r_{0n})}\frac{q}{q^2+\beta_2^2}\Big]\times\frac{1}{q+\nu r_{0n}^2}. \quad (24)$$

By a straightforward calculus we obtain the following function-finite Hankel transform pairs:

$$f(r)=\frac{R_1\Omega_1(R_2^2-r^2)}{r(R_2^2-R_1^2)},$$

$$f_H(r_{1n})=\int_{R_1}^{R_2}rf(r)B_1(rr_{1n})dr=-\frac{2\Omega_1}{\pi r_{1n}^2},$$

$$g(r)=\frac{R_2\Omega_2(r^2-R_1^2)}{r(R_2^2-R_1^2)},$$

$$g_H(r_{1n})=\int_{R_1}^{R_2}rg(r)B_1(rr_{1n})dr=\frac{2\Omega_2J_1(R_1r_{1n})}{\pi r_{1n}^2J_1(R_2r_{1n})},$$

$$h(r)=\frac{U_1}{ln(R_2/R_1)}ln(R_2/r), \quad (25)$$

$$h_H(r_{0n})=\int_{R_1}^{R_2}rh(r)B_0(rr_{0n})dr=-\frac{2U_1}{\pi r_{0n}^2},$$

$$k(r)=\frac{U_2}{ln(R_2/R_1)}ln(r/R_1),$$

$$k_H(r_{0n})=\int_{R_1}^{R_2}rk(r)B_0(rr_{0n})dr=\frac{2U_2J_0(R_1r_{0n})}{\pi r_{0n}^2J_0(R_2r_{0n})}.$$

Generally, for a finite Hankel transform

$a_H(r_{in})=\int_{R_1}^{R_2}ra(r)B_i(rr_{in})dr, i=0,1,$

the inverse Hankel transform is [12]

$$a(r)=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_{in}^2B_i(rr_{in})J_i^2(R_2r_{in})}{J_i^2(R_1r_{in})-J_i^2(R_2r_{in})}a_H(r_{in}), \quad (26)$$
$$i=0,1.$$

Applying the inverse Hankel transform to Eqs. (23) and (24) and using (25) and (26) we obtain the Laplace transforms of the functions $\omega(r,t)$ and $u(r,t)$ in the following forms

$$\bar{\omega}(r,q)=\frac{R_1\Omega_1(R_2^2-r^2)}{r(R_2^2-R_1^2)}\frac{\alpha_1}{q^2+\alpha_1^2}$$
$$+\frac{R_2\Omega_2(r^2-R_1^2)}{r(R_2^2-R_1^2)}\frac{\alpha_2}{q^2+\alpha_2^2}$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_{1n})B_1(rr_{1n})}{J_1^2(R_1r_{1n})-J_1^2(R_2r_{1n})}-\Big[\frac{\alpha_1\Omega_1q}{q^2+\alpha_1^2}$$
$$-\frac{\alpha_2\Omega_2J_1(R_1r_{1n})}{J_1(R_2r_{1n})}\frac{q}{q^2+\alpha_2^2}\Big]\frac{1}{q+\nu r_{1n}^2},$$

respectively,

$$\bar{u}(r,q)=\frac{U_1ln(R_2/r)}{ln(R_2/R_1)}\frac{\beta_1}{q^2+\beta_1^2}$$
$$+\frac{U_2ln(r/R_1)}{ln(R_2/R_1)}\frac{\beta_2}{q^2+\beta_2^2}$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_{0n})B_0(rr_{0n})}{J_0^2(R_1r_{0n})-J_0^2(R_2r_{0n})}\Big[\frac{\beta_1U_1q}{q^2+\beta_1^2}$$
$$-\frac{\beta_2U_2J_0(R_1r_{0n})}{J_0(R_2r_{0n})}\frac{q}{q^2+\beta_2^2}\Big]\frac{1}{q+\nu r_{0n}^2}. \quad (27)$$

Applying the inverse Laplace transform to Eqs. (27) and (28) and using the convolution theorem, we find the velocity fields

$$\omega(r,t)=\frac{R_1\Omega_1(R_2^2-r^2)}{r(R_2^2-R_1^2)}\sin(\alpha_1t)$$
$$+\frac{R_2\Omega_2(r^2-R_1^2)}{r(R_2^2-R_1^2)}\sin(\alpha_2t)$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_{1n})B_1(rr_{1n})}{J_1^2(R_1r_{1n})-J_1^2(R_2r_{1n})}$$
$$\Big[\alpha_1\Omega_1\int_0^t\cos\alpha_1(t-s)e^{-\nu sr_{1n}^2}ds-$$
$$-\alpha_2\Omega_2\frac{J_1(R_1r_{1n})}{J_1(R_2r_{1n})}\int_0^t\cos\alpha_2(t-s)e^{-\nu sr_{1n}^2}ds\Big],$$

and

$$u(r,t)=\frac{U_1 ln(R_2/r)}{ln(R_2/R_1)}\sin(\beta_1 t)$$
$$\frac{U_2 ln(r/R_1)}{ln(R_2/R_1)}\sin(\beta_2 t)$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_{0n})B_0(rr_{0n})}{J_0^2(R_1r_{0n})-J_0^2(R_2r_{0n})}$$
$$\left[\beta_1 U_1\int_0^t \cos\beta_1(t-s)e^{-\nu s r_{0n}^2}ds\right.$$
$$\left.-\beta_2U_2\frac{J_0(R_1r_{0n})}{J_0(R_2r_{0n})}\int_0^t\cos\beta_2(t-s)e^{-\nu sr_{0n}^2}ds\right]. \quad (28)$$

Using the following result

$$\int_0^t \cos a(t-s)e^{-bs}ds \quad (29)$$
$$=\frac{1}{a^2+b^2}(a\sin(at)+b\cos(at)-be^{-bt}),$$

we can write the functions $\omega(r,t)$ and $u(r,t)$ under the forms

$$\omega(r,t)=\omega_s(r,t)+\omega_t(r,t), \quad (30)$$

where

$$\omega_s(r,t)=\frac{R_1\Omega_1(R_2^2-r^2)}{r(R_2^2-R_1^2)}\sin(\alpha_1 t)$$
$$+\frac{R_2\Omega_2(r^2-R_1^2)}{r(R_2^2-R_1^2)}\sin(\alpha_2 t)$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_{1n})B_1(rr_{1n})}{J_1^2(R_1r_{1n})-J_1^2(R_2r_{1n})}$$
$$\left[\frac{\alpha_1\Omega_1}{\alpha_1^2+(\nu r_{1n}^2)^2}(\alpha_1\sin(\alpha_1 t)+\nu r_{1n}^2\cos(\alpha_1 t))\right.$$
$$-\frac{\alpha_2\Omega_2}{\alpha_2^2+(\nu r_{1n}^2)^2}\frac{J_1(R_1r_{1n})}{J_1(R_2r_{1n})}$$
$$\left.(\alpha_2\sin(\alpha_2 t)+\nu r_{1n}^2\cos(\alpha_2 t))\right],$$

$$\omega_t(r,t)=-\nu\pi\sum_{n=1}^{\infty}\frac{r_{1n}^2J_1^2(R_2r_{1n})B_1(rr_{1n})}{J_1^2(R_1r_{1n})-J_1^2(R_2r_{1n})}$$
$$\left[\frac{\alpha_1\Omega_1}{\alpha_1^2+(\nu r_{1n}^2)^2}-\frac{\alpha_2\Omega_2}{\alpha_2^2+(\nu r_{1n}^2)^2}\frac{J_1(R_1r_{1n})}{J_1(R_2r_{1n})}\right]$$
$$e^{-\nu t r_{1n}^2}, \quad (31)$$

respectively,

$$u(r,t)=u_s(r,t)+u_t(r,t), \quad (32)$$

where

$$u_s(r,t)=\frac{U_1 ln(R_2/r)}{ln(R_2/R_1)}\sin(\beta_1 t)$$
$$+\frac{U_2 ln(r/R_1)}{ln(R_2/R_1)}\sin(\beta_2 t)$$
$$+\pi\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_{0n})B_0(rr_{0n})}{J_0^2(R_1r_{0n})-J_0^2(R_2r_{0n})}$$
$$\left[\frac{\beta_1U_1}{\beta_1^2+(\nu r_{0n}^2)^2}(\beta_1\sin(\beta_1 t)+\nu r_{0n}^2\cos(\beta_1 t))\right.$$
$$-\frac{\beta_2U_2}{\beta_2^2+(\nu r_{0n}^2)^2}\frac{J_0(R_1r_{0n})}{J_0(R_2r_{0n})}$$
$$\left.(\beta_2\sin(\beta_2 t)+\nu r_{0n}^2\cos(\beta_2 t))\right].$$

$$u_t(r,t)=-\nu\pi\sum_{n=1}^{\infty}\frac{r_{0n}^2J_0^2(R_2r_{0n})B_0(rr_{0n})}{J_0^2(R_1r_{0n})-J_0^2(R_2r_{0n})}$$
$$\left[\frac{\beta_1U_1}{\beta_1^2+(\nu r_{0n}^2)^2}-\frac{\beta_2U_2}{\beta_2^2+(\nu r_{0n}^2)^2}\frac{J_0(R_1r_{0n})}{J_0(R_2r_{0n})}\right] \quad (33)$$
$$e^{-\nu t r_{0n}^2}.$$

The starting solutions (32) and (35), presented as the sum

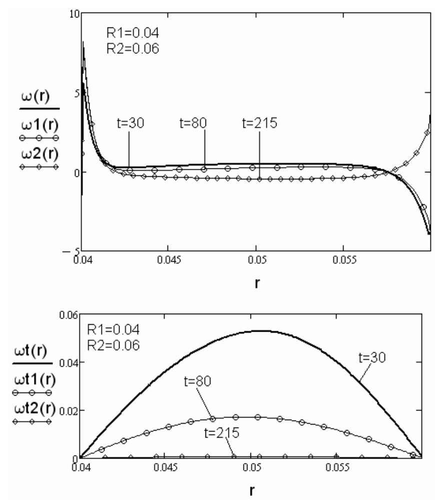

Figure 1. The variation of ω and ωt with r for different values of t and R1=0.04, R2=0.06

of the steady-state solutions $\omega_s(r,t),\ u_s(r,t)$ and the transient solutions $\omega_t(r,t),\ u_t(r,t)$ satisfy both the partial differential equations (2) and (3) and all imposed initial and boundary conditions. They describe the motion of the fluid for any time after its initiation. After the time in which the transients disappear, the starting solutions tend to the steady-state solutions (33) and (36), which are periodic in time.

4. Calculation of the shear stress

The shear stresses $\tau_1(r,t)=S_{r\theta}(r,t)$ and $\tau_2(r,t)=S_{rz}(r,t)$ can be determined from (4), (5) and (32)–(37). After a straightforward calculation we find

$$\tau_1(r,t)=\tau_{1s}(r,t)+\tau_{1t}(r,t), \quad (34)$$

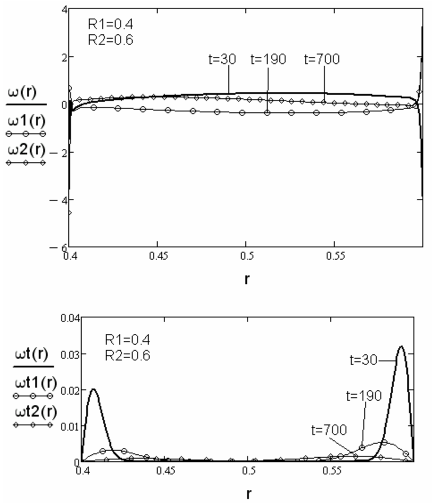

Figure 2. The variation of ω and ω_t with r for different values of t and R1=0.4, R2=0.6

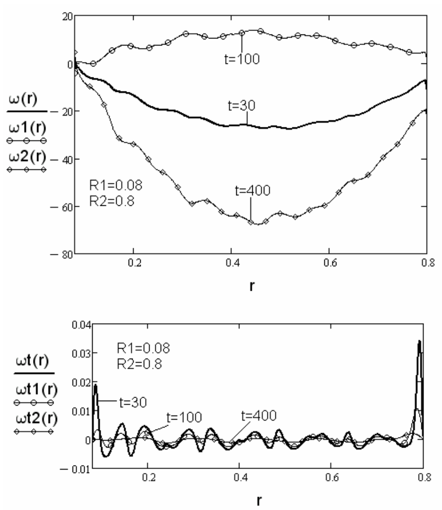

Figure 3. The variation of ω and ω_t with r for different values of t and R1=0.08, R2=0.8

where

$$\tau_{1s}(r,t) = -\frac{2\mu R_1 R_2^2 \Omega_1}{r^2(R_2^2 - R_1^2)}\sin(\alpha_1 t) + \frac{2\mu R_1^2 R_2 \Omega_2}{r^2(R_2^2 - R_1^2)}\sin(\alpha_2 t) - \pi\mu \sum_{n=1}^{\infty} \frac{r_{1n} J_1^2(R_2 r_{1n}) B_{21}(r r_{1n})}{J_1^2(R_1 r_{1n}) - J_1^2(R_2 r_{1n})} \left[\frac{\alpha_1 \Omega_1}{\alpha_1^2 + (\nu r_{1n}^2)^2}(\alpha_1 \sin(\alpha_1 t) + \nu r_{1n}^2 \cos(\alpha_1 t)) - \frac{\alpha_2 \Omega_2}{\alpha_2^2 + (\nu r_{1n}^2)^2}\frac{J_1(R_1 r_{1n})}{J_1(R_2 r_{1n})} (\alpha_2 \sin(\alpha_2 t) + \nu r_{1n}^2 \cos(\alpha_2 t))\right], \tag{35}$$

$$\tau_{1t}(r,t) = \nu\pi\mu \sum_{n=1}^{\infty} \frac{r_{1n}^3 J_1^2(R_2 r_{1n}) B_{21}(r r_{1n})}{J_1^2(R_1 r_{1n}) - J_1^2(R_2 r_{1n})} \left[\frac{\alpha_1 \Omega_1}{\alpha_1^2 + (\nu r_{1n}^2)^2} - \frac{\alpha_2 \Omega_2}{\alpha_2^2 + (\nu r_{1n}^2)^2}\frac{J_1(R_1 r_{1n})}{J_1(R_2 r_{1n})}\right] e^{-\nu t r_{1n}^2},$$

and

$$B_{21}(r r_{1n}) = J_2(r r_{1n}) Y_1(R_1 r_{1n}) - J_1(R_1 r_{1n}) Y_2(r r_{1n}), \tag{36}$$

respectively,

$$\tau_2(r,t) = \tau_{2s}(r,t) + \tau_{2t}(r,t), \tag{37}$$

where

$$\tau_{2s}(r,t) = -\frac{\mu U_1}{r ln(R_2/R_1)}\sin(\beta_1 t) + \frac{\mu U_2}{r ln(R_2/R_1)}\sin(\beta_2 t) - \pi\mu \sum_{n=1}^{\infty} \frac{r_{0n} J_0^2(R_2 r_{0n}) B_{10}(r r_{0n})}{J_0^2(R_1 r_{0n}) - J_0^2(R_2 r_{0n})} \left[\frac{\beta_1 U_1}{\beta_1^2 + (\nu r_{0n}^2)^2}(\beta_1 \sin(\beta_1 t) + \nu r_{0n}^2 \cos(\beta_1 t)) - \frac{\beta_2 U_2}{\beta_2^2 + (\nu r_{0n}^2)^2}\frac{J_0(R_1 r_{0n})}{J_0(R_2 r_{0n})} (\beta_2 \sin(\beta_2 t) + \nu r_{0n}^2 \cos(\beta_2 t))\right]. \tag{38}$$

$$\tau_{2t}(r,t) = \nu\pi\mu \sum_{n=1}^{\infty} \frac{r_{0n}^3 J_0^2(R_2 r_{0n}) B_{10}(r r_{0n})}{J_0^2(R_1 r_{0n}) - J_0^2(R_2 r_{0n})} \left[\frac{\beta_1 U_1}{\beta_1^2 + (\nu r_{0n}^2)^2} - \frac{\beta_2 U_2}{\beta_2^2 + (\nu r_{0n}^2)^2}\frac{J_0(R_1 r_{0n})}{J_0(R_2 r_{0n})}\right] e^{-\nu t r_{0n}^2}. \tag{39}$$

and

$$B_{10}(r r_{0n}) = J_1(r r_{0n}) Y_0(R_1 r_{0n}) - J_0(R_1 r_{0n}) Y_1(r r_{0n}). \tag{40}$$

5. Conclusion

The main purpose of this paper is to provide exact solutions for the velocity field $\mathbf{v}(r,t) = \omega(r,t)\mathbf{e}_\theta + u(r,t)\mathbf{e}_z$ and the shear stresses $\tau_1 = S_{r\theta}(r,t)$ and $\tau_2 = S_{rz}(r,t)$, corresponding to the non-steady flow of a Newtonian fluid between two circular cylinders. The motion is produced by the longitudinal and torsional oscillations of both cylinders. These solutions are obtained by means of the Laplace and finite Hankel transforms and are presented as a sum of

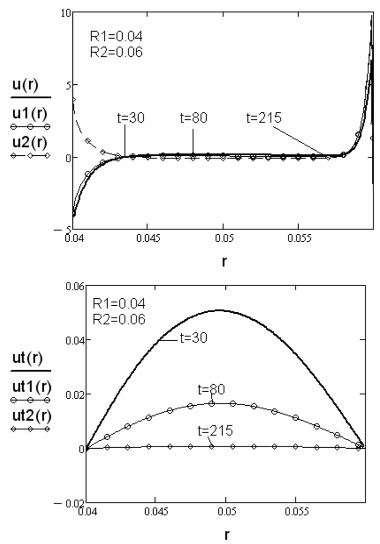

Figure 4. The variation of u and u_t with r for different values of t and R1=0.04, R2=0.06

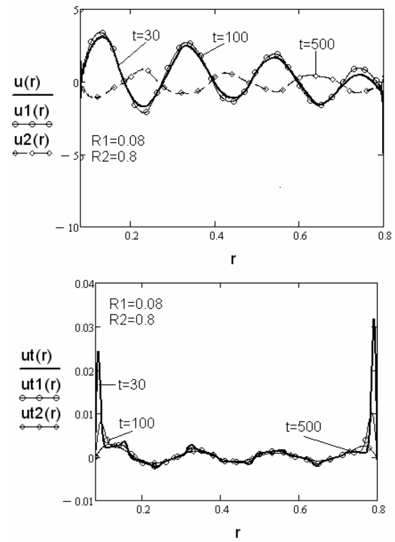

Figure 6. The variation of u and u_t with r for different values of t and R1=0.08, R2=0.8

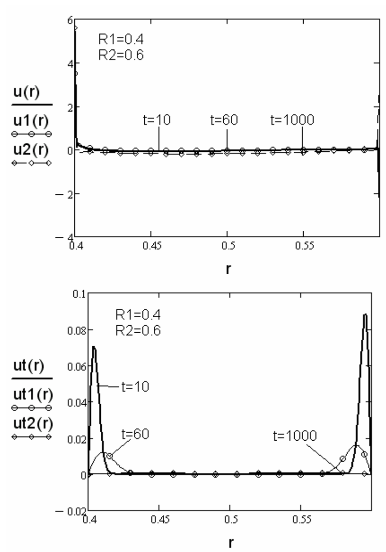

Figure 5. The variation of u and u_t with r for different values of t and R1=0.4, R2=0.6

the steady-state and transient solutions. They describe the motion of the fluid for low and high values of the time. For high values of t, when the transient solutions disappear, the motion is described in the corresponding steady-state solutions, which are periodic in time. Straightforward computation shows that $w(r,t)$ and $u(r,t)$ given by (29) and (30) satisfy both the associate partial differential equations (2) and (3) and all imposed initial and boundary conditions. Finally, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity fields $w(r,t)$ and $u(r,t)$ given by Eqs. (32–34) and (35–37) have been drawn against r for different values of t. In order to determine the time required to attain the steady-state the diagrams of the transient velocities $w_t(r,t)$ and $u_t(r,t)$ given by Eqs. (34) and (37) have been drawn against for different values of t. From these diagrams we can determine those values of t after which this transient components of the velocity can be neglected.

Figs.1–3 show the influence of time and radii R_1, R_2 on the circumferential velocity $w(r,t)$ and the transient circumferential velocity $w_t(r,t)$. Figs. 4–6 show the influence of time and radii R_1, R_2 on the longitudinal velocity $u(r,t)$ and the transient longitudinal velocity $u_t(r,t)$. From these figures it is clearly seen that the time required to attain the steady-state have lower values if the radii R_1 and R_2 have low values. This time increases if R_1 or R_2 increases.

The graphics have been plotted for $\Omega_1 = 5$, $\Omega_2 = 2$, $U_1 = 6$, $U_2 = 4$, $\mu = 0.915 * 10^{-}3$ (the distilled water at $24°$), $\alpha_1 = 2, \alpha_2 = 1, \beta_1 = 2, \beta_1 = 1, \rho = 1000$.

The units of the parameters in Figs.1–6 are taken from IS units and the roots r_{0n} and r_{1n} have been approximated with $\frac{n\pi}{R_2-R_1}$ [15].

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