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A Brief Investigation of a Stieltjes Transform in a Class of Boehmians

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Abstract: Various integral transforms have been extended to various spaces of Boehmians. In this article, we discuss the Stieltjes transform in a class of Boehmians. The presented transform preserves many properties of the classical transform in the space of Boehmians.

Keywords: Stieltjes Transform; Schwartz Space; Boehmian; Weak Topology.
[**AMS Classif:**] Primary 54C40, 14E20; Secondary 46E25, 20C20

1 Introduction

Let $p\,(p\neq 0)$ be a nonnegative complex number. Then, for all τ in the plant cut, the classical Stieltjes transform is defined by $[10]$

$$\boldsymbol{\xi}_{st}^{al}\phi\,(\tau) := \int_0^\infty \frac{\phi\,(x)}{(\tau+x)^p}\,\mathrm{d}x. \tag{1}$$

If $p>0$, $\phi\,(x)$ is a locally integrable on $\mathscr{I}, 0<x<\infty$, the integral (1) converges, and the limits $\phi\,(x\pm 0)$ exist, then the transform inversion formula is recovered from (1) as

$$\tfrac{1}{2}\left(\phi\,(x+0)+\phi\,(x-0)\right) = \lim_{\eta\to 0+} \tfrac{-1}{2\pi i}\int_0^x \mathrm{d}x \int_{c_{\eta x}} (x+\tau)^{p-1}\,\boldsymbol{\xi}_{st}^{al}\phi\,(\tau)\,\mathrm{d}\tau.$$

where $c_{\eta x}$ is a contour in the plane cut from $-x-i\eta$ to $x+i\eta$. In $[18, \text{Section } 4.2]$, Zemanian has extended the Stieltjes transform to the linear space, $\acute{\mathscr{M}}_{a,b}$, of generalized functions, which are distributions in the sense of Zemanian, when assigned the weak topology. In a different perfomance, Pandy in $[10]$ has extended the theory of Stieltjes transform to the dual space $\mathscr{S}'_\alpha\,(\mathscr{I})$, $\mathscr{S}_\alpha\,(\mathscr{I})$ of all infinitely smooth complex-valued functions $\phi\,(x)$ over $\mathscr{I}$ where

$$\gamma_k\,(\phi) = \sup_{0<x<\infty}(1+x)^\alpha \left|x^k D_x^k \phi\,(x)\right| < \infty, \tag{2}$$

for any fixed $k\,(k=0,1,2,\ldots)$ and α being arbitrary but fixed real number, $D_x \equiv \dfrac{\mathrm{d}}{\mathrm{d}x}$. The topology of $\mathscr{S}_\alpha\,(\mathscr{I})$ is generated by the sequence of seminorms $(\gamma_k\,(\phi))_0^\infty$ which makes $\mathscr{S}_\alpha\,(\mathscr{I})$ a locally convex Hausdörff topological vector space, $\boldsymbol{\tau}\,(\mathscr{I}) \subset \mathscr{S}_\alpha\,(\mathscr{I}), \boldsymbol{\tau}\,(\mathscr{I})$ is the Schwartz space of test functions of compact support. The topology of $\boldsymbol{\tau}\,(\mathscr{I})$ is, indeed, stronger than that induced on $\boldsymbol{\tau}\,(\mathscr{I})$ by $\mathscr{S}'_\alpha\,(\mathscr{I})$, and the restriction of any element of $\mathscr{S}_\alpha\,(\mathscr{I})$ to $\boldsymbol{\tau}\,(\mathscr{I})$ is in $\boldsymbol{\tau}'\,(\mathscr{I})$, the dual space of Schwartz distributions.

For a non-negative $\tau, \tau\neq 0, (\tau+x)^{-p} \in \mathscr{S}_\alpha\,(\mathscr{I}), Rep>\alpha$, the distributional Stieltjes transform $\boldsymbol{\xi}_{st}^{di}\,(\tau)$ of $\phi \in \mathscr{S}'_\alpha\,(\mathscr{I})$ is therefore defined by

$$\boldsymbol{\xi}_{st}^{di}\phi\,(\tau) \overset{\Delta}{=} \left\langle \phi\,(x), (\tau+x)^{-p}\right\rangle, \tag{3}$$

where, $Rep>\alpha, \tau$ belongs to the complex plane cut along the negative real axis including the origion.

Theorem 1 $\left(\text{Analyticity of } \boldsymbol{\xi}_{st}^{di}\right)$ Let $\phi \in \mathscr{S}'_\alpha\,(\mathscr{I})$, $Rep>\alpha$; then $\boldsymbol{\xi}_{st}^{di}\phi\,(\tau)$ is differentiable and

$$D_\tau^k \boldsymbol{\xi}_{st}^{di}\phi\,(\tau) = \left\langle \phi\,(x), (-1)^k (p)_k (\tau+x)^{-(p+k)}\right\rangle, \tag{4}$$

where $(p)_k = p\,(p+1)\ldots(p+k-1)$.

On a more general spaces than distributions, Roopkumar in $[15]$, has discussed some variant of the Stieltjes transform on certain space of Boehmians. The Stieltjes transform of a Boehmian is a usual Boehmian

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and is well defined, consistent with the distributional Stieltjes transform and has all desired properties. In this paper, we discuss the a generalization of Stieltjes transform in [15] and obtain some of its properties.

This paper is organized as follows: the Stieltjes transform is reviewed in Section 1. Section 2 presents the general construction of Boehmian spaces. The Boehmian spaces are described in Section 3. The extended Stieltjes transform and its properties are obtained in Section 4.

More about Stieltjes transforms is available in $[8,9,10,13,14,15]$.

2 Stieltjes Transforms for Boehmians

One of the most youngest generalizations of functions, and more particularly of distributions, is the theory of Boehmians. The idea of construction of Boehmians was initiated by the concept of regular operators introduced by Boehme [6]. Regular operators form a subalgebra of the field of Mikusinski operators and they include only such functions whose support is bounded from the left. In a concrete case, the space of Boehmians contains all regular operators, all distributions and some objects which are neither operators nor distributions.

The construction of Boehmians is similar to the construction of the field of quotients and in some cases, it gives just the field of quotients. On the other hand, the construction is possible where there are zero divisors, such as the space C (the space of continous functions) with the pointwise additions and convolution. See; for further construction, $[1-7,15-17]$.

Before we proceed we introduce the following definition.

Definition 2 (Main Definition) Let ϕ and ω be integrable functions defined on $(0,\infty)$. Between ϕ and ω define the mapping $\curlywedge$ by the integral

$$(\phi \curlywedge \omega)(\tau) = \int_0^\infty \frac{\phi\left(\tau y^{-1}\right)}{y^p}\omega(y)\,\mathrm{d}y. \tag{5}$$

Definition 3 Let ϕ and ω be integrable functions defined on $(0,\infty)$. The Mellin type convolution product between ϕ and ω is given by [18] as

$$(\phi \curlyvee \omega)(x) = \int_0^\infty \frac{\phi\left(xy^{-1}\right)}{y}\omega(y)\,\mathrm{d}y. \tag{6}$$

The product $\curlyvee$ satisfies the following properties [13]

$(i)\,(\phi \curlyvee \psi)(t) = (\psi \curlyvee \phi)(t)$;

$(ii)\,((\phi+\psi)\curlyvee \varphi)(t) = (\phi \curlyvee \varphi)(t) + (\psi \curlyvee \varphi)(t)$;

$(iii)\,(\alpha\phi \curlyvee \psi)(t) = \alpha(\psi \curlyvee \phi)(t), \alpha$ is a complex number ;

$(vi)\,((\phi \curlyvee \psi)\curlyvee \varphi)(t) = (\phi \curlyvee (\psi \curlyvee \varphi))(t)$.

We refer to [18] for more details.

In what follows, we construct two spaces of Boehmians by aid of the equations (5) and (6). The space obtained from (5) is denoted by

$$\boldsymbol{b}\left((\mathscr{S}_\alpha,\curlyvee),(\boldsymbol{\tau},\curlyvee),\Delta,\mathscr{I}\right),$$

where Δ is the collection of delta sequances (δ_n) satisfying the properties

$\Delta_1 : \int_0^\infty \delta_n(x)\,\mathrm{d}x = 1$;

$\Delta_2 : |\delta_n(x)| < M, M \in \mathbb{R}, M > 0$;

$\Delta_3 : \lim supp\delta_n(x) \subseteq [a_n,b_n], b_n,a_n \to 0$ as $n \to \infty$;

$\Delta_4 : (\delta_n(x)) \in \boldsymbol{\tau}(\mathscr{I})$.

The space obtained from (5) and (6) will be denoted by

$$\boldsymbol{b}\left((\mathscr{S}_\alpha,\curlywedge),(\boldsymbol{\tau},\curlyvee),\Delta,\mathscr{I}\right).$$

Let us initiate the construction of $\boldsymbol{b}\left((\mathscr{S}_\alpha,\curlywedge),(\boldsymbol{\tau},\curlyvee),\Delta,\mathscr{I}\right)$ since the construction of the space $\boldsymbol{b}\left((\mathscr{S}_\alpha,\curlyvee),(\boldsymbol{\tau},\curlyvee),\Delta,\mathscr{I}\right)$ follows from similar technique.

Theorem 4 Let $\phi,\psi \in \mathscr{S}_\alpha(\mathscr{I})$. Then, we have

$$\boldsymbol{\xi}_{st}^{al}(\phi \curlyvee \omega)(\tau) = \left(\boldsymbol{\xi}_{st}^{al}\phi \curlywedge \omega\right)(\tau). \tag{7}$$

Let ϕ,ψ be arbitrary in the space $\mathscr{S}_\alpha(\mathscr{I})$; then, by using (1), we write

$$\boldsymbol{\xi}_{st}^{al}(\phi \curlyvee \omega)(\tau) = \int_0^\infty\int_0^\infty \frac{\phi\left(xy^{-1}\right)}{y}\omega(y)\,\mathrm{d}y\frac{1}{(\tau+x)^p}\mathrm{d}x.$$

By change of variables, $xy^{-1} = \xi$, $\mathrm{d}x = y\mathrm{d}\xi$, we get that

$$\boldsymbol{\xi}_{st}^{al}(\phi \curlyvee \omega)(\tau) = \int_0^\infty\int_0^\infty \phi(\xi)\,\omega(y)\frac{1}{(\tau+\xi y)^p}\mathrm{d}y\mathrm{d}\xi.$$

By using Fubini's theorem, we obtain

$$\boldsymbol{\xi}_{st}^{al}(\phi \curlyvee \omega)(\tau) = \int_0^\infty \frac{\omega(y)}{y^p}\int_0^\infty \phi(\xi)\frac{1}{(\tau y^{-1}+\xi)^p}\mathrm{d}\xi\,\mathrm{d}y. \tag{8}$$

Hence, Equation 8 can be written as

$$\boldsymbol{\xi}_{st}^{al}(\phi \curlyvee \omega)(\tau) = \int_0^\infty \frac{\omega(y)}{y^p}\boldsymbol{\xi}_{st}^{al}\phi\left(\tau y^{-1}\right)\mathrm{d}y.$$

This completes the proof of the theorem.

Theorem 5 Let $\phi \in \mathscr{S}_\alpha(\mathscr{I})$ and $\varphi,\psi \in \boldsymbol{\tau}(\mathscr{I})$. Then, we have

$$(\phi \curlywedge (\varphi \curlyvee \psi))(\tau) = ((\phi \curlywedge \varphi)\curlywedge \psi)(\tau), \tag{9}$$

for all $\tau \in \mathscr{I}$.

Let $\phi \in \mathscr{S}_\alpha(\mathscr{I})$ and $\psi,\varphi \in \boldsymbol{\tau}(\mathscr{I})$ be arbitrary. Then using (5) and (6) and Fubini's theorem yield

$$(\phi \curlywedge (\varphi \curlyvee \psi))(\tau) = \int_0^\infty \frac{\phi\left(\tau y^{-1}\right)}{y^p}\left(\int_0^\infty \frac{\varphi\left(yt^{-1}\right)}{t}\psi(t)\,\mathrm{d}t\right)\mathrm{d}y$$

$$\text{i.e.} \qquad = \int_0^\infty \frac{\phi\left(\tau y^{-1}\right)}{y^p}\int_0^\infty \frac{\varphi\left(yt^{-1}\right)}{t}\psi(t)\,\mathrm{d}t\mathrm{d}y. \tag{1}$$

Applying change of variables in (10) and rearranging imply

$$(\phi \curlywedge (\varphi \curlyvee \psi))(\tau) = \int_0^\infty \frac{\psi(t)}{t^p} \int_0^\infty \frac{\phi\left(\left(\tau t^{-1}\right)w^{-1}\right)}{w^p} \varphi(w)\, \mathrm{d}w \mathrm{d}t. \quad (11)$$

By (6), Equation 11 is written as

$$(\phi \curlywedge (\varphi \curlyvee \psi))(\tau) = \int_0^\infty \frac{(\phi \curlywedge \varphi)\left(\tau t^{-1}\right)}{t^p} \psi(t)\, \mathrm{d}t.$$

Once again, by (6) we complete the proof of the theorem.

Theorem 6 Let $\phi \in \mathscr{S}_\alpha(\mathscr{I})$; then $\boldsymbol{\xi}_{st}^{al}\phi \in \mathscr{S}_\alpha(\mathscr{I})$.

Detailed proof is as follows : Let $\phi \in \mathscr{S}_\alpha(\mathscr{I})$, then by using (1) and (2) we get

$$\gamma_k(\phi) \le \sup_{\mathscr{I}} (1+\tau)^\alpha \left|\tau^k\right| \int_0^\infty \left| D_\tau^k \frac{\phi(x)}{(\tau+x)^p}\right| \mathrm{d}x.$$

Hence, computations yield

$$\begin{aligned}\gamma_k(\phi) &\le \sup_{\mathscr{I}} (1+\tau)^\alpha \left|\tau^k\right| \int_0^\infty \left|\phi(x)(-1)^k(p)_k(\tau+x)^{-(p+k)}\right| \mathrm{d}x \\ &\le \sup_{\mathscr{I}} (1+\tau)^\alpha \left|\tau^{-p}\right| \int_0^\infty \frac{p(p+k)}{\left|(\tau+x)^{(p+k)}\right|} |\phi(x)|\, \mathrm{d}x,\end{aligned}$$

where $(p)_k = p(p+1)\ldots(p+k-1) < p(p+k)$.

Hence, the hypothesis that $\phi \in \mathscr{S}_\alpha(\mathscr{I})$ implies that $\gamma_k(\phi) < \infty$ for any fixed $k, k = 0, 1, 2, \ldots$.

This complete the proof of the theorem.

Let us follow the abstract construction of $\boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$.

Theorem 7 Let $\phi \in \mathscr{S}_\alpha(\mathscr{I})$ and $\varphi \in \boldsymbol{\tau}(\mathscr{I})$. Then, $\phi \curlywedge \varphi \in \mathscr{S}_\alpha(\mathscr{I})$.

Proof For $\phi \in \mathscr{S}_\alpha(\mathscr{I})$ and $\varphi \in \boldsymbol{\tau}(\mathscr{I})$ and $\lim supp\varphi \subset (a,b), 0 < a < b$. Then, we have

$$\begin{aligned}\sup_{\mathscr{I}} (1+\tau)^\alpha \left|\tau^k D_\tau^k (\phi \curlywedge \varphi)(\tau)\right| &= \sup_{\mathscr{I}} \int_a^b (1+\tau)^\alpha \left|\tau^k D_\tau^k \frac{\phi\left(\tau y^{-1}\right)}{y^p}\right| |\varphi(y)|\, \mathrm{d}y \\ \text{i.e.} \qquad &\le A \int_a^b |\varphi(y)|\, \mathrm{d}y,\end{aligned}$$

where A is certain positive constant.

Hence, the fact that $\varphi \in \boldsymbol{\tau}(\mathscr{I})$ completes the proof of the theorem.

Theorem 8 Let $\phi_1, \phi_2 \in \mathscr{S}_\alpha(\mathscr{I})$ and $\phi_1, \phi_2 \in \boldsymbol{\tau}(\mathscr{I})$. Then, we have

(1) $(\phi_1 \curlywedge \phi_2)(y) = (\phi_2 \curlywedge \phi_1)(y)$.
(2) $((\phi_1 + \phi_2) \curlywedge \varphi_1)(y) = (\phi_1 \curlywedge \varphi_1)(y) + (\phi_2 \curlywedge \varphi_1)(y)$.
(3) Let $\phi_n \to \phi$ in $\mathscr{S}_\alpha(\mathscr{I})$ and $\varphi \in \boldsymbol{\tau}(\mathscr{I})$; then $\phi_n \curlywedge \varphi \to \phi \curlywedge \varphi$ as $n \to \infty$.

Proof of Equations 1,2 and 3 follows from simple integration.

Hence the theorem is completely proved.

Theorem 9 Let $\phi \in \mathscr{S}_\alpha(\mathscr{I})$ and $(\delta_n) \in \Delta$; then $(\phi \curlywedge \delta_n)(\tau) \to \phi(\tau)$ as $n \to \infty$.

Proof Let the hypothesis of the theorem satisfies for some ϕ and (δ_n). Then, for a compact subset K of $\mathscr{I}$ and by using the property Δ_1 of delta sequences we have that

$$\begin{aligned}\gamma_k(\phi \curlywedge \delta_n - \phi) &= \sup_{\mathscr{I}} (1+\tau)^\alpha \left|\tau^k D_\tau^k (\phi \curlywedge \delta_n - \phi)(\tau)\right| \\ &\le \sup_{\mathscr{I}} (1+\tau)^\alpha \int_{a_n}^{b_n} \left|\tau^k D_\tau^k \left(\frac{\phi\left(\tau y^{-1}\right)}{y^p} - \phi(\tau)\right)\right| |\delta_n(y)|\, \mathrm{d}y.\end{aligned}$$

Since the mapping $h(y) = \dfrac{\phi\left(\tau y^{-1}\right)}{y^p} - \phi(\tau)$ is a member of $\mathscr{S}_\alpha(\mathscr{I})$ for every choice of $y \in \mathscr{I}$ it follows that

$$\gamma_k(\phi \curlywedge \delta_n - \phi) \le \gamma_k(h) \int_{a_n}^{b_n} |\delta_n(y)|\, \mathrm{d}y.$$

By the property Δ_2 of delta sequences we write

$$\gamma_k(\phi \curlywedge \delta_n - \phi) \le M\gamma_k(h)(b_n - a_n)$$

where M in certain positive constant.

Hence, the property Δ_3 of delta sequences implies

$$\phi \curlywedge \delta_n - \phi \to 0$$

as $n \to \infty$ in the topology of $\mathscr{S}_\alpha(\mathscr{I})$.

This completes the proof of the theorem.

Thus, the space $\boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ describes a Boehmian space.

We define the sum and multiplication by a scalar in $\boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ as :

$$\left[\frac{\phi_n}{\delta_n}\right] + \left[\frac{g_n}{\varphi_n}\right] = \left[\frac{\phi_n \curlywedge \varphi_n + g_n \curlywedge \delta_n}{(\delta_n) \curlyvee \varphi_n}\right]$$

and

$$\rho\left[\frac{\phi_n}{\delta_n}\right] = \left[\frac{\rho\phi_n}{\delta_n}\right],$$

ρ is a complex number.

Between $\mathscr{S}_\alpha(\mathscr{I})$ and $\boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ there is a canonical embedding expressed as

$$x \to \frac{x \curlywedge \delta_n}{\delta_n} \text{ as } n \to \infty.$$

The operation $\curlywedge$ can be extended to $\boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I}) \times \mathscr{S}_\alpha(\mathscr{I})$ by

$$\frac{x_n}{\delta_n} \curlywedge t = \frac{x_n \curlywedge t}{\delta_n}.$$

Convergence in $\boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ is defined as follows :

δ convergence A sequence $(\beta_n) \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ is said to be δ

convergent to $\beta \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ if there can be a delta sequence (δ_n) such that

$$(\beta_n \curlywedge \delta_n), (\beta \curlywedge \delta_n) \in \mathscr{S}_\alpha(\mathscr{I})$$

and

$$(\beta_n \curlywedge \delta_k) \to (\beta \curlywedge \delta_k)$$

as $n \to \infty$, in $\mathscr{S}_\alpha(\mathscr{I}), \forall k, n \in \mathbb{N}$.

Δ **convergence** A sequence $(\beta_n) \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ is said to be Δ **convergent** to $\beta \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ if there can be a $(\delta_n) \in \Delta$ such that

$$(\beta_n - \beta) \curlywedge \delta_n \in \mathscr{S}_\alpha(\mathscr{I}),$$

$$\forall n \in \mathbb{N}, \text{ and } (\beta_n - \beta) \curlywedge \delta_n \to 0 \text{ as } n \to \infty \text{ in } \mathscr{S}_\alpha(\mathscr{I}).$$

Similary, the space $\boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ can be established.

Addition, scalar multiplications and $\curlyvee$ in the space $\boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ is defined as :

$$\left[\frac{\phi_n}{\delta_n}\right] + \left[\frac{g_n}{\varphi_n}\right] = \left[\frac{\phi_n \curlyvee \varphi_n + g_n \curlyvee \delta_n}{\delta_n \curlyvee \varphi_n}\right]$$

and

$$\rho \left[\frac{\phi_n}{\delta_n}\right] = \left[\frac{\rho \phi_n}{\delta_n}\right]$$

and

$$\frac{x_n}{\delta_n} \curlyvee t = \frac{x_n \curlyvee t}{\delta_n},$$

ρ is a complex number, respectively.

3 ξ_{st}^{ge} Transform of Boehmians

Definition 10 Let $\left[\frac{(\phi_n)}{(\delta_n)}\right] \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$. Then, we define the extension of $\boldsymbol{\xi}_{st}^{al}$ as

$$\boldsymbol{\xi}_{st}^{ge} \left[\frac{\phi_n}{\delta_n}\right] = \left[\frac{\boldsymbol{\xi}_{st}^{al} \phi_n}{\delta_n}\right] \tag{12}$$

in the space $\boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$.

Definition 10 is clearly well-defined by Theorem 6. Defailed proof is omitted.
Let us now derive some properties of $\boldsymbol{\xi}_{st}^{ge}$.

Theorem 11 Let $\beta_1, \beta_2 \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ then $\boldsymbol{\xi}_{st}^{ge}(\beta_1 \curlyvee \beta_1) = \boldsymbol{\xi}_{st}^{ge} \beta_1 \curlywedge \beta_2$.

Proof Assume the requirements of the theorem are satisfied for some $\beta_1, \beta_2 \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$, then there are $(\phi_n), (\kappa_n) \in \mathscr{S}_\alpha(\mathscr{I})$ and $(\varphi_n), (\delta_n) \in \Delta$ such that $\beta_1 = \left[\frac{\phi_n}{\varphi_n}\right]$ and $\beta_2 = \left[\frac{\kappa_n}{\delta_n}\right]$.
Therefore, we write

$$\boldsymbol{\xi}_{st}^{ge}(\beta_1 \curlyvee \beta_1) = \boldsymbol{\xi}_{st}^{ge}\left(\left[\frac{\phi_n \curlyvee \kappa_n}{\varphi_n \curlyvee \delta_n}\right]\right)$$

$$\text{i.e.} \quad = \left[\frac{\boldsymbol{\xi}_{st}^{al}(\phi_n \curlyvee \kappa_n)}{\varphi_n \curlyvee \delta_n}\right]$$

$$\text{i.e.} \quad = \left[\frac{\boldsymbol{\xi}_{st}^{al} \phi_n \curlywedge \kappa_n}{\varphi_n \curlyvee \delta_n}\right]$$

$$\text{i.e.} \quad = \left[\frac{\boldsymbol{\xi}_{st}^{al} \phi_n}{\varphi_n}\right] \curlywedge \left[\frac{\kappa_n}{\delta_n}\right].$$

This completes the proof of the theorem.

Theorem 12The transform $\boldsymbol{\xi}_{st}^{ge}$ defines a linear mapping from $\boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ into $\boldsymbol{b}((\mathscr{S}_\alpha, \curlywedge), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$.
Proof is straightforward from the definitions.

Theorem 13 Let $\left[\frac{\phi_n}{\delta_n}\right] \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ and $\delta \in \boldsymbol{\tau}(\mathscr{I})$. Then we have

$$\boldsymbol{\xi}_{st}^{ge}\left(\left[\frac{\phi_n}{\delta_n}\right] \curlyvee \delta\right) = \left[\frac{\boldsymbol{\xi}_{st}^{al} \phi_n}{\delta_n}\right] \curlywedge \delta.$$

Proof By applying (12) for $\left[\frac{\phi_n}{\delta_n}\right] \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ and $\delta \in \boldsymbol{\tau}(\mathscr{I})$ yields

$$\boldsymbol{\xi}_{st}^{ge}\left(\left[\frac{\phi_n}{\delta_n}\right] \curlyvee \delta\right) = \left[\frac{\boldsymbol{\xi}_{st}^{al}(\phi_n \curlyvee \delta)}{\delta_n}\right].$$

By Theorem 3 and once again by (12) we get

$$\boldsymbol{\xi}_{st}^{ge}\left(\left[\frac{\phi_n}{\delta_n}\right] \curlyvee \delta\right) = \left[\frac{\boldsymbol{\xi}_{st}^{al} \phi_n \curlywedge \delta}{\delta_n}\right] = \left[\frac{\boldsymbol{\xi}_{st}^{al} \phi_n}{\delta_n}\right] \curlywedge \delta.$$

This completes the proof of the theorem.

Theorem 14 The transform $\boldsymbol{\xi}_{st}^{ge}$ is consistent with $\boldsymbol{\xi}_{st}^{al}(\mathscr{I}) : \mathscr{S}_\alpha(\mathscr{I}) \to \mathscr{S}_\alpha(\mathscr{I})$.

Proof For every $\phi \in \mathscr{S}_\alpha(\mathscr{I})$, let $\beta \in \boldsymbol{b}((\mathscr{S}_\alpha, \curlyvee), (\boldsymbol{\tau}, \curlyvee), \Delta, \mathscr{I})$ be its representative, then we have $\beta = \left[\frac{\phi \curlyvee \delta_n}{\delta_n}\right], \forall n \in \mathbb{N}, (\delta_n) \in \Delta$. For all $n \in \mathbb{N}$ its clear that (δ_n) is independent from the representative.

We also have

$$\boldsymbol{\xi}_{st}^{ge}(\beta) = \boldsymbol{\xi}_{st}^{ge}\left(\left[\frac{\phi \curlyvee \delta_n}{\delta_n}\right]\right) = \left[\frac{\boldsymbol{\xi}_{st}^{al}(\phi \curlyvee \delta_n)}{\delta_n}\right] = \left[\frac{\boldsymbol{\xi}_{st}^{al} \phi \curlywedge \delta_n}{\delta_n}\right] = \left[\frac{\delta_n}{\delta_n}\right] \curlywedge \boldsymbol{\xi}_{st}^{al} \phi$$

which is the representative of $\boldsymbol{\xi}_{st}^{al} \phi$ in the space $\mathscr{S}_\alpha(\mathscr{I})$.
Hence the proof is completed.

Theorem 15 The mappings ξ_{st}^{ge} are continuous with respect to δ and Δ convergence.

Proof of this theorem is available in many papers of the same author. We prefer we omit the details.

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