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An Explicit Formula for Bernoulli Numbers in Terms of Stirling Numbers of the Second Kind

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Abstract: In the paper, the authors recover an explicit formula for computing Bernoulli numbers in terms of Stirling numbers of the second kind.

Keywords: explicit formula; Bernoulli number; Stirling number of the second kind; Bell polynomial of the second kind

1 Introduction

It is well known that Bernoulli numbers B_k for $k \ge 0$ may be generated by

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!} \tag{1}$$

for $|x| < 2\pi$. See [1, p. 48]. In combinatorics, Stirling numbers of the second kind $S(n,k)$ for $n \ge k \ge 0$ may be computed by

$$S(n,k) = \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n \tag{2}$$

and may be generated by

$$\frac{(e^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}. \tag{3}$$

See [1, p. 206]. Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ are defined by

$$\mathrm{B}_{n,k}(x_1, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n \ge k \ge 1$, See [1, p. 134, Theorem A].

The aim of this paper is to recover an explicit formula for computing Bernoulli numbers B_n in terms of Stirling numbers of the second kind $S(n,k)$.

The main results may be summarized as the following theorem.

Theorem 1 *For $n \ge k \ge 0$, we have*

$$\mathrm{B}_{n,k}(0, 1, \dots, 1) = \sum_{i=0}^{k} (-1)^i \binom{n}{i} S(n-i, k-i) \tag{4}$$

and

$$\mathrm{B}_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2}\right) = \frac{n!}{(n+k)!} \sum_{i=0}^{k} (-1)^{k-i} \binom{n+k}{k-i} S(n+i, i). \tag{5}$$

For $n \ge 0$, we have

$$B_n = \sum_{i=0}^{n} (-1)^i \frac{\binom{n+1}{i+1}}{\binom{n+i}{i}} S(n+i, i). \tag{6}$$

2 Proof of Theorem 1

In combinatorics, Faà di Bruno formula may be described in terms of Bell polynomials of the second kind

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$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f\circ g(x) = \sum_{k=1}^{n} f^{(k)}(g(x))\mathrm{B}_{n,k}\bigl(g'(x),g''(x),\dots,g^{(n-k+1)}(x)\bigr). \quad (7)$$

See [1, p. 139, Theorem C]. It is easy to see that

$$\frac{x}{e^x-1}=\frac{1}{\int_0^1 e^{xt}\,\mathrm{d}t}.$$

Applying in (7) the functions $f(y)=\frac{1}{y}$ and $y=g(x)=\int_0^1 e^{xt}\,\mathrm{d}t$ results in

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{e^x-1}\right)&=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{\int_0^1 e^{xt}\,\mathrm{d}t}\right)\\
&=\sum_{k=1}^{n}(-1)^k\frac{k!}{\left(\int_0^1 e^{xt}\,\mathrm{d}t\right)^{k+1}}\\
&\quad\times \mathrm{B}_{n,k}\left(\int_0^1 te^{xt}\,\mathrm{d}t,\int_0^1 t^2e^{xt}\,\mathrm{d}t,\dots,\int_0^1 t^{n-k+1}e^{xt}\,\mathrm{d}t\right)\\
&\to\sum_{k=1}^{n}(-1)^k k!\mathrm{B}_{n,k}\left(\int_0^1 t\,\mathrm{d}t,\int_0^1 t^2\,\mathrm{d}t,\dots,\int_0^1 t^{n-k+1}\,\mathrm{d}t\right)\\
&=\sum_{k=1}^{n}(-1)^k k!\mathrm{B}_{n,k}\left(\frac12,\frac13,\dots,\frac{1}{n-k+2}\right)\end{aligned}$$

as $x\to0$. On the other hand, differentiating n times on both sides of (1) leads to

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{e^x-1}\right)=\sum_{k=n}^{\infty}B_k\frac{x^{k-n}}{(k-n)!}\to B_n,\quad x\to0.$$

As a result, we obtain

$$B_n=\sum_{k=1}^{n}(-1)^k k!\mathrm{B}_{n,k}\left(\frac12,\frac13,\dots,\frac{1}{n-k+2}\right). \quad (8)$$

In [1, p. 133], it was listed that

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k=\sum_{n=k}^{\infty}\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!} \quad (9)$$

for $n\ge k\ge0$. Letting $x_1=0$ and $x_m=1$ for $m\ge2$ in (9) and employing (3) give

$$\begin{aligned}\sum_{n=k}^{\infty}\mathrm{B}_{n,k}(0,1,\dots,1)\frac{t^n}{n!}&=\frac{1}{k!}\left(\sum_{m=2}^{\infty}\frac{t^m}{m!}\right)^k=\frac{1}{k!}(e^t-1-t)^k\\
&=\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(e^t-1)^i t^{k-i}\\
&=\sum_{i=0}^{k}\frac{(-1)^{k-i}}{(k-i)!}\sum_{j=i}^{\infty}S(j,i)\frac{t^{k+j-i}}{j!}.\end{aligned}$$

This implies that

$$\begin{aligned}\mathrm{B}_{n,k}(0,1,\dots,1)&=n!\sum_{i=0}^{k}\frac{(-1)^{k-i}}{(k-i)!}\frac{S(n-k+i,i)}{(n-k+i)!}\\
&=\sum_{i=0}^{k}(-1)^{k-i}\binom{n}{k-i}S(n-k+i,i)\\
&=\sum_{i=0}^{k}(-1)^{i}\binom{n}{i}S(n-i,k-i).\end{aligned}$$

The formula (4) follows.

By virtue of

$$\mathrm{B}_{n,k}\left(\frac{x_2}{2},\frac{x_3}{3},\dots,\frac{x_{n-k+2}}{n-k+2}\right)=\frac{n!}{(n+k)!}\mathrm{B}_{n+k,k}(0,x_2,\dots,x_{n+1}),$$

see [1, p. 136], and the formula (4), we obtain

$$\begin{aligned}\mathrm{B}_{n,k}\left(\frac12,\frac13,\dots,\frac{1}{n-k+2}\right)&=\frac{n!}{(n+k)!}\mathrm{B}_{n+k,k}(0,1,\dots,1)\\
&=\frac{n!}{(n+k)!}\sum_{i=0}^{k}(-1)^i\binom{n+k}{i}S(n+k-i,k-i),\end{aligned}$$

from which, the formula (5) follows.

Substituting (5) into (8) leads to

$$\begin{aligned}B_n&=\sum_{k=1}^{n}\frac{k!n!}{(n+k)!}\sum_{i=0}^{k}(-1)^i\binom{n+k}{k-i}S(n+i,i)\\
&=\sum_{k=1}^{n}\sum_{i=0}^{k}(-1)^i\frac{\binom{k}{i}}{\binom{n+i}{i}}S(n+i,i)\\
&=\sum_{i=0}^{n}\frac{(-1)^i}{\binom{n+i}{i}}S(n+i,i)\sum_{k=i}^{n}\binom{k}{i}\\
&=\sum_{i=0}^{n}\frac{(-1)^i}{\binom{n+i}{i}}\binom{n+1}{i+1}S(n+i,i),\end{aligned}$$

which may be rewritten as the formula (6). The proof of Theorem 1 is complete.

3 Remarks

Finally we list several remarks on something to do with our main results.

Remark 1 *The formula (5) may be alternatively proved as follows.*

Taking $x_m = \frac{1}{m+1}$ for all $m \in \mathbb{N}$ in (9) and utilizing (3) yield

$$\begin{aligned}\sum_{n=k}^{\infty} \mathrm{B}_{n,k}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n-k+2}\right)\frac{t^n}{n!} &= \frac{1}{k!}\left[\sum_{m=1}^{\infty}\frac{t^m}{(m+1)!}\right]^k \\ &= \frac{1}{k!}\left(\frac{e^t-1-t}{t}\right)^k = \frac{1}{k!}\left(\frac{e^t-1}{t}-1\right)^k \\ &= \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\left(\frac{e^t-1}{t}\right)^\ell \\ &= \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\frac{\ell!}{t^\ell}\sum_{i=\ell}^{\infty}S(i,\ell)\frac{t^i}{i!} \\ &= \sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!}\sum_{i=\ell}^{\infty}S(i,\ell)\frac{t^{i-\ell}}{i!}.\end{aligned}$$

This implies that

$$\mathrm{B}_{n,k}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n-k+2}\right) = n!\sum_{\ell=0}^{k}\frac{(-1)^{k-\ell}}{(k-\ell)!(n+\ell)!}S(n+\ell,\ell).$$

The formula (5) follows.

Remark 2 *In [1, p. 220] and [3, pp. 559–560], the following explicit formula for computing Bernoulli numbers B_n in terms of Stirling numbers of the second kind $S(n,k)$ was presented: For $n \ge 0$, we have*

$$B_n = \sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}S(n,k). \tag{10}$$

Recently, four alternative proofs for the formula (10) were supplied in [4] and [11].

The first formula for Bernoulli numbers B_n listed in [2] is

$$B_n = \sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n, \quad n \ge 0, \tag{11}$$

which is a special case of the general formula [8, (2.5)]. We observe that the formula (11) is equivalent to the one (10). In all, we may collect at least seven alternative proofs for the formula (10) or (11) in the references [2], [3], [4], [8], and [11].

Remark 3 *In [6, p. 1128, Corollary], among other things, it was found that, for $k \ge 1$,*

$$B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k\sum_{i=1}^{k-1}\frac{A_{2(k-i)}}{2(k-i)+1}, \tag{12}$$

where A_m is defined by

$$\sum_{m=1}^{n} m^k = \sum_{m=0}^{k+1} A_m n^m.$$

In [5, Theorem 3.1], it was presented that Bernoulli numbers B_{2k} may be computed by

$$\begin{aligned}B_{2k} = 1 &+ \sum_{m=1}^{2k-1}\frac{S(2k+1,m+1)S(2k,2k-m)}{\binom{2k}{m}} \\ &- \frac{2k}{2k+1}\sum_{m=1}^{2k}\frac{S(2k,m)S(2k+1,2k-m+1)}{\binom{2k}{m-1}}\end{aligned} \tag{13}$$

for $k \in \mathbb{N}$. In [10, Theorem 1.4], among other things, it was discovered that

$$B_{2k} = \frac{(-1)^{k-1}k}{2^{2k-2}(2^{2k}-1)}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-i-1}(-1)^{i+\ell}\binom{2k}{\ell}(k-i-\ell)^{2k-1}.$$

for $n \in \mathbb{N}$.

Remark 4 *The object of the paper [2], motivated by the paper [8], is to set matters straight by presenting a bibliography, including 33 references, on explicit formulas for Bernoulli numbers and to show how one can easily manufacture expressions for Bernoulli numbers.*

In [2, p. 48, (11)], it was deduced that

$$B_n = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\frac{n!}{(n+j)!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^{n+j}$$

for $n \ge 0$. This may be rearranged as the form of the formula (6).

On 21 January 2014, the authors searched out that the formula (6) was ever derived in [7, p. 59] and [13, p. 140] by different tools from Faà di Bruno formula (7).

For more information on the history and literature of explicit formulas for computing Bernoulli numbers, please refer to [2], [7], [8], [12], and [13] and plenty references therein.

Remark 5 *This paper is a revised version of the preprint [9].*

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