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Homotopy Analysis Wiener-Hermite Expansion Method for Solving Stochastic Differential Equation

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Abstract: This paper introduces a new technique called Homotopy analysis Wiener Hermite expansion (HAM-WHE) which considered as an extension to Wiener Hermite expansion linked with perturbation technique WHEP. The WHEP technique uses the Wiener Hermite expansion and perturbation technique to solve a class of nonlinear partial differential equations with a perturbed nonlinearity. The homotopy perturbation method (HPM) was used instead of the conventional perturbation methods which generalizes the WHEP technique such that it can be applied to stochastic differential equations without the necessary of presence of the small parameter. For more generalizing, the homotopy analysis method (HAM) is used instead of HPM; since HAM contains the control parameter to guarantee the convergence of the solution and HPM is only a special case of HAM obtained at $\hbar = -1$ .The proposed technique is applied on stochastic quadratic nonlinear diffusion problem to obtain some approximation orders of mean and variance with making comparisons with HAM and homotopy-WHE to testify the method of analysis using symbolic computation software Mathematica. The current work extends the use of WHEP for solving stochastic nonlinear differential equations.

Keywords: Stochastic nonlinear Diffusion equation; Homotopy analysis method; WHEP technique; Convergence-controller parameter

1 Introduction

In many practical situations, it is appropriate to assume that the nonlinear term affecting the phenomena under study is small enough; then its intensity is controlled by means of a frank small parameter, say ε. Relevant examples in this sense appear for instance in epidemiology [1,2]. In addition to these considerations, diffusion models with nonlinear perturbations can also consider the introduction of a forcing term in order to model external aspects which can become very complex, such as: the environment in biology; unexpected material changes in the surrounding medium in physics; and foreign political events that can affect the markets where an investment has been ordered in finance. Stochastic differential equations based on the white noise process provide a powerful tool for dynamically modeling these complex and uncertain aspects. El-Tawil used the Wiener-Hermite expansion together with perturbation theory (WHEP) technique to solve a perturbed nonlinear stochastic diffusion equation [3]. The technique has been developed to be applied on non-perturbed differential equations using the homotopy perturbation method and is called homotopy-WHEP [4]. The homotopy-WHEP technique is used in solving nonlinear diffusion equation with stochastic non homogeneity [5]. In this paper the homotopy analysis method (HAM) will be used instead of HPM to obtain some approximation orders of mean and variance for quadratic nonlinear diffusion equation under stochastic non homogeneity. The homotopy analysis method (HAM) is an analytical technique for solving non linear differential equations. HAM proposed by Liao in 1992, [6], the technique is superior to the traditional perturbation methods in that it leads to convergent series solutions of strongly nonlinear problems, independent of any small or large physical parameter associated with the problem, [7]. The HAM provides a more viable alternative to non perturbation techniques such as the Adomian decomposition method (ADM) [8] and other techniques that cannot guarantee the convergence of the solution series and may be only valid for weakly nonlinear problems, [7]. We note here that He's homotopy perturbation method (HPM), [9] is only a special case of the HAM [6]. Indeed Liao [10] makes a compelling case

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that the Adomian decomposition method, the Lyapunov artificial small parameter method and the-expansion method are nothing but special cases of the HAM. In recent years; this method has been successfully employed to solve many in science and engineering [11,12,13,14, 15,16,17,18,19]. HAM was used in solving nonlinear stochastic diffusion models with nonlinear losses [20,21]. The HAM-WHEP is applied to find the mean and variance of the stochastic quadratic nonlinear equation with $\sigma n(x;\omega)$ as non homogeneity given by [22]

$$\frac{\partial u(t,x;\omega)}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \varepsilon u^2 + \sigma n(x;\omega); (t,x) \in (0,\infty) \times (0,\ell),$$
$$u(t,0;\omega)=0, u(t,\ell;\omega)=0 \text{ and } u(0,\omega)=\phi(x). \quad (1)$$

where $u(t,x;\omega)$ is the diffusion process, ε is a deterministic scale for the nonlinear term. And ω is a random outcome for a triple probability space (Ω,A,P) where Ω is a sample space, A is a σ-algebra associated with Ω and P is a probability measure. The current work also deals with the solution of 2D stochastic quadratic nonlinear equation with as $\sigma n(x;\omega)$ non homogeneity which has the following important properties

$$En(x;\omega) = 0,$$
$$En(x_1;\omega)n(x_2;\omega) = \delta(x_1 - x_2). \quad (2)$$

where E denotes the ensemble average (mean) operator, δ (-) is the Dirac delta function it can represent several relations.

2 HAM Technique.

A presentation of the standard HAM for deterministic problems can be found in [6,7].The following subsection is a brief description of HAM.To describe the basic ideas of HAM, we consider the following differential equation:

$$N[u(t,x)] = 0 \quad (3)$$

where N is a nonlinear operator x and t denote independent variables, and is an unknown function. By means of generalizing the traditional homotopy method, Liao [6,7] constructs the zero-order deformation equation

$$(1-q)L[(\phi(t,x;q) - u_0(t,x)] = q\hbar H(t,x)N[\phi(t,x;q)]. \quad (4)$$

where $q \in [0,1]$ denotes the embedding parameter, $\hbar$ is an auxiliary parameter and L is an auxiliary linear operator. The HAM is based on a kind of continuous mapping $u(t,x) \to \phi(t,x;q)$,$\phi(t,x;q)$ is an unknown function, $u_0(t,x)$ is an initial guess of $u(t,x)$ and $H(t,x)$ denotes a non-zero auxiliary function. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, equation (4) becomes respectively

$$\phi(t,x;0) = u_0(t,x), \phi(t,x;1) = u(t,x) \quad (5)$$

Thus as q increases from 0 to 1, the solution $\phi(t,x;q)$ varies from the initial guess $u_0(t,x)$ to the solution $u(t,x)$. In topology, this kind of variation is the called deformation, equation (4)construct the homotopy $\phi(t,x;q)$. Having the freedom to choose the auxiliary parameter$\hbar$, the auxiliary function $H(t,x)$, the initial approximation$u_0(t,x)$, and the auxiliary linear operator L, we can assume that all of them are properly chosen so that the solution$\phi(t,x;q)$ of the zero-order deformation equation (4) exists for $0 < q \leq 1$. Expanding $\phi(t,x;q)$ in the Taylor series with respect to q, one has

$$\phi(t,x;q) = u_0(t,x) + \sum_{m=1}^{\infty} u_m(t,x)q^m \quad (6)$$

where

$$u_m(t,x) = \frac{1}{m!}\frac{\partial^m \phi(t,x;q)}{\partial q^m}\bigg|_{q=0} \quad (7)$$

Assume that the auxiliary parameter$\hbar$, the auxiliary function $H(t,x)$, the initial approximation $u_0(t,x)$ and the auxiliary linear operator L are so properly chosen that the series (6) converges at $q = 1$ and

$$\phi(t,x;1) = u_0(t,x) + \sum_{m=1}^{\infty} u_m(t,x), \quad (8)$$

Which must be one of the solutions of the original nonlinear equation, as proved by Liao [6]. As $\hbar = -1$ and $H(t,x) = 1$ (4)becomes

$$(1-q)L[(\phi(t,x;q) - u_0(t,x)] + qN[\phi(t,x;q)] = 0, \quad (9)$$

This is mostly used in the homotopy-perturbation method. According to definition (7),the governing equation and the corresponding initial condition of u_m(t,x) can be deduced from the zero-order deformation equation (4). Define the vector $\vec{u_m}(t,x) = \{u_0(t,x), u_1(t,x), .., u_m(t,x)\}$ Differentiating equation (4) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the mth -order deformation equation:

$$L[u_m(t,x) - \chi_m u_{m-1}(t,x)] = \hbar H(t,x)R(u_{m-1}), \quad (10)$$

where

$$R(u_{m-1}) = \frac{1}{m-1!}\frac{\partial^{m-1} N[\phi(t,x;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \quad (11)$$

and

$$\chi_m = \begin{cases} 0 \ \ when\ m \leq 1 \\ 1 \ \ otherwise \end{cases}, \quad (12)$$

The solution is computed as:

$$u(t,x) = \sum_{i=0}^{\infty} u_i(t,x). \quad (13)$$

It should be emphasized that $u_m t,x$ for $m \geq 1$ is governed by the linear equation (10) with linear boundary conditions that come from the original problem, which can be solved by the symbolic computation software such as Mathematica, Maple, and Matlab

3 Application of HAM for Solving Stochastic Quadratic Nonlinear Diffusion Equation

HAM will be used to find mean and variance of stochastic quadratic nonlinear diffusion problem (1) like follows. The auxiliary linear operator chosen as

$$L[\phi(t,x;q)] = \frac{\partial \phi(t,x;q)}{\partial t} - \frac{\partial^2 \phi(t,x;q)}{\partial x^2}. \qquad (14)$$

We have many choices in guessing the initial approximation together with its initial conditions which greatly affects the consequent approximation. The choice of $u_0(t,x)$ is a design problem which can be taken as follows:

$$u_0(t,x) = \sum_{n=0}^{\infty} B_n e^{\beta_n t} \sin \frac{n\pi}{\ell} x, \qquad (15)$$

$$B_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \sin \frac{n\pi}{\ell} x dx.$$

One can notice that the selected value function satisfies the initial and boundary conditions and it depends on the parameter B_n which is totally free .One can also notice thatB_n selection could control the solution convergence. Furthermore, we define the nonlinear operator as

$$N[\phi(t,x;q)] = \frac{\partial \phi(t,x;q)}{\partial t} - \frac{\partial^2 \phi(t,x;q)}{\partial x^2} + \varepsilon[\phi(t,x;q)]^2 - \sigma n(x;\omega) \qquad (16)$$

We construct the zero-order deformation equation,

$$(1-q)L[(u_m(t,x) - \chi_m u_{m-1}(t,x)] = q\hbar H(t,x) R(u_{m-1}). \qquad (17)$$

The mth -order deformation equation for $m \geq 1$ and $H(t,x) = 1$ is

$$L[(u_m(t,x) - \chi_m u_{m-1}(t,x)] = \hbar R(\mathbf{u}_{m-1}) \qquad (18)$$

subject to boundary conditions

$$u_m(t,0) = 0, u_m(t,\ell) = 0, \qquad (19)$$

and initial condition

$$u_m(0,x) = 0, \qquad (20)$$

where

$$R(\mathbf{u}_{m-1}) = \frac{\partial u_{m-1}(t,x)}{\partial t} - \frac{\partial^2 u_{m-1}(t,x)}{\partial x^2} + \varepsilon \sum_{i=0}^{m-1} u_{m-1-i}(t,x) u_i(t,x)) - (1-\chi_m)\sigma\delta(x-x_1). \qquad (21)$$

Now the solution of the mth -order deformation equation (18) for $m \geq 1$ becomes

$$L[(u_m(t,x) - \chi_m u_{m-1}(t,x)] = \hbar[\frac{\partial u_{m-1}(t,x)}{\partial t} - \frac{\partial^2 u_{m-1}(t,x)}{\partial x^2} + \varepsilon \sum_{i=0}^{m-1} u_{m-1-i}(t,x) u_i(t,x) - (1-\chi_m)\sigma\delta(x-x_1)]. \qquad (22)$$

The first order approximation is obtained by substituting with $m = 1$ in (18) as follows

$$L[u_1(t,x)] = \hbar R(u_0) \qquad (23)$$

where

$$R(u_0) = \frac{\partial u_0(t,x)}{\partial t} - \frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon u_0^2 - \sigma n(x;\omega); \qquad (24)$$

then

$$L[u_1(t,x)] = \hbar[\frac{\partial u_0(t,x)}{\partial t} - \frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon u_0^2 - \sigma n(x;\omega)] \qquad (25)$$

The approximated first order solution of (25) can be obtained using Eigen function expansion as follows:

$$u_1(t,x) = \sum_{n=0}^{\infty} I_{n,1}(t) \sin \frac{n\pi}{\ell} x,$$

where

$$I_{n,1}(t) = \int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau)} F_{n,1}(\tau) d\tau,$$

$$F_{n,1}(t) = \frac{2\hbar}{\ell} \int_0^{\ell} [\frac{\partial u_0(t,x)}{\partial t} - \frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon u_0^2 - \sigma n(x;\omega)] \sin \frac{n\pi}{\ell} x dx. \qquad (26)$$

The ensemble average of the first order approximation is

$$E[u_1(t,x)] = \sum_{n=0}^{\infty} E(I_{n,1}(t)) \sin \frac{n\pi}{\ell} x,$$

where

$$E(I_{n,1}(t)) = \int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau)} E(F_{n,1}(\tau)) d\tau$$

$$E(F_{n,1}(t)) = \frac{2\hbar}{\ell} \int_0^{L} [\frac{\partial u_0(t,x)}{\partial t} - \frac{\partial^2 u_0(t,x)}{\partial x^2} + \varepsilon u_0^2] \sin \frac{n\pi}{\ell} x dx,$$

then $E[u_1(t,x)]$ becomes:

$$E[u_1(t,x)] = \frac{e^{-\pi^2 t} h \ \mathrm{Sin}[\pi x](3(-1+e^{t(\pi^2+\beta_n)})\pi(\pi^2+2\beta_n))}{3(\pi^3+2\pi\beta_n)} + \frac{e^{-\pi^2 t} h \mathrm{Sin}[\pi x](-8\varepsilon + 8e^{t(\pi^2+2\beta n)}\varepsilon)}{3(\pi^3+2\pi\beta_n)} \qquad (27)$$

The covariance of the first order solution can have the following expression

$$Cov[u_1(t,x_1),u_1(t,x_2)] = E[(u_1(t,x_1)-Eu_1(t,x_1))(u_1(t,x_2)-Eu_1(t,x_2))] = \sum_{n=1}^{\infty}(I_{n,1}(t)-EI_{n,1}(t))\sin\frac{n\pi}{\ell}x_1)(\sum_{m=1}^{\infty}(I_{m,1}(t) -EI_{m,1}(t))\sin\frac{m\pi}{\ell}x_2)] \quad (28)$$

where Cov denotes the covariance operator. The covariance is obtained from the following final express

$$Cov(u_1(t,x_1),u_1(t,x_2)) = \frac{4\hbar^2\sigma^2}{\ell^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi}{\ell}x_1\sin\frac{m\pi}{\ell}x_2(\int_0^{\ell}\sin\frac{n\pi}{\ell}x\sin\frac{m\pi}{\ell}xdx)(\int_0^t\int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau_1)}e^{(\frac{-m\pi}{\ell})^2(t-\tau_2)}d\tau_1 d\tau_2).$$

$$Cov(u_1(t,x_1),u_1(t,x_2)) = \frac{2(1-e^{-\pi^2 t})^2h^2Sin[\pi x_1]Sin[\pi x_2]}{\pi^4} \quad (29)$$

The variance of the first order solution can have the following expression

$$Var[u_1(t,x)] = E[u_1(t,x)-Eu_1(t,x)]^2 = E[(\sum_{n=1}^{\infty}(I_{n,1}(t)-EI_{n,1}(t))\sin\frac{n\pi}{\ell}x)]^2 \quad (30)$$

where Var denotes the variance operator.The variance can then be obtained from equation (29) by setting

$$Var[u_1(t,x)] = \frac{4\hbar^2\sigma^2}{\ell^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi}{\ell}x\sin\frac{m\pi}{\ell}x(\int_0^{\ell}\sin\frac{n\pi}{\ell}x\sin\frac{m\pi}{\ell}xdx)(\int_0^t\int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau_1)}e^{(\frac{-m\pi}{\ell})^2(t-\tau_2)}d\tau_1 d\tau_2)$$

$$Var[u_1(t,x)] = \frac{2(1-e^{-\pi^2 t})^2h^2Sin[\pi x]^2}{\pi^4} \quad (31)$$

The second order approximation is obtained by substituting with $m=2$ in (18) as follows

$$L[u_2(t,x)-u_1(t,x)] = \hbar R(u_1) \quad (32)$$

where

$$R(u_1) = \frac{\partial u_1(t,x)}{\partial t} - \frac{\partial^2 u_1(t,x)}{\partial x^2} + \varepsilon\sum_{i=0}^{1}u_{1-i}(t,x)u_i(t,x) = \frac{\partial u_1(t,x)}{\partial t} - \frac{\partial^2 u_1(t,x)}{\partial x^2} + 2\varepsilon u_0(t,x)u_1(t,x) \quad (33)$$

Substituting by (33) in (32) we get

$$L[u_2(t,x)-u_1(t,x)] = \hbar[\frac{\partial u_1(t,x)}{\partial t} - \frac{\partial^2 u_1(t,x)}{\partial x^2} +2\varepsilon u_0(t,x)u_1(t,x)] \quad (34)$$

The solution of (34) can be obtained using Eigen function expansion as follows:

$$u_2(t,x) = u_1(t,x) + \sum_{n=0}^{\infty}I_{n,2}(t)\sin\frac{n\pi}{\ell}x,$$

where

$$I_{n,2}(t) = \int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau)}F_{n,2}(\tau)d\tau,,$$

$$F_{n,2}(t) = \frac{2\hbar}{\ell}\int_0^{\ell}[\frac{\partial u_1(t,x)}{\partial t} - \frac{\partial^2 u_1(t,x)}{\partial x^2} +2\varepsilon u_0(t,x)u_1(t,x)]\sin\frac{n\pi}{\ell}xdx. \quad (35)$$

The ensemble average of the second order solution can be obtained as

$$E[u_2(t,x)] = E[u_1(t,x)] + \sum_{n=0}^{\infty}E(I_{n,2}(t))\sin\frac{n\pi}{\ell}x,$$

where

$$E(I_{n,2}(t)) = \int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau)}E(F_{n,2}(\tau))d\tau,$$

$$E(F_{n,2}(t)) = \frac{2\hbar}{\ell}\int_0^{\ell}[\frac{\partial}{\partial t}E[u_1(t,x)] - \frac{\partial^2}{\partial x^2}E[u_1(t,x)] +\varepsilon u_0(t,x)E[u_1(t,x)]]\sin\frac{n\pi}{\ell}xdx,$$

then $E[u_2(t,x)]$ becomes

$$E(F_{n,2}(t)) = \frac{e^{-\pi^2 t}h(3(-1+e^{t(\pi^2+\beta_n)})\pi(\pi^2+2\beta_n)}{3(\pi^3+2\pi\beta n)} + \frac{-8\varepsilon+8e^{t(\pi^2+2\beta_n)}\varepsilon)Sin[\pi x]}{3(\pi^3+2\pi\beta n)} + (e^{-\pi^2 t}h^2(-128\beta_n\varepsilon^2-(\pi^2+3\beta_n)(9\pi^2\beta_n(\pi^2+2\beta_n)+ 72\pi\beta_n\varepsilon-16\varepsilon(3\pi^3+6\pi\beta_n+8\varepsilon))+ e^{t\beta_n}(128e^{t(\pi^2+2\beta_n)}\beta_n\varepsilon^2+(\pi^2+3\beta_n) (9e^{\pi^2 t}\pi^2\beta_n(\pi^2+2\beta_n)+72e^{t(\pi^2+\beta_n)}\pi\beta_n\varepsilon- 16\varepsilon(3\pi^3+6\pi\beta_n+8\varepsilon))))Sin[\pi x])/ (9\pi^2\beta n(\pi^2+2\beta_n)(\pi^2+3\beta_n)) \quad (36)$$

The covariance of the second order solution has the following expression

$$Cov(u_2(t,x_1),u_2(t,x_2)) = E[(u_2(t,x_1)-Eu_2(t,x_1))(u_2(t,x_2)-Eu_2(t,x_2))] = E[(\sum_{n=1}^{\infty}(I_{n,1}(t)-EI_{n,1}(t))\sin\frac{n\pi}{\ell}x_1 + \sum_{n=1}^{\infty}(I_{n,2}(t)-EI_{n,2}(t))\sin\frac{n\pi}{\ell}x_1)$$

$$Cov(u_2(t,x_1),u_2(t,x_2) = Cov(u_1(t,x_1),u_1(t,x_2) + E[(\sum_{n=1}^{\infty}(I_{n,1}(t)-EI_{n,1}(t))\sin\frac{n\pi}{\ell}x_1 + \sum_{n=1}^{\infty}(I_{n,2}(t) - EI_{n,2}(t))\sin\frac{n\pi}{\ell}x_1) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}E((I_{n,2}(t)-EI_{n,2}(t)) (I_{m,2}(t)-EI_{m,2}(t)))\sin\frac{n\pi}{\ell}x_1\sin\frac{m\pi}{\ell}x_2$$

$$Cov(u_2(t,x_1),u_2(t,x_2) = \frac{1}{\pi^8}2h^2(3\left((-1+e^{-\pi^2t})^2\right)\pi^4 -1/(\pi^2+\beta_n)\,2e^{-2\pi^2t}\pi^3(2\left((-1+e^{-\pi^2t})^2\right)\pi^2 +2\left((-1+e^{-\pi^2t})^2\right)\beta_n+(-1+e^{\pi^2t}) (-1+e^{t(\pi^2+\beta_n)})\pi\varepsilon-(-1+e^{t(\pi^2+\beta_n)})\pi^3t\varepsilon) +e^{t\beta_n}(128e^{t(\pi^2+2\beta_n)}\beta_n\varepsilon^2+(\pi^2+3\beta_n) -4e^{-2\pi^2t}(\pi(-4\left((-1+e^{-\pi^2t})^2\right)h^2\pi +\frac{(1-e^{\pi^2t}+\pi^2t)((-1+e^{t\beta_n})\pi^4+e^{t\beta n}(-1+e^{\pi^2t})\beta_n{}^2)\varepsilon}{\beta_n(\pi^2+\beta_n)}) -\frac{16e^{2t\beta_n}h^2(1-e^{\pi^2t}+\pi^2t)^2\varepsilon^2Sin[\pi x]^2}{\pi^2}))Sin[\pi x_1]Sin[\pi x_2] \quad (37)$$

$$Cov(u_1(t,x_1),u_2(t,x_2) = \frac{1}{\pi^5}2h^2(\left((-1+e^{-\pi^2t})^2\right)\pi -1/(\pi^2+\beta_n)\,e^{-2\pi^2t}(2\left((-1+e^{-\pi^2t})^2\right)\pi^2 +2\left((-1+e^{-\pi^2t})^2\right)\beta_n+(-1+e^{\pi^2t})(-1+e^{t(\pi^2+\beta n)})\pi\varepsilon -(-1+e^{t(\pi^2+\beta n)})\pi^3t\varepsilon))Sin[\pi x_1]Sin[\pi x_2] \quad (38)$$

In this manner, we can have more results of $E[u_m(t,x)]$ and $Var[u_m(t,x)]$ obtained at $m=3,4,\ldots$. The final expression of mean of the 3rd order solution will be. $E[u(t,x)] = \sum_{m=0}^{M}E[u_m(t,x)]=u_0(t,x) + E[u_1(t,x)] + E[u_2(t,x)]+E[u_3(t,x)]$

$$E[u(t,x)] = \frac{1}{9}(9e^{t\beta n} + \frac{6e^{-\pi^2t}h(3(-1+e^{t(\pi^2+\beta_n)})\pi(\pi^2+2\beta_n)-8\varepsilon+8e^{t(\pi^2+2\beta_n)}\varepsilon)}{\pi^3+2\pi\beta_n} +(e^{-\pi^2t}h^2(-128\beta_n\varepsilon^2-(\pi^2+3\beta_n)(9\pi^2\beta_n(\pi^2+2\beta_n)+ 72\pi\beta_n\varepsilon-16\varepsilon(3\pi^3+6\pi\beta_n+8\varepsilon))+e^{t\beta n}(128e^{t(\pi^2+2\beta n)}\beta_n\varepsilon^2 +(\pi^2+3\beta_n)(9e^{\pi^2t}\pi^2\beta_n(\pi^2+2\beta_n)+72e^{t(\pi^2+\beta_n)}\pi\beta_n\varepsilon -16\varepsilon(3\pi^3+6\pi\beta_n+8\varepsilon)))))/ (\pi^2\beta_n(\pi^2+2\beta_n)(\pi^2+3\beta_n)))\mathrm{Sin}[\pi x]. \quad (39)$$

Since $u(t,x)=\sum_{i=0}^{N}u_i(t,x)$ Then the final expression of the variance of the 2nd order solution will be $Var\left(\sum_{i=1}^{N}u_i(t,x)\right) = \sum_{i=1}^{N}Var[u_i(t,x)]+\left(\sum_{i=1}^{N}\sum_{j\neq i}^{N}Cov[u_i(t,x),u_j(t,x)]\right)$

$$Var[u(t,x)] = Var[u_1(t,x)] + Var[u_2(t,x)]+2Cov[u_1(t,x),u_2(t,x)] \quad (40)$$

$$Var[u(t,x)] = h^2t(4.+h(4.-1.7149t)+ h^2(1.-0.735t+0.12t^2))$$

$$Var[u(t,x)] = \frac{1}{\pi^8}4h^2\mathrm{Sin}\left((-1+e^{-\pi^2t})^2\right) (3\left((-1+e^{-\pi^2t})^2\right)\pi^4 - 1/(\pi^2+\beta_n)\,2e^{-2\pi^2t}\pi^3(2\left((-1+e^{-\pi^2t})^2\right)\pi^2 +2\left((-1+e^{-\pi^2t})^2\right)\beta_n+(-1+e^{\pi^2t}) (-1+e^{t(\pi^2+\beta_n)})\pi\varepsilon-(-1+e^{t(\pi^2+\beta_n)})\pi^3t\varepsilon)- 2e^{-2\pi^2t}(\pi(-4\left((-1+e^{-\pi^2t})^2\right)h^2\pi +\frac{1}{\beta_n(\pi^2+\beta_n)}(1-e^{\pi^2t}+\pi^2t)((-1+e^{t\beta_n})\pi^4+ e^{t\beta n}(-1+e^{\pi^2t})\beta_n{}^2)\varepsilon) -\frac{16e^{2t\beta_n}h^2(1-e^{\pi^2t}+\pi^2t)^2\varepsilon^2Sin[\pi x]^2}{\pi^2})) \quad (41)$$

4 WHE technique

As a consequence of the completeness of the Wiener-Hermite set[29], any arbitrary stochastic process can be expanded in terms of the Weiner-Hermite polynomial set and this expansion converges to the original stochastic process with probability one. The solution function $u(t,x,\omega)$ can be expanded in terms of Wiener-Hermite functions [26] as:

$$u(t,x;\omega) = u^{(0)}\left(t,x\right)+\int_{-\infty}^{\infty}u^{(1)}\left(t,x;x_1\right)H^{(1)}\left(x_1\,;\omega\right)dx_1 +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u^{(2)}\left(t,x;x_1,x_2\right)H^{(2)}\left(x_1\,,x_2;\omega\right)dx_1dx_2 +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u^{(3)}(t,x;x_1,x_2,x_3)H^{(3)}(x_1,x_2,x_3;\omega)dx_1dx_2dx_3 +\ldots \quad (42)$$

The first term in the expansion (42) is the non-random part or ensemble mean of the function. The first two terms represent the normally distributed (Gaussian) part of the solution. Higher terms in the expansion depart more and more from the Gaussian form[30]. The Gaussian approximation is usually a bad approximation for nonlinear problems, especially when high order statistics are concerned [31]. The components $u^{(i)}(t,x_1,x_2,\ldots,x_i)$ are called the (deterministic) kernels of the Wiener-Hermite expansion of $u(t,x)$. They are functions of time and space variables and fully account for the time dependence of $u(t,x)$ as well as for its statistical properties [32].ω is a random output of a triple probability space (Ω,A,P) , where Ω is a sample space, B is a σ-algebra associated with Ω and P is a probability measure. For simplicity ω will be dropped later on The function $H^{(n)}(t,x_1,x_2,...,x_n)$ is the nth order Wiener-Hermite time-independent functional which is defined for 1D continuous problem as [28]:

$$H^{(n)}(x_1,x_2,...,x_n) = \delta^{n/2}(0)e^{\frac{1}{2}\sum_{i=1}^{n}\xi^2(x_i)}\prod_{k=1}^{n}\left(\frac{-\partial}{\partial\xi(x_k)}\right) e^{\frac{-1}{2}\sum_{i=1}^{n}\xi^2(x_i)} \quad (43)$$

Where ξ_k is a denumerable set of independent Gaussian random variables with zero mean and unit variance, and is the Dirac delta function. The WH functional form a complete set [29]and they satisfy the following recurrence relation for $n \geq 1$

$$H^{(n)}(x_1,x_2,...x_n) = H^{(n-1)}(x_1,x_2,...x_{n-1}).H^{(1)}(x_n) - \sum_{m=1}^{n-1} H^{(n-2)}(x_{n_1},x_{n_2},...x_{i_{n-2}}).\delta(t_{n-m}-t_n), n \geq 2 \quad (44)$$

With $H^{(0)} = 1$ and $H^{(1)}(x_1) = N(x_1)$: the white noise. By construction, the Wiener-Hermite functions are symmetric in their arguments and are statistically orthonormal w.r.t the weighting function $e^{\frac{1}{2}\sum_{i=1}^{n}\xi^2(x_i)}$, i.e $E[H^{(i)}H^{(j)}] = 0 \forall i \neq j$. The average of almost all Wiener-Hermite functions vanishes, particularly, $E[H^{(i)}] = 0$ for $i \geq 1$. The expectation and variance will be $E[u(t,x)] = u^{(0)}(t,x)$
$Var[u(t,x)] = \int_{-\infty}^{\infty}[u^{(1)}(t,x;x_1)]^2dx_1 +$
$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[u^{(2)}(t,x;x_1,x_2)]^2dx_1dx_2 + ...$ The WHE method can be elementary used in solving stochastic differential equations by expanding the solution as well as the stochastic input processes via the WHE. The resultant equation is more complex than the original one due to being a stochastic integro-differential equation. Taking a set of ensemble averages together with using the statistical properties of the WHE functions, a set of deterministic integro-differential equations are obtained in the deterministic kernels $u^{(i)}(t,x_1,x_2,...,x_i)$ To obtain approximate solutions of these deterministic kernels, one can use perturbation theory in the case of having a perturbed system depending on a small parameter . Expanding the kernels as a power series of , another set of simpler iterative equations in the kernel series components is obtained. This is the main algorithm of the WHEP algorithm, [27]. The technique was successfully applied to several nonlinear stochastic equations; see for example [22].

5 HAM-WHE

Step 1: Applying the Wiener-Hermite expansion The first order solution can be obtained when Consider equation (1) and searching for the Gaussian part of the solution process, $u(t,x,\omega)$ can be expanded as:

$$u(t,x;\omega) = u^{(0)}(t,x) + \int_0^x u^{(1)}(t,x;x_1)H^{(1)}(x_1)dx_1 \quad (45)$$

which is a stochastic integro-differential equation in the deterministic kernels $u^{(k)}(-)$ where $u^{(0)}(t,x)$ and $u^{(1)}(t,x)$ are deterministic kernels to be evaluated, substituting in the original equation (1) we get:

$$\frac{\partial u^{(0)}(t,x)}{\partial t} + \int_0^x \frac{\partial u^{(1)}(t,x;x_1)}{\partial t}H^{(1)}(x_1)dx_1 = [\frac{\partial^2 u^{(0)}(t,x)}{\partial x^2} + \int_0^x \frac{\partial^2 u^{(1)}(t,x;x_1)}{\partial t}H^{(1)}(x_1)dx_1] - \varepsilon\left(u^{(0)}(t,x) + \int_0^x u^{(1)}(t,x;x_1),H^{(1)}(x_1)\mathrm{d}x_1\right)^2 + \sigma n(x;\omega), \quad (46)$$

Performing the direct average of (46), we get the following set of deterministic equation:

$$i)\frac{\partial u^{(0)}(t,x)}{\partial t} = \frac{\partial^2 u^{(0)}(t,x)}{\partial x^2} - \varepsilon\left(u^{(0)}(t,x)\,]\right)^2 - \varepsilon\int_0^x [u^{(1)}(t,x;x_1)]^2dx_1,$$

$$u^{(0)}(t,0) = 0, u^{(0)}(t,\ell) = 0, u^{(0)}(0,x) = \varphi(x). \quad (47)$$

Multiplying equation (46) by $H^{(1)}(x_2)$, taking the average with using the statistical properties of Wiener-Hermite polynomials [23]. And letting $x_2 \to x_1$ in the result, we get the following set of deterministic equation:

$$ii)\frac{\partial u^{(1)}(t,x;x_1)}{\partial t} = \frac{\partial^2 u^{(1)}(t,x;x_1)}{\partial x^2} - 2\varepsilon u^{(0)}(t,x)u^{(1)}(t,x;x_1) + \sigma\delta(x-x_1)$$

$$u^{(0)}(t,0;x_1) = 0, u^{(0)}(t,\ell;x_1) = 0, u^{(0)}(0,x,x_1) = 0. \quad (48)$$

The general expression of first order mean is obtained by taking the average of $u(t,x)$ that expanded in equation (55) we get:

$$\mu[u(t,x)] = E[u(t,x)] = u^{(0)}(t,x) \quad (49)$$

The general expression of first order variance is:

$$Var[u(t,x)] = E[((u,(t,x) - \mathrm{Eu}(t,x)]^2 = [\int_0^x\int_0^x u^{(1)}(t,x;x_1)u^{(1)}(t,x;x_2)E(H^{(1)}(x_1)H^{(1)}(x_2))dx_2dx_1] = \int_0^x [u^{(1)}(t,x;x_1)]^2dx_1 \quad (50)$$

Step 2: Using HAM in solving the nonlinear integral-differential equations (47and 48) separately as follows: The auxiliary linear operator for equation (47) is defined as

$$L[u^{(0)}(t,x)] = \frac{\partial u^{(0)}(t,x)}{\partial t} - \frac{\partial^2 u^{(0)}(t,x)}{\partial x^2} \tag{51}$$

with

$$u_0^{(0)}(t,x) = \sum_{n=0}^{\infty} B_n e^{\beta_n t} \sin\frac{n\pi}{\ell}x, \tag{52}$$

$$B_n = \frac{2}{\ell}\int_0^{\ell} \phi(x)\sin\frac{n\pi}{\ell}x dx$$

Furthermore, we define the nonlinear operator as

$$N[u^{(0)}(t,x)] = \frac{\partial u^{(0)}(t,x)}{\partial t} - \frac{\partial^2 u^{(0)}(t,x)}{\partial x^2} + \varepsilon[u^{(0)}(t,x)]^2 + \varepsilon\int_0^x [u^{(1)}(t,x;x_1)]^2 dx_1 \tag{53}$$

We construct the zero-order deformation equation,

$$(1-q)L[(u_m^{(0)}(t,x) - \chi_m u_{m-1}^{(0)}(t,x)] = q\hbar H(t,x)R(u_{m-1}^{(0)}) \tag{54}$$

The mth-order deformation equation for $m \geq 1$ and $H(t,x) = 1$ is

$$L[(u_m^{(0)}(t,x) - \chi_m u_{m-1}^{(0)}(t,x)] = \hbar R(u_{m-1}^{(0)}) \tag{55}$$

Subject to boundary conditions

$$u_m^{(0)}(t,0) = 0, u_m^{(0)}(t,\ell) = 0 \tag{56}$$

and initial condition

$$u_m^{(0)}(0,x) = 0 \tag{57}$$

where χ_m is defined by (12) and

$$R(u_{m-1}^{(0)}) = \frac{\partial u_{m-1}^{(0)}(t,x)}{\partial t} - \frac{\partial^2 u_{m-1}^{(0)}(t,x)}{\partial x^2} + \varepsilon(\sum_{i=0}^{m-1} u_{m-1-i}^{(0)}(t,x)u_i^{(0)}(t,x)) + \varepsilon\int_0^x (\sum_{i=0}^{m-1} u_{m-1-i}^{(1)}(t,x;x_1)u_i^{(1)}(t,x;x_1))dx_1 \tag{58}$$

Now the mth order deformation equation (55) for $m \geq 1$ becomes

$$L[(u_m^{(0)}(t,x) - \chi_m u_{m-1}^{(0)}(t,x)] = \hbar[\frac{\partial u_{m-1}^{(0)}(t,x)}{\partial t} - \frac{\partial^2 u_{m-1}^{(0)}(t,x)}{\partial x^2} + \varepsilon(\sum_{i=0}^{m-1} u_{m-1-i}^{(0)}(t,x)u_i^{(0)}(t,x)) + \varepsilon\int_0^x (\sum_{i=0}^{m-1} u_{m-1-i}^{(1)}(t,x;x_1)u_i^{(1)}(t,x;x_1))dx_1; \tag{59}$$

The first correction is obtained by substituting with $m = 1$ in (55) as follows:

$$L[u_1^{(0)}(t,x)] = \hbar R(u_1^{(0)}(t,x)) \tag{60}$$

where

$$R(u_1^{(0)}(t,x)) = \frac{\partial u_0^{(0)}(t,x)}{\partial t} - \frac{\partial^2 u_0^{(0)}(t,x)}{\partial x^2} + \varepsilon[u_0^{(0)}(t,x)]^2 + \varepsilon\int_0^x [u_0^{(1)}(t,x;x_1)]^2 dx_1; \tag{61}$$

then

$$L[u_1^{(0)}(t,x)] = \hbar[\frac{\partial u_0^{(0)}(t,x)}{\partial t} - \frac{\partial^2 u_0^{(0)}(t,x)}{\partial x^2} + \varepsilon[u_0^{(0)}(t,x)]^2 + \varepsilon\int_0^x [u_0^{(1)}(t,x;x_1)]^2 dx_1], \tag{62}$$

The approximated first correction solution of (62) can be obtained using Eigen function expansion as follows:

$$u_1^{(0)}(t,x) = \sum_{n=0}^{\infty} I_{n,1}^{(0)}(t)\sin\frac{n\pi}{\ell}x$$

where

$$I_{n,1}^{(0)}(t) = \int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau)} F_{n,1}^{(0)}(\tau)d\tau$$

$$F_{n,1}^{(0)}(t) = \frac{2\hbar}{L}\int_0^L [\frac{\partial u_0^{(0)}(t,x)}{\partial t} - \frac{\partial^2 u_0^{(0)}(t,x)}{\partial x^2} + \varepsilon[u_0^{(0)}(t,x)]^2 + \varepsilon\int_0^x [u_0^{(1)}(t,x;x_1)]^2 dx_1]\sin\frac{n\pi}{\ell} \tag{63}$$

The auxiliary linear operator for equation (48) is defined as

$$L[u^{(1)}(t,x;x_1)] = \frac{\partial u^{(1)}(t,x;x_1)}{\partial t} - \frac{\partial^2 u^{(1)}(t,x;x_1)}{\partial x^2} \tag{64}$$

with

$$u_0^{(1)}(t,x;x_1) = \sum_{n=0}^{\infty} I_{n,0}^{(1)}(t)\sin\frac{n\pi}{\ell}x$$

where

$$I_{n,0}^{(1)}(t) = \int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau)} F_{n,0}^{(1)}(\tau)d\tau$$

$$F_{n,0}^{(1)}(t) = \frac{2\sigma}{\ell}\sin\frac{n\pi}{\ell}x_1 \tag{65}$$

Defining the nonlinear operator as

$$N[u^{(1)}(t,x)] = \frac{\partial u^{(1)}(t,x;x_1)}{\partial t} - \frac{\partial^2 u^{(1)}(t,x;x_1)}{\partial x^2} + 2\varepsilon(u^{(0)}(t,x)u^{(1)}(t,x;x_1); \tag{66}$$

We construct the zero-order deformation equation,

$$(1-q)L[(u_m^{(1)}(t,x) - \chi_m u_{m-1}^{(1)}(t,x)] = q\hbar H(t,x)R(u_{m-1}^{(1)}). \tag{67}$$

The mth-order deformation equation for $m \geq 1$ and $H(t,x) = 1$ is

$$L[(u_m^{(1)}(t,x) - \chi_m u_{m-1}^{(1)}(t,x)] = \hbar R(u_{m-1}^{(1)}) \tag{68}$$

Subject to boundary conditions

$$u_m^{(1)}(t,0;x_1) = 0, u_m^{(1)}(t,\ell;x_1) = 0 \tag{69}$$

and initial condition

$$u_m^{(0)}(0,x;x_1) = 0 \tag{70}$$

where

$$R(u_{m-1}^{(1)}) = \frac{\partial u_{m-1}^{(1)}(t,x;x_1)}{\partial t} - \frac{\partial^2 u_{m-1}^{(1)}(t,x;x_1)}{\partial x^2} + 2\varepsilon(\sum_{i=0}^{m-1} u_{m-1-i}^{(0)}(t,x)u_i^{(1)}(t,x;x_1)) - \sigma\delta(x-x_1) \tag{71}$$

Now the mth-order deformation equation (68) for $m \geq 1$ becomes

$$L[(u_m^{(1)}(t,x;x_1) - \chi_m u_{m-1}^{(1)}(t,x;x_1)] = \hbar[\frac{\partial u_{m-1}^{(1)}(t,x;x_1)}{\partial t} - \frac{\partial^2 u_{m-1}^{(1)}(t,x;x_1)}{\partial x^2} + 2\varepsilon(\sum_{i=0}^{m-1} u_{m-1-i}^{(0)}(t,x)u_i^{(1)}(t,x;x_1)) - \sigma\delta(x-x_1)]; \tag{72}$$

The first correction is obtained by substituting with $m = 1$ in (68) as follows

$$L[u_1^{(1)}(t,x;x_1)] = \hbar R(u_0^{(1)}(t,x;x_1)) \tag{73}$$

where

$$R(u_0^{(1)}(t,x;x_1)) = \frac{\partial u_0^{(1)}(t,x;x_1)}{\partial t} - \frac{\partial^2 u_0^{(1)}(t,x;x_1)}{\partial x^2} + 2\varepsilon(u_0^{(0)}(t,x)u_0^{(1)}(t,x;x_1)) - \sigma\delta(x-x_1); \tag{74}$$

then

$$L[u_1^{(1)}(t,x;x_1)] = \hbar[\frac{\partial u_0^{(1)}(t,x;x_1)}{\partial t} - \frac{\partial^2 u_0^{(1)}(t,x;x_1)}{\partial x^2} + 2\varepsilon(u_0^{(0)}(t,x)u_0^{(1)}(t,x;x_1)) - \sigma\delta(x-x_1)], \tag{75}$$

The approximated first correction solution of (75) can be obtained using Eigen function expansion as follows

$$u_1^{(1)}(t,x;x_1) = \sum_{n=0}^{\infty} I_{n,1}^{(1)}(t)\sin\frac{n\pi}{\ell}x$$

where

$$I_{n,1}^{(1)}(t) = \int_0^t e^{(\frac{-n\pi}{\ell})^2(t-\tau)} F_{n,1}^{(1)}(\tau)d\tau$$

$$F_{n,1}^{(1)}(t) = \frac{2\hbar}{\ell}\int_0^L [\frac{\partial u_0^{(1)}(t,x;x_1)}{\partial t} - \frac{\partial^2 u_0^{(1)}(t,x;x_1)}{\partial x^2} + 2\varepsilon(u_0^{(0)}(t,x)u_0^{(1)}(t,x;x_1))]\sin\frac{n\pi}{\ell}xdx - \frac{2\hbar\sigma}{\ell}\sin\frac{n\pi}{\ell}x_1, \tag{76}$$

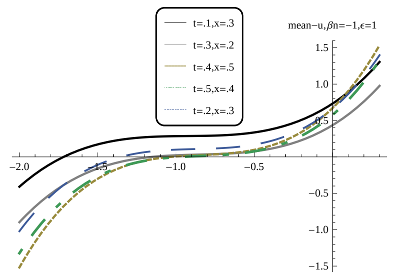

Fig. 1: The change of the mean u with parameter $\hbar$ at different t and x values.

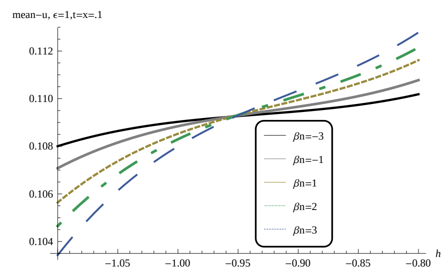

Fig. 2: The change of the mean u with parameter $\hbar$ at different β_n values, $\varepsilon = 1$ and $t = x = 0.1$..

The ensemble average of the first order first correction approximation is obtained by substituting in (49)

$$E[u_1(t,x)] = u^{(0)}(t,x) = u_0^{(0)}(t,x) + u_1^{(0)}(t,x) \tag{77}$$

The variance of the first order first correction approximation is obtained by substituting in (50)

$$Varu_1(t,x) = \int_0^x [u^{(1)}(t,x;x_1)]^2 dx_1 = \int_0^x [u_0^{(1)}(t,x;x_1) + u_1^{(1)}(t,x;x_1)]^2 dx_1 \tag{78}$$

where *Var* denotes the variance operator. Similarly the second, third and fourth corrections are obtained by substituting in equations (55 and 68) by $m = 2,3,4$.

6 Result analysis

In the following figures, results of HAM technique are shown first followed by HAM-WHEP results finally comparisons between them.

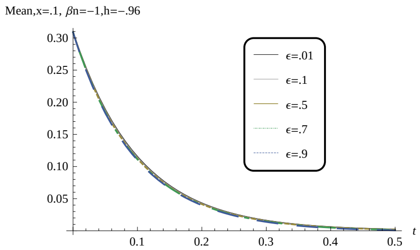

Fig. 3: The change of the mean u with time t at different ε values, $x = 0.1$, $\beta_n = -1$ and $\hbar = -0.96$.

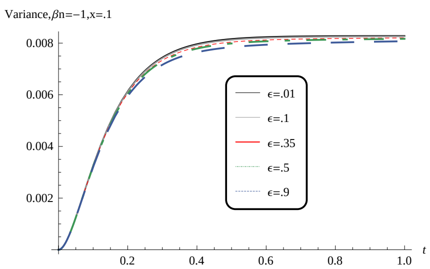

Fig. 4: The change of the variance u with time t at different ε values, $x = 0.1$, $\beta_n = -1$ and $\hbar = -0.96$.

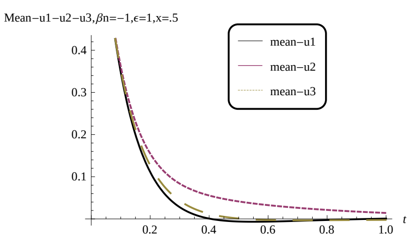

Fig. 5: Mean comparison between first u_1, second u_2 and third order u_3 approximations with time t at $x = 0.1$, $\beta_n = -1$ and $\hbar = -0.96$.

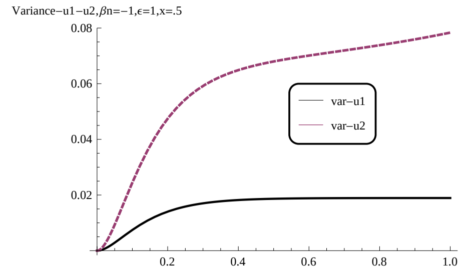

Fig. 6: Variance comparison between first and second approximations u_1, u_2 with time t at $x = 0.1$, $\beta_n = -1$and $\hbar = -0.96$

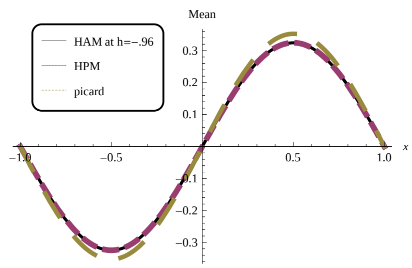

Fig. 7: Mean comparison between HAM first order at $\hbar = -0.96$, HPM first order and Picard first order at $t = 0.1$.

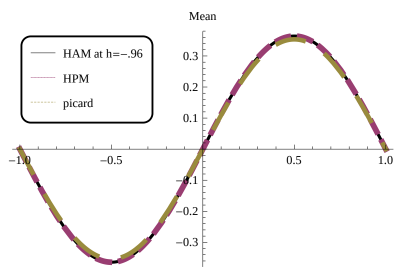

Fig. 8: Mean comparison between HAM second order at $\hbar = -0.96$, HPM second order and Picard second order at $t = 0.1$.

6.1 HAM Results

Results of the solution of 2D stochastic quadratic nonlinear diffusion model using HAM technique are shown at $\sigma = 1, \ell = 1, \beta_n = -1, n = 1, \varepsilon = 1, \Phi(x) = \sin\frac{n\pi}{\ell}x$. Figure 1 shows the $\hbar$-curve of third order approximation of mean for different values of time t and space variable x at $\sigma = 1, \ell = 1, \beta_n = -1, n = 1, \varepsilon = 1, \Phi(x) = \sin\frac{n\pi}{\ell}x$.Figure 2 shows the $\hbar$-curve of third order approximation of mean for different β_n values. According to these $\hbar$-curves, it is easy to discover that the valid region of is a horizontal line segments $-1.1 \le \hbar \le -0.9$, thus $\hbar = -0.96$. Figures 3 and 4 show mean and variance with time t for different ε values respectively. Figure 5 shows mean comparison between first, second and third order approximations; figure 6 shows variance comparison between first and second order approximations.Figures 7 and 8 show mean

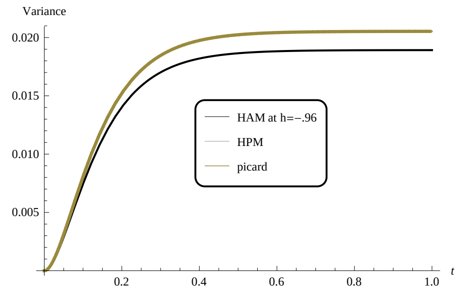

Fig. 9: Variance comparison between HAM first order at $\hbar = -0.96$, HPM first order and Picard first order at $t = 0.1$.

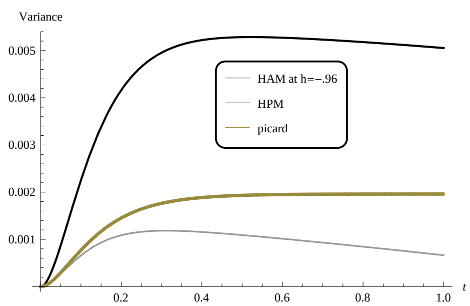

Fig. 10: Variance comparison between HAM second order at $\hbar = -0.96$, HPM second order and Picard second order at $t = 0.1$.

comparison between HAM, HPM and Picard [19] methods for first, second and third order approximations.

Table 1: Mean comparison between HAM second order at $\hbar = -0.96$, HPM second order and Picard second order at $t = 0.1$, $\beta_n = -1$ and $\varepsilon = 1$.

x	HAM	HPM	Picard
-1	-4.5E-17	-4.5E-17	-4.3E-17
-0.6	-0.34622	-0.34622	-0.33616
-0.2	-0.21398	-0.21398	-0.20776
0.2	0.213978	0.213978	0.207758
0.6	0.346223	0.346223	0.336159
1	4.46E-17	4.46E-17	4.33E-17

Tables 1 and 2 show the comparison between Homotopy analysis method, Homotopy perturbation method and Picard method. these tables shows the results between three methods are closed. Figures 9 and 10 illustrate variance comparison between HAM, HPM and Picard methods for first and second approximations only. We should note to the inability of computing high order approximations of mean and variance because of huge computations required. Comparisons among results of the computations of mean and variance illustrates that the results of three methods are very close from each other.

Table 2: Variance comparison between HAM first order at $\hbar = -0.96$, HPM first order and Picard first order at $x = 0.1$, $\beta_n = -1$ and $\varepsilon = 1$.

t	HAM	HPM	Picard
0	0	0	0
0.2	0.01403	0.01403	0.015224
0.4	0.018199	0.018199	0.019747
0.6	0.018821	0.018821	0.020422
0.8	0.018908	0.018908	0.020517
1	0.01892	0.01892	0.02053

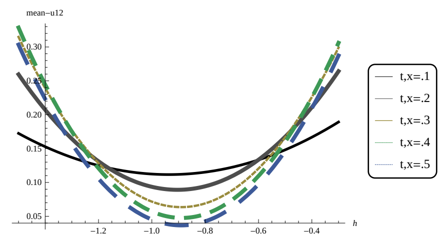

Fig. 11: The change of the mean of first order second correction approximation u_1^2 with parameter $\hbar$ at different t, x values, $\varepsilon = 1$ and $\beta_n = -1$.

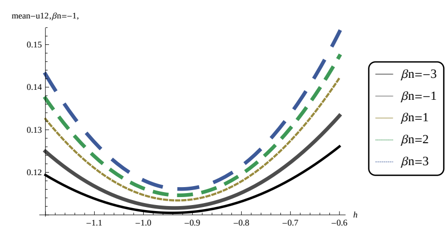

Fig. 12: The change of the mean of first order second correction approximation u_1^2 with parameter $\hbar$ at different β_n values and $t = x = 0.1$.

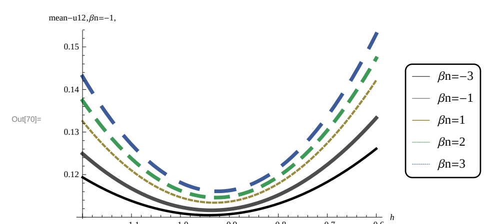

Fig. 13: The change of the variance of first order second correction approximation u_1^2 with parameter $\hbar$ at different t and x values.

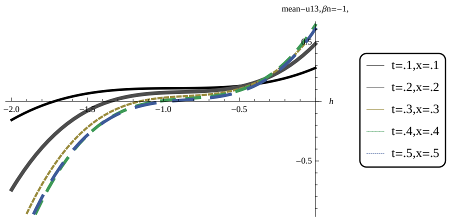

Fig. 14: The change of the mean of first order third correction approximation u_1^3 with parameter $\hbar$ at different t and x values.

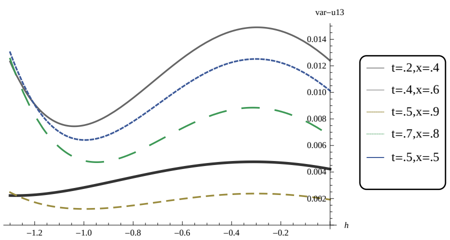

Fig. 15: The change of the variance of first order third correction approximation u_1^3 with parameter $\hbar$ at different t and x values.

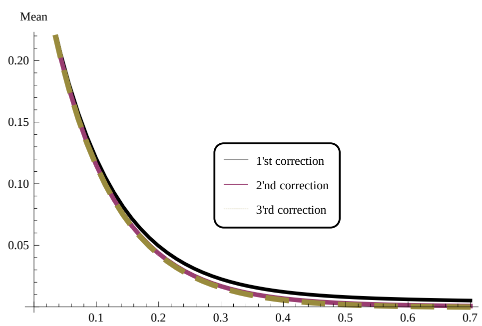

Fig. 16: The change of the first order second correction mean u_2^1 with time t at different ε values $x = 0.1$ and $\hbar = -0.96$.

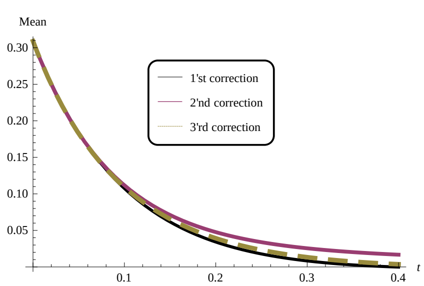

Fig. 17: The change of the first order third correction mean u_3^1 with time t at different ε values, $x = 0.1$ and $\hbar = -0.96$.

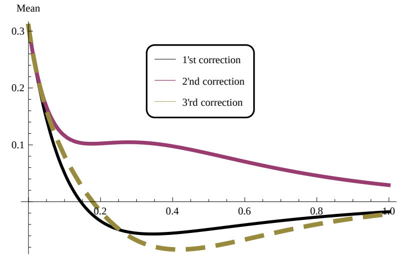

Fig. 18: The change of the first order second correction variance u_2^1 with time t at different ε values, $x = 0.1$ and $\hbar = -0.3$.

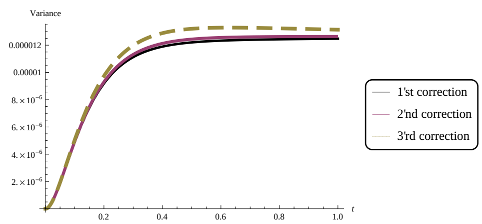

Fig. 19: The change of the first order second correction variance u31 with time t at different ε values, $x = 0.1$ and $\hbar = -0.3$.

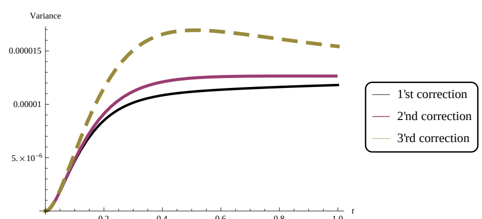

Fig. 20: The change of first order variance of first, second and third corrections. Comparison between the different corrections for $\varepsilon = 0.5$ and $x = 0.1$.

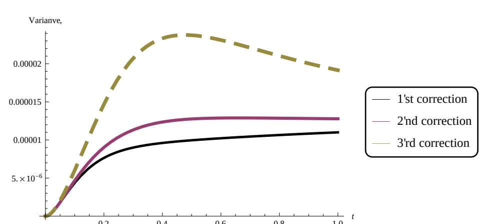

Fig. 21: he change of first order variance of first, second and third corrections. Comparison between the different corrections for $\varepsilon = 1$ and $x = 0.1$

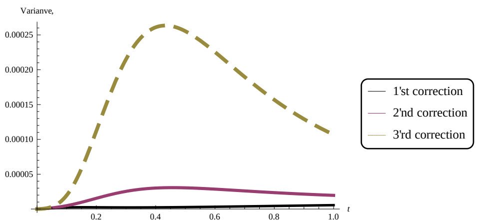

Fig. 22: he change of first order variance of first, second and third corrections. Comparison between the different corrections for $\varepsilon = 5$ and $x = 0.1$

6.2 HAM-WHE Results

Results of the solution of 2D stochastic quadratic nonlinear diffusion model using HAM-WHE technique are shown at $\sigma = 1, \ell = 1, \beta_n = -1, n = 1, \varepsilon = 1, \Phi(x) = \sin\frac{n\pi}{\ell}x$. Figures 11 and 13 show the $\hbar$-curves of mean and variance of first order second correction for different time and space values. Figure 12 shows the Plot of $\hbar$-curve of mean of first order second correction at different β_n values at $t = x = 0.1$.Figures 14 and 15 shows the Plot of $\hbar$ -curves of mean and variance of first order third correction for different time and space values. Figures 16, 17 and 18 show first order mean of first, second and third correction for different values of ε for $\varepsilon = 0.1$, $\varepsilon = 0.5$ and $\varepsilon = 5$. For small value of nonlinearity strength $\varepsilon = 0.1, 1$, the divergence of solution occurred in later interval after $t = 0.7$, but for large value of ε the mean of the solution diverges at $t = 0.1$ as indicated in figure 18. Figures 19, 20, 21 and 22 show first order variance of first, second and third correction for different values of for $\varepsilon = 0.1$, $\varepsilon = 0.5$ and $\varepsilon = 5$. For small value of nonlinearity strength $\varepsilon = 0.1, 1$ the divergence of solution occurred in later interval after $t = 0.7$, but for large value of ε the mean of the solution diverges after $t = 0.1$ as indicated in figure 22. We can say that it's a good result since in (WHEP and homotopy-WHEP) we couldn't use high values of ε without explosion of the solution in a small time interval.

7 Conclusions and Discussion

In this paper, the HAM-WHEP is proposed and used to give a statistical analytic solution of the stochastic diffusion equations. The application of this method has two steps, the first step indicated the approximation of the stochastic model using the first order series of the Wiener Hermite expansion of the stochastic solution process and the second step presented the application of the homotopy analysis method (HAM) to approximate the deterministic system which reduced from the first step using the statistic- al properties of WHE. The solution obtained by means of the HAM is an infinite power series for appropriate initial approximation, which can be, in turn, expressed in a closed form.

Different from all other analytic methods, the HAM-WHE provides us with a simple way to adjust and control the convergence region of the series solution by means of the auxiliary parameter $\hbar$. Thus the auxiliary parameter $\hbar$ plays an important role within the frame of the HAM so also the HAM-WHE which can be determined by the so called $\hbar$-curves. As shown in figures 1 and 2 we can see that the valid $\hbar$ region using HAM is $-1 < \hbar < -0.9$.and using HAM-WHE the interval is $-0.98 < \hbar < -0.92$, as shown in figure 11. The results demonstrate reliability and efficiency of the HAM-WHE method. From the results of two steps, some cases studies indicated some corrections of the approximation process for the statistical moments of the solution process, we can say that this is the first time to apply HAM-WHE method on stochastic problems and we found that it's easier than WHEP and more general than HPM and homotopy-WHEP since HPM is a special case of HAM obtained at and its results is accurate.

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