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A Fixed Point Theorem in Generalized Metric Spaces

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Abstract: In this paper, we state and prove a generalization of Ciric fixed point theorem[1] in generalized metric space by using a quasi-contractive map. Result presented in this paper generalize and extend well known fundamental metrical fixed point theorems in the literature (Banach [2], Kannan [3], Nadler [4], Reich [5], etc.) in the setting of generalized metric spaces.

Keywords: Fixed point, G-metric space, Quasi-contraction

1 Introduction and Preliminaries

In 2006, Mustafa and Sims [6] introduced the concept of G-metric spaces to overcome fundamental flaws in Dhage's theory of generalized metric spaces as follows:

Definition 1 *Let X be a non-empty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms: for all $x, y, z, a \in X$,*
$(G1)$ $G(x,y,z) = 0$ *if* $x = y = z$;
$(G2)$ $G(x,x,y) > 0$ *with* $x \neq y$;
$(G3)$ $G(x,x,y) \leq G(x,y,z)$ *with* $z \neq y$;
$(G4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...$;
$(G5)$ $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$;
then the function G is called a Generalized metric or more specifically a G-metric on X and the pair (X,G) is called a G-metric space.

Definition 2 *[6] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ in X, is said to be a G-Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer N such that $G(x_n,x_m,x_k) < \epsilon$, for all $n,m,k \geq N$; i.e., if $G(x_n,x_m,x_k) \to 0$ as $n,m,k \to \infty$.*

Definition 3 (6) *Let (X,G) be a G-metric space. A sequence $\{x_n\}$ in X, is said to be G-convergent to a point $x \in X$ if $lim_{m,n\to\infty} G(x,x_n,x_m) = 0$, i.e., for each $\epsilon > 0$, there exists a positive integer N such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$.*

Definition 4 (6) *A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in X.*

Definition 5 (6) *A G-metric space (X,G) is called a symmetric G-metric space if $G(x,y,y) = G(x,x,y)$, for all $x,y \in X$.*

Motivated by the work of Mustafa and Sims [6,7], various researchers (see, e.g., [8-10]) have proved number of well known results in G-metric spaces.

2 Main Result

In this section, we introduce quasi-contraction mappings in G-metric spaces as follows:

Definition 6 *A mapping $T : X \to X$ of a G-metric space X into itself is said to be quasi-contraction iff there exists a number q, $0 \leq q < 1$ such that*
$G(Tx,Ty,Ty) \leq q\, max\{G(x,y,y), G(x,Tx,Tx),$
$G(y,Ty,Ty), G(x,Ty,Ty), G(y,Tx,Tx)\}$.

Definition 7 (1) *Let T be a mapping of G-metric space X into itself. For $A \subseteq X$, define*
(i) $\delta(A) = sup\{G(a,b,c) : a,b,c \in A\}$ *and*
(ii) *for each* $x \in X$,
$O(x,n) = \{x, Tx, T^2x, T^3x, ..., T^nx\}, n = 1,2,3,...$
and $O(x,\infty) = \{x, Tx, T^2x, T^3x, ...\}$.
A space (X,G) is said to be T-orbitally complete iff every Cauchy sequence which is contained in $O(x,\infty)$ for some $x \in X$ converges in X.

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Theorem 1 *Let (X,G) be a G-metric space. Suppose that $T:X\to X$ is a quasi-contraction and X is T-orbitally complete. Then we have*
(i) T has a unique fixed point.
(ii) $lim_{n\to\infty}T^n x = z$ for all $x\in X$.
(iii) $G(T^n x,z,z)=\frac{q^n}{1-q}G(x,Tx,Tx)$ for all $x\in X$ and $n\in N$.

Proof. For each $x\in X$ and $0\le i\le n-1, 0\le j\le n$, we have
$G(T^i x,T^j x,T^j x)=G(TT^{i-1}x,TT^{j-1}x,TT^{j-1}x)$
$\le q\ max\{G(T^{i-1}x,T^{j-1}x,T^{j-1}x),$
$G(T^{i-1}x,TT^{i-1}x,TT^{i-1}x),$
$G(T^{j-1}x,TT^{j-1}x,TT^{j-1}x),$
$G(T^{i-1}x,TT^{j-1}x,TT^{j-1}x),$
$G(T^{j-1}x,TT^{i-1}x,TT^{i-1}x)\}$
$=q\ max\{G(T^{i-1}x,T^{j-1}x,T^{j-1}x),$
$G(T^{i-1}x,T^i x,T^i x),G(T^{j-1}x,T^j x,T^j x),$
$G(T^{i-1}x,T^j x,T^j x),G(T^{j-1}x,T^i x,T^i x)\}$
$\le q\delta[O_T(x,n)]$
where
$\delta[O_T(x,n)]$ = max $\{G(T^i x,T^j x,T^j x):0\le i,j\le n\}$.
Since $0\le q<1$, there exists $h_n(x)\le n$ such that
$G(x,T^{h_n x}x,T^{h_n x}x)=\delta[O_T(x,n)]$.
Then we have
$G(x,T^{h_n x}x,T^{h_n x}x)\le$
$G(x,Tx,Tx)+G(Tx,T^{h_n x}x,T^{h_n x}x)$
$\le G(x,Tx,Tx)+q\delta[O_T(x,n)]$
= $G(x,Tx,Tx)+qG(x,T^{h_n x}x,T^{h_n x}x)$.
It implies that
$G(x,T^{h_n x}x,T^{h_n x}x)\le\frac{1}{1-q}G(x,Tx,Tx)$ ...(1)
For all $n,m\ge 1$ and $n<m$, it follows from the quasi contractive condition of T and (1) that
$G(T^n x,T^m x,T^m x)=$
$G(TT^{n-1}x,T^{m-n+1}T^{n-1}x,T^{m-n+1}T^{n-1}x)$
$\le q.\delta(O_T(T^{n-1}x,m-n+1))$
= $q.G(T^{n-1}x,T^{m-n+1}T^{n-1}x,T^{m-n+1}T^{n-1}x)$
= $q.G(TT^{n-2}x,T^{m-n+2}T^{n-2}x,T^{m-n+2}T^{n-2}x)$
$\le q^2\delta(O_T(T^{n-2}x,m-n+2))$
...
$\le q^n\delta[O_T(x,m)]$
$\le\frac{q^n}{1-q}G(x,Tx,Tx)$...(A)
This gives $\{T^n x\}$ is a Cauchy sequence in X. Since X is T-orbitally complete, there exists z belongs to X such that $lim_{n\to\infty}T^n x=z$. ...(2)
By using the quasi contractive condition, we get
$G(z,Tz,Tz)=0$
$\le G(z,T^{n+1}x,T^{n+1}x)$
$+q\ max\{G(T^n x,z,z),G(T^n x,TT^n x,TT^n x),$
$G(z,Tz,Tz),G(T^n x,Tz,Tz),G(z,TT^n x,TT^n x)\}$...(3)
Taking limit as $n\to\infty$ in (3) and using (2), we get
$G(z,Tz,Tz)\le qG(z,Tz,Tz)$.
Since $0\le q<1$, we obtain $G(z,Tz,Tz)=0$. This gives, T has a fixed point $z\in X$.
To prove uniqueness of fixed point, let w be another fixed point of T. Then by using quasi-contractive condition on T, we have
$G(z,w,w)=G(Tz,Tw,tw)$
$\le q\ max\{G(z,w,w),G(z,Tz,Tz),G(w,Tw,Tw),$
$G(z,Tw,Tw),G(w,Tz,Tz)\}$
$G(z,w,w)\le qG(z,w,w)$
a contradiction, hence $z=w$. This proves uniqueness of fixed point. Also, by taking limit as $n\to\infty$ in (A), we have
$G(T^n x,z,z)=\frac{q^n}{1-q}G(x,Tx,Tx)$.
Hence result follows.

3 Conclusion

Result presented in this paper generalize and extend well known fundamental metrical fixed point theorems in the literature (Banach [2], Kannan [3], Nadler [4], Reich [5], etc.) in the setting of generalized metric spaces.

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References

[1] L. B. Ciric, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., **45**, 267-273 (1974).
[2] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, Fund. Math., **3**, 133-181 (1922).
[3] R. Kannan, Some results on fixed points, Bull. Cal. Math. Soc., **60**, 71-76 (1968).
[4] S. B. Nadler, Multi-valued contraction mappings, Pacific J. Math., **30**(2), 475-488 (1969).
[5] S. Reich, Some remarks concerning contraction mappings, Oanad. Math. Bull., **14**, 121-124 (1971).
[6] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal., **7**, 289-297 (2006).
[7] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Applications, Article ID 917175, 10 pages, Vol. 2009.
[8] S. Manro, S. Kumar and S.S. Bhatia, Weakly compatible maps of type (A) in G-metric spaces, Demonstratio Mathematica, **45**(4), 901-908 (2012).
[9] S. Manro, Fixed point theorems in G-metric space using E.A. property, J. Mathematics., **1**(1), 1015-1025 (2012).
[10] S. Manro, S. S. Bhatia, S. Kumar and C. Vetro, A common fixed point theorem for two weakly compatible pairs in G-metric spaces using the property E.A, Fixed Point Theory and Applications 2013, 2013:41.

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