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A Common Fixed Point Theorem for Fuzzy Maps under Nonexpansive type Condition

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Abstract: In this paper, we obtain a common fixed point theorem for a sequence of fuzzy mappings satisfying a contractive definition involving nonexpansive mapping.

Keywords: Fuzzy sets, common fixed point, fuzzy mapping, nonexpansive mapping.

1 Introduction

Fixed point theorems are very important tools for providing evidence of the existence and uniqueness of solutions to various mathematical models. The literature of the last four decades flourishes with results which discover fixed points of self and nonself nonlinear operators in the metric spaces. The first important result on fixed points for contractive type mappings was the well-known Banach contraction principle [1] appeared in explicit form in Banach's thesis in 1922, where it was used and established the existence of a solution for an integral equation.

In 1965, L.A. Zadeh[2] introduced the concept of a fuzzy set as a new way to represent vagueness in every day life. The study of fixed point theorems in fuzzy mathematics was investigated by Weiss [3], Butnariu [4], Singh and Talwar [5], Mihet [6], Qiu et al. [7], and Beg and Abbas [8] and many others.

Heilpern [9] first used the concept of fuzzy mappings to prove the Banach contraction principle for fuzzy (approximate quantity-valued) mappings on a complete metric linear spaces. The result obtained by Heilpern [9] is a fuzzy analogue of the fixed point theorem for multivalued mappings of Nadler et al. [10]. Bose and Sahani [11], Vijayaraju and Marudai [12], improved the result of Heilpern. In some earlier work, Watson and Rhoades [13],[14] proved several fixed point theorems involving a very general contractive definition.

In this paper, we prove a common fixed point theorem for sequence of fuzzy mappings using a more general contractive condition involving nonexpansive mapping. Our results extend and generalized the corresponding results of Bose and Sahani [11], Vijayaraju and Mohanraj [12] and Rhoades [15],[16].

2 Preliminaries

In this paper, we shall use the terminology and notations of Heilpern[9]. Heilpern gave some fundamental results related to fuzzy mappings.

Throughout this paper, let (X,d) be a complete linear metric space. A fuzzy set in X is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x \in X$, then the function value $A(x)$ is called the grade of membership of x in A. The α-level set of A is denoted by

$$A_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0,1]$$
$$A_0 = \overline{\{x : A(x) > 0\}}$$

where $\overline{B}$ denotes the closure of a set B.

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Definition 2.1. A fuzzy set A is said to be an approximate quantity if and only if A_α is compact and convex for each $\alpha \in [0,1]$ and $\sup_{x\in X} A(x) = 1$, when A is an approximate quantity and $A(x_0) = 1$ for some $x_0 \in X$, A is identified with an approximation of x_0.
The collection of all fuzzy sets in X is denoted by $F(X)$ and $W(X)$ is the sub-collection of all appropriate quantities.

Definition 2.2. Let $A,B \in W(X), \ \alpha \in [0,1]$. Then

$$D_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y);$$
$$H^\alpha(A,B) = dist(A_\alpha, B_\alpha);$$
$$D(A,B) = \sup_\alpha D_\alpha(A,B).$$

where 'dist' is the Hausdorff distance .The function D_α is called an α-distance and H a distance between A and B. Note that D_α is a nondecreasing function of α.

Definition 2.3. Let $A,B \in W(X)$. Then A is said to be more accurate than B, denoted by $A \subset B$, iff $A(x) \le B(x)$ for each $x \in X$.
The relation $\subset$ induces a partial ordering on the family $W(X)$.

Definition 2.4. Let X be an arbitrary set and Y be any metric space. F is called a fuzzy mapping if and only if F is a mapping from the set X into $W(Y)$.
A fuzzy mapping F is a fuzzy subset of $X \times Y$ with membership function $F(x,y)$. The function value $F(x,y)$ is the grade of membership of y in $F(x)$. Note that each fuzzy mapping is a set valued mapping.

Let $A \in F(X), \ B \in F(Y)$. The fuzzy set $F(A)$ in $F(Y)$ is defined by

$$F(A)(y) = \sup_{x\in X}(F(x,y) \wedge A(x)), \ y \in Y$$

Lee [17] proved the following.

Lemma 2.1. Let (X,d) be a complete linear metric space, F is a fuzzy mapping from X into $W(X)$ and $x_0 \in X$, then there exists an $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

The following two lemmas are due to Heilpern [9].

Lemma 2.2. Let $a,B \in W(X), \ \alpha \in [0,1]$ and $D_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x,y)$, where $A_\alpha = \{x : A(x) \ge \alpha\}$, then

$$D_\alpha(x,A) \le d(x,y) + D_\alpha(y,A), \ \text{ for each } x,y \in X.$$

Lemma 2.3. Let $H_\alpha(A,B) = dist(A_\alpha, B_\alpha)$, where 'dist' is the Hausdorff distance. If $\{x_0 \subset A\}$, then $D_\alpha(x_0,B) \le H_\alpha(A,B)$ for each $B \in W(X)$.

Rhoades [15] proved the following common fixed point theorem involving a very general contractive condition, for fuzzy mappings on complete linear metric space. He proved the following theorem.

Theorem 2.1. Let (X,d) be a complete linear metric space and let F, G be fuzzy mappings from X into $W(X)$ satisfying

$$H(Fx,Gy) \le Q(m(x,y)), \ \text{ for all } x,y \in X,$$

where

$$m(x,y) = \max\{d(x,y), D_\alpha(x,Fx), D_\alpha(y,Gy), \frac{1}{2}[D_\alpha(x,Gy) + D_\alpha(y,Fx)]\}.$$

Q is a real-valued function defined on D, the closure of the range of d, satisfying the following three conditions:

(a)$0 < Q(s) < s$ for each $s \in D\backslash\{0\}$ and $Q(0) = 0$,
(b)Q is non-decreasing on D, and
(c)$g(s) = s/s - Q(s)$ is non-increasing on $D\backslash\{0\}$.

Then there exists a point z in X such that $\{z\} \subset Fz \cap Gz$.

In [16] Rhoades, generalized the result of Theorem 2.1 for sequence of fuzzy mappings on complete linear metric space. He proved the following theorem.

Theorem 2.2. Let g be a nonexpansive sel mapping of a complete linear metric space (X,d). Let $\{F_i\}$ be a sequence of fuzzy mappings from X into $W(X)$. For each pair of fuzzy mappings F_i, F_j and for any $x \in X$, $\{u_x\} \subset F_i(x)$, there exists a $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that $D(\{u_x\},\{v_y\}) \le Q(m(x,y))$, where

$$m(x,y) = \max\{d(g(x),g(u_x)), d(g(y),g(v_y)), d(g(x),g(y)), \frac{1}{2}[d(g(x),g(v_y)) + d(g(y),g(u_x))]\}$$

and Q satisfying the conditions (a)-(c) of Theorem 2.1. Then there exists $\{p\} \subset \cap_{i\in N} F_i(p)$.

In this paper, we prove the result of above for common fixed point for sequence of fuzzy mappings of non-expansive condition.

3 Main Result

Theorem 3.1. Let g be a nonexpansive sel mapping of a complete linear metric space (X,d). Let $\{F_i\}$ be a sequence of fuzzy mappings from X into $W(X)$. For each pair of fuzzy mappings F_i, F_j and for any

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$x \in X$, $\{u_x\} \subset F_i(x)$, there exists a $\{v_y\} \subset F_j(y)$ for all $y \in X$ such that

$$\begin{aligned} D(\{u_x\},\{v_y\}) \leq \max\{&d(g(x),g(u_x)),\\ &d(g(y),g(v_y))\\ &,d(g(x),g(y))\\ &,\frac{1}{2}[d(g(x),g(v_y))+d(g(y),g(u_x))]\}\\ -w(\max\{&d(g(x),g(u_x)),d(g(y),g(v_y)),\\ &d(g(x),g(y))\\ &,\frac{1}{2}[d(g(x),g(v_y))+d(g(y),g(u_x))]\}) \end{aligned} \tag{1}$$

for all $x,y \in X$, $w : R^+ \to R^+$ be a continuous function such that $0 < w(r) < r$ for all $r > 0$.
Then there exists $\{p\} \subset \cap_{i\in N} F_i(p)$, i.e. p is a common fixed point of the sequence of fuzzy mappings.

Proof. Let $x_0 \in X$. Then we can choose $x_1 \in X$ such that $\{x_1\} \subset F_1(x_0)$ by Lemma 2.1. From the hypothesis, there exists an $x_2 \in X$ such that $\{x_2\} \subset F_2(x_1)$ and from 1, we have

$$\begin{aligned} D(\{x_1\},\{x_2\}) \leq \max\{&d(g(x_0),g(x_1)),\\ &d(g(x_1),g(x_2)),\\ &d(g(x_0),g(x_1))\\ &,\frac{1}{2}[d(g(x_1),g(x_1))\\ &+d(g(x_0),g(x_2))]\}\\ -w(\max\{&d(g(x_0),g(x_1)),\\ &d(g(x_1),g(x_2)),\\ &d(g(x_0),g(x_1))\\ &,\frac{1}{2}[d(g(x_1),g(x_1))\\ &+d(g(x_0),g(x_2))]\})\\ \leq \max\{&d(x_0,x_1),d(x_1,x_2),\\ &d(x_0,x_1),\\ &\frac{1}{2}[d(x_0,x_2)]\}\\ -w(\max\{&d(x_0,x_1),d(x_1,x_2),\\ &d(x_0,x_1),\\ &\frac{1}{2}[d(x_0,x_2)]\}) \end{aligned}$$

(Since g is a nonexpansive self mapping.)

It is clear that $d(x_0,x_2)$ is not maximum. Thus

$$\begin{aligned} d(x_1,x_2) &= D(\{x_1\},\{x_2\})\\ &\leq \max\{d(x_0,x_1),d(x_1,x_2)\}\\ &-w(\max\{d(x_0,x_1),d(x_1,x_2)\}) \end{aligned}$$

which implies that

$$d(x_1,x_2) \leq d(x_0,x_1) - w(d(x_0,x_1)) \tag{2}$$

Similarly

$$d(x_2,x_3) \leq d(x_1,x_2) - w(d(x_1,x_2)) \tag{3}$$

Inductively, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \subset F_{n+1}(x_n)$ and

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_n) - w(d(x_{n-1},x_n)) \tag{4}$$

Adding 2-4, we get

$$\sum_{i=0}^{n} w(d(x_i,x_{i+1})) \leq d(x_0,x_1) - d(x_n,x_{n+1}) \leq d(x_0,x_1).$$

Therefore

$$\sum_{i=0}^{n} w(d(x_i,x_{i+1})) < \infty \text{ and } \lim_{n\to\infty} w(d(x_n,x_{n+1})) = 0. \tag{5}$$

Now suppose that $\{x_n\}$ is not a Cauchy sequence, then there is an $\varepsilon > 0$ such that for each positive even integer $2k$, there exists positive even integer $2m > 2n > 2k$ such that

$$d(x_2m,x_2n) < \varepsilon. \tag{6}$$

Also, for each $2k$, we may find the least $2m$ exceeding $2n$ such that

$$d(x_{2n},x_{2m-2}) < \varepsilon. \tag{7}$$

Since $d(x_n,x_{n+1})$ is a decreasing sequence of non-negative terms, it converges, call the limit z. Suppose that $z > 0$. Then, since w is continuous, $\lim_{n\to\infty} w(d(x_n,x_{n+1})) = w(z)$. But $\lim_{n\to\infty} w(d(x_n,x_{n+1})) = 0$. Hence $w(z) = 0$, which is a contradiction to the fact that $0 < w(p) < p$. Hence

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{8}$$

Now

$$\begin{aligned} d(x_2m,x_2n) \leq\ & d(x_{2n},x_{2m-2}) + d(x_{2m-2},x_{2m-1})\\ &+ d(x_{2m-1},x_{2m}). \end{aligned} \tag{9}$$

Using 6-9, we obtain

$$d(x_2m,x_2n) \to \varepsilon, \text{ as } k \to \infty. \tag{10}$$

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Again applying 1, we get

$$\begin{aligned}
d(x_{2m+1},x_{2n+2}) &= D(\{x_{2m+1}\},\{x_{2n+2}\}) \\
&\le \max\{d(g(x_{2m}),g(x_{2m+1})), \\
&\quad d(g(x_{2n+1}),g(x_{2n+2})),d(g(x_{2m}),g(x_{2n+1})) \\
&\quad ,\frac{1}{2}[d(g(x_{2n+1}),g(x_{2m+1})) \\
&\quad +d(g(x_{2m}),g(x_{2n+2}))]\} \\
&- w(\max\{d(g(x_{2m}),g(x_{2m+1})), \\
&\quad d(g(x_{2n+1}),g(x_{2n+2})),d(g(x_{2m}),g(x_{2n+1})) \\
&\quad ,\frac{1}{2}[d(g(x_{2n+1}),g(x_{2m+1})) \\
&\quad +d(g(x_{2m}),g(x_{2n+2}))]\}) \\
&\le \max\{d(x_{2m},x_{2m+1}), \\
&\quad d(x_{2n+1},x_{2n+2}),d(x_{2m},x_{2n+1}) \\
&\quad ,\frac{1}{2}[d(x_{2n+1},x_{2m+1}) \\
&\quad +d(x_{2m},x_{2n+2})]\} \\
&- w(\max\{d(x_{2m},x_{2m+1}),d(x_{2n+1},x_{2n+2}), \\
&\quad d(x_{2m},x_{2n+1}) \\
&\quad ,\frac{1}{2}[d(x_{2n+1},x_{2m+1}) \\
&\quad +d(x_{2m},x_{2n+2})]\}) \qquad (11)
\end{aligned}$$

Note that

$$\begin{aligned}
d(x_{2m},x_{2n+1})-d(x_{2m},x_{2n}) &\le d(x_{2n},x_{2n+1}), \\
d(x_{2m+1},x_{2n+1})-d(x_{2m},x_{2n+1}) &\le d(x_{2m},x_{2m+1}), \\
d(x_{2m},x_{2n+2})-d(x_{2m},x_{2n+1}) &\le d(x_{2n+1},x_{2n+2}) \\
d(x_{2m+1},x_{2n+2})-d(x_{2m+1},x_{2n+1}) &\le d(x_{2n+1},x_{2n+2})
\end{aligned}$$

which implies that

$$\begin{aligned}
&d(x_{2m},x_{2n+1}) \to \varepsilon, \quad d(x_{2m+1},x_{2n+1}) \to \varepsilon, \\
&d(x_{2m},x_{2n+2}) \to \varepsilon, \quad d(x_{2m+1},x_{2n+2}) \to \varepsilon, \quad \text{as } k \to \infty
\end{aligned}$$

Therefore from 11 taking the limit as $k \to \infty$, we get

$$\varepsilon \le \max\{\varepsilon,\varepsilon,\varepsilon,\frac{1}{2}[\varepsilon+\varepsilon]\} - w(\max\{\varepsilon,\varepsilon,\varepsilon,\frac{1}{2}[\varepsilon+\varepsilon]\}),$$

which gives a contradiction. Thus $\{x_n\}$ is a Cauchy sequence and since X is complete, it converges to some $p \in X$.

Let F_m be an arbitrary member of $\{F_i\}$. Since $\{x_n\} \subset F_m(x_{n-1})$, by Lemma 2.1, there exists a $v_n \in X$ such that $\{v_n\} \subset F_m(p)$ for all n.

Applying 1 again, we have

$$\begin{aligned}
d(x_n,v_n) &= D(\{x_n\},\{v_n\}) \\
&\le \max\{d(g(x_{n-1}),g(x_n)), \\
&\quad d(g(p),g(v_n)), \\
&\quad d(g(x_{n-1}),g(p)) \\
&\quad ,\frac{1}{2}[d(g(x_{n-1}),g(v_n)) \\
&\quad +d(g(p),g(x_n))]\} \\
&- w(\max\{d(g(x_{n-1}),g(x_n)), \\
&\quad d(g(p),g(v_n)), \\
&\quad d(g(x_{n-1}),g(p)) \\
&\quad ,\frac{1}{2}[d(g(x_{n-1}),g(v_n)) \\
&\quad +d(g(p),g(x_n))]\}) \\
&\le \max\{d(x_{n-1},x_n),d(p,v_n), \\
&\quad d(x_{n-1},p) \\
&\quad ,\frac{1}{2}[d(x_{n-1},v_n) \\
&\quad +d(p,x_n)]\} \\
&- w(\max\{d(x_{n-1},x_n),d(p,v_n), \\
&\quad d(x_{n-1},p) \\
&\quad ,\frac{1}{2}[d(x_{n-1},v_n) \\
&\quad +d(p,x_n)]\})
\end{aligned}$$

Suppose that $\lim v_n \neq p$ and taking limit as $n \to \infty$, we get

$$d(p,v_n) \le d(p,v_n) - w(d(p,v_n)).$$

Since w is continuous, we get a contradiction. Therefore $\lim v_n = p$. Since F_m is arbitrary, $\{p\} \subset \cap_{i=1}^{\infty} F_i(p)$.

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