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Legendre–Gauss–Lobatto Pseudo–spectral Method for One–Dimensional Advection–Diffusion Equation

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Abstract: In this paper, we present a Legendre pseudo–spectral method based on a Legendre–Gauss–Lobatto zeros with the aid of tensor product formulation for solving one–dimensional parabolic advection–diffusion equation with constant parameters subject to a given initial condition and boundary conditions. First, we introduce an approximation to the unknown function by using differentiation matrices and its derivatives with respect to x and t. Secondly , we convert our problem to a linear system of equations to unknowns at the collocation points, then solve it. Finally, several examples are given and the numerical results are shown to demonstrate the efficiency of the proposed technique.

Keywords: One-dimensional parabolic partial differential equation, Spectral method, Legendre Pseudo–spectral method, Legendre Differentiation matrices, Kronecker product

1 Introduction

In this paper, we are concerned with an efficient numerical approximation scheme of the mathematical model of a physical phenomena involving the one–dimensional time–dependent advection–diffusion equation of the form

$$\frac{\partial u}{\partial t}+\beta\frac{\partial u}{\partial x}-\alpha\frac{\partial^2 u}{\partial x^2}=f(x,t), \tag{1}$$

where $x\in(a,b)\subseteq\mathbb{R}, t\in(0,T], T>0$, associated with initial condition and Dirichlet boundary conditions, respectively:

$$u(x,0)=u_0(x), \qquad \forall x, \tag{2}$$

$$u(a,t)=g_1(t), \quad u(b,t)=g_2(t), \quad \forall t, \tag{3}$$

where $f(x,t), u_0(x), g_1(t)$ and $g_2(t)$ are known functions, whereas u is the unknown function. Note that α and β are considered to be positive constants quantifying the diffusion and advection processes, respectively.

One–dimensional version of the partial differential equations which describe advection–diffusion of quantities such as mass, heat, energy, vorticist, etc [1,2]. Equation (1) has been used to describe heat transfer in a draining film [3], water transfer in soils [4], dispersion of tracers in porous media [5], the intrusion of salt water into fresh water aquifers, the spread of pollutants in rivers and streams [6], the dispersion of dissolved material in estuaries and coastal seas [7], contaminant dispersion in shallow lakes [8], the absorption of chemicals into beds [9], the spread of solute in a liquid flowing through a tube, long–range transport of pollutants in the atmosphere [10], forced cooling by fluids of solid material such as windings in turbo generators [11], thermal pollution in river systems [12], flow in porous media [13] and dispersion of dissolved salts in groundwater [14].

In the present contribution, we construct the solution using the pseudo–spectral techniques [15,16] with Legendre basis. Pseudo–spectral methods are powerful approach for numerical solution of partial differential equations [17,18,19], which can be traced back to 1970s [20]. If one wants to solve an ordinary or partial differential equation to high accuracy on a simple domain and if the data defining the problem are smooth, then pseudo–spectral methods are usually the best tool. They can often achieve 10 digits of accuracy where a finite difference scheme or a finite element method would get 3 or 4. At lower accuracies, they demand less computer memory than the alternatives.

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In pseudo–spectral methods [21], there are basically two steps to obtaining a numerical approximation to a solution of differential equation. First, an appropriate finite or discrete representation of the solution must be chosen. This may be done by polynomial interpolation of the solution based on some suitable nodes. However, it is well known that the Lagrange interpolation polynomial based on equally spaced points does not give a satisfactory approximation to general smooth functions. In fact, as the number of collocation points increases, interplant polynomials typically diverge. This poor behavior of the polynomial interpolation can be avoided for smoothly differentiable functions by removing the restriction to equally spaced collocation points. Good results are obtained by relating the collocation points to the structure of classical orthogonal polynomials, such as the well-known Legendre-Gauss-Lobatto points. The second step is to obtain a system of algebraic equations from discretization of the original equation. In the case of differential equations, this second step involves finding an approximation for the differential operator (see [20]).

Many authors have considered this technique to solve many problems. In [22,23], pseudospectral scheme to approximate the optimal control problems. Also, a Legendre pseudospectral Penalty scheme used for solving time–domain Maxwells equations [24]. The method of Hermite pseudospectral scheme is used for Dirac equation [25], and nonlinear partial differential equations [26], respectively. In [27], multidomain pseudospectral method for nonlinear convection–diffusion equations was presented. Finally, [28] also pseudospectral methods used in Quantum and Statistical Mechanics.

The organization of the rest of this article is as follows. In Section 2, we present some preliminaries and drive some tools for discretizing the introduced problem. Section 3 summarizes the application of pseudo–spectral Legendre method to the solution of the problem (1)–(3). As a result a set of algebraic linear equations are formed and a solution of the considered problem is discussed. In Section 4, we present some numerical examples to demonstrate the effectiveness of the proposed method.

2 Preliminaries and Notations

The well-known Legendre polynomials [29,30] are defined on the interval [-1,1] and can be determined with the aid of the following recurrence formulas:

$$L_0(z)=1, \quad L_1(z)=z,$$
$$L_{i+1}(z)=\frac{2i+1}{i+1}zL_i(z)-\frac{i}{i+1}L_{i-1}(z), \quad i\geq 1, \tag{4}$$

Let $L_N(z)$ denote the Legendre polynomial of order N, then the Legendre–Gauss–Lobatto nodes (**LGL**) nodes will be $z_0^{(N)},...,z_N^{(N)}$, where these nodes defined by $z_0^{(N)}=-1, z_N^{(N)}=1$ and for $\{z_i^{(N)}\}_{i=1}^{N-1}$ are the zeros of $L_N'(z)$. Unfortunately, there are no explicit formulas for the **LGL** nodes is known. However, they can be computed numerically [31].

Let $\{\phi_i^{(N)}(z)\}_{i=0}^N$ be the Lagrange polynomials based on **LGL** nodes, that are expressed as [32,33]:

$$\phi_j^{(N)}(z)=\prod_{i=0,i\neq j}^{N}\frac{z-z_i^{(N)}}{z_j^{(N)}-z_i^{(N)}}, \; j=0,...,N, \tag{5}$$

with the Kronecker property

$$\phi_j^{(N)}(z_k^N)=\delta_{jk}=\begin{cases}0, & j\neq k,\\ 1, & j=k.\end{cases}$$

It is more convenient to consider an alternative expression [32,33], for $j=0,...,N$,

$$\phi_j^{(N)}(z_k^N)=\frac{1}{N(N+1)L_N(z_j^{(N)})}\frac{(1-z^2)L_N'(z)}{z-z_j^{(N)}} \tag{6}$$

Any defined function f on the interval $[-1,1]$ may be approximated by Lagrange polynomials as

$$f(z)\simeq\sum_{i=0}^{N}c_i\phi_i^{(N)}(z), \tag{7}$$

where $c_i=\{f(z_i^{(N)})\}_{i=0}^N$. Equation (7) will be exact when f is a polynomial of degree at most N. Equation (7) can be expressed in the following matrix form

$$f(z)\simeq\boldsymbol{\Phi}^{(N)}\mathbf{F},$$

where $\boldsymbol{\Phi}^{(N)}=\left[\phi_0^{(N)}(z),...,\phi_N^{(N)}(z)\right]$ and $\mathbf{F}=[f(z_0^{(N)}),...,f(z_N^{(N)})]^T$. The first derivative to equation (7) can be expressed as

$$f'(z)\simeq\sum_{i=0}^{N}c_i\phi_i'^{(N)}(z), \tag{8}$$

where $\phi_i'^{(N)}(z)$ is a polynomial of degree $N-1$, which can be written as

$$\phi_i'^{(N)}(z)=\sum_{k=0}^{N}\phi_i'^{(N)}(z_k^{(N)})\phi_k^{(N)}(z), \quad i=0,...,N. \tag{9}$$

Equation (9) can be expressed in the following matrix form:

$$\frac{d}{dz}\boldsymbol{\Phi}^{(N)}(z)=\boldsymbol{\Phi}^{(N)}(z)\mathbf{D}_{N+1}, \tag{10}$$

where $\mathbf{D}_{N+1}$ is the so–called differentiation matrix with dimension $N+1$. From the last two equations (9,10) we get $[\mathbf{D}_{N+1}]_{i,k}=\phi_i'^{(N)}(z_k^{(N)})$. The entries of the differentiation matrix can be defined for **LGL** points (cf.

[33]) as the following

$$[\mathbf{D}_{N+1}]_{i,k} = \begin{cases} \dfrac{L_N(z_i^{(N)})}{L_N(z_k^{(N)})}\dfrac{1}{z_i^{(N)}-z_k^{(N)}}, & i \neq k, \\ -\dfrac{N(N+1)}{4}, & i=k=0, \\ \dfrac{N(N+1)}{4}, & i=k=N, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Now, we introduce the second order differentiation matrix as $\mathbf{D}^2_{N+1}$ which is the derivative to differentiation matrix $\mathbf{D}_{N+1}$. The entries to the second order differentiation matrix can be defined for **LGL** points (cf. [34]) as the following

$$[\mathbf{D}^2_{N+1}]_{i,j} = \begin{cases} 2[\mathbf{D}_{N+1}]_{i,k}\Big([\mathbf{D}_{N+1}]_{i,i} - \dfrac{1}{z_i^{(N)}-z_k^{(N)}}\Big), & i \neq k \\ -\sum\limits_{i=0, i\neq k}^{N} [\mathbf{D}^2_{N+1}]_{i,k}, & i=k. \end{cases} \quad (12)$$

Also, any defined function $h(x)$ on an arbitrary interval $[a,b]$ may be approximated by making transformation from $z \in [-1,1]$ to $x \in [a,b]$ as:

$$h(x) \simeq \sum_{i=0}^{N} h(x_i^{(N)})\phi_i^{(N)}(\frac{2}{b-a}(x-a)-1), \quad (13)$$

where $x_i^{(N)} = \{\frac{b-a}{2}(z_i^{(N)}+1)+a\}_{i=0}^{N}$ are the shifted **LGL** nodes associated with interval $[a,b]$. Equation (13) can be expressed in the following matrix form:

$$h(x) \simeq \Phi^{(N)}_{[a,b]}(x)\mathbf{H}. \quad (14)$$

In view of equations (10) and (13), we conclude that

$$\frac{d^i}{dx^i}\Phi^{(N)}_{[a,b]}(x) = (\frac{2}{b-a})^i \Phi^{(N)}_{[a,b]}(x)\mathbf{D}^i_{N+1}, \quad (15)$$

For an arbitrary N and M, any function of two variables $u : [a,b]\times[c,d] \to \mathbf{R}$ may be approximated by

$$u(x,y) \simeq \sum_{i=0}^{N}\sum_{j=0}^{M} U_{i,j}\phi_i^{(N)}(\frac{2}{b-a}(x-a)-1) .\phi_j^{(M)}(\frac{2}{d-c}(y-c)-1), \quad (16)$$

where

$$U_{i,j} = u\Big(\frac{b-a}{2}(z_i^{(N)}+1)+a, \frac{d-c}{2}(z_j^{(M)}+1)+c\Big). \quad (17)$$

Equation (16) can be expressed based on Kronecker product in the following matrix form:

$$u(x,y) \simeq \Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[c,d]}(y)\Big)\mathbf{U}, \quad (18)$$

where $\mathbf{U}$ is the $(N+1)(M+1)$ vector as the following form:

$$\mathbf{U} = [U_{0,0},...,U_{0,M} \mid ... \mid U_{N,0},...,U_{N,M}]^T \quad (19)$$

The previous representations that are based on Kronecker product, provide some simplification in calculations when we deal with our original problem. Also by using first and second differentiation matrices we can approximate relative derivatives of any function from its expansion as we can see next. For example let u be approximated as in (18), now we can write the first derivative to u with respect to x as the following:

$$\begin{aligned} u_x(x,y) &\simeq \Big(\frac{d}{dx}\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[c,d]}(y)\Big)\mathbf{U} \\ &= \frac{2}{b-a}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}_{N+1}\otimes\Phi^{(M)}_{[c,d]}(y)\Big)\mathbf{U} \\ &= \frac{2}{b-a}\Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[c,d]}(y)\Big) \\ &\quad .\Big(\mathbf{D}_{N+1}\otimes\mathbf{I}_{M+1}\Big)\mathbf{U}. \end{aligned} \quad (20)$$

In a similar way, we can conclude that the first derivative to u with respect to y as the following:

$$u_y(x,y) \simeq \frac{2}{d-c}\Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[c,d]}(y)\Big)\Big(\mathbf{I}_{M+1}\otimes\mathbf{D}_{M+1}\Big)\mathbf{U}. \quad (21)$$

3 Legendre Pseudo–spectral Approximation

In order to solve problem (1)–(3), we approximate $u(x,t)$ as:

$$u(x,t) \simeq \Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[0,T]}(t)\Big)\mathbf{U}, \quad (22)$$

where the positive and integer numbers N and M are discretization parameters corresponding to space and time dimensions, respectively. Also we will consider $\{x_i\}_{i=0}^{N}$ and $\{t_j\}_{j=0}^{M}$ as the **LGL** nodes corresponding to the intervals $[a,b]$ and $[0,T]$, respectively.

By using (22) and differentiation matrices, we can write the derivatives to $u(x,t)$ as the following

$$u_x(x,t) \simeq \frac{2}{b-a}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}_{N+1}\otimes\Phi^{(M)}_{[0,T]}(t)\Big)\mathbf{U}, \quad (23)$$

$$u_{xx}(x,t) \simeq \frac{4}{(b-a)^2}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}^2_{N+1}\otimes\Phi^{(M)}_{[0,T]}(t)\Big)\mathbf{U}, \quad (24)$$

$$u_t(x,t) \simeq \frac{2}{T}\Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[0,T]}(t)\mathbf{D}_{M+1}\Big)\mathbf{U}. \quad (25)$$

Now, by substituting from the previous equations in equation (1), we obtain

$$\begin{aligned} &\Big[\frac{2}{T}\Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[0,T]}(t)\mathbf{D}_{M+1}\Big) \\ &+ \frac{2\beta}{b-a}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}_{N+1}\otimes\Phi^{(M)}_{[0,T]}(t)\Big) \\ &- \frac{4\alpha}{(b-a)^2}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}^2_{N+1}\otimes\Phi^{(M)}_{[0,T]}(t)\Big)\Big]\mathbf{U} = f(x,t). \end{aligned} \quad (26)$$

Now, for $1 < i < N-1$ and $1 < j < M$, we collocate the above equation at the collocation points $\{(x_i,t_j)\}_{i,j}$. Note that these collocation points are the interior points not lie in initial or boundary conditions. After collocating, equation (26) becomes:

$$\Big[\frac{2}{T}\Big(e_{i+1}^{N+1}\otimes e_{j+1}^{M+1}\mathbf{D}_{M+1}\Big)+\frac{2\beta}{b-a}\Big(e_{i+1}^{N+1}\mathbf{D}_{N+1}\otimes e_{j+1}^{M+1}\Big) - \frac{4\alpha}{(b-a)^2}\Big(e_{i+1}^{N+1}\mathbf{D}_{N+1}^2\otimes e_{j+1}^{M+1}\Big)\Big]\mathbf{U}_1 = f(x_i,t_j), \quad i=1,\cdots,N-1, \quad j=1,\cdots,M, \tag{27}$$

where e_k^p is the k^{th} row of $p\times p$ identity matrix. Equation (27) can be represented in the following matrix form using identity matrix:

$$\Big[\frac{2}{T}\Big([\mathbf{I}]_2^N\otimes[\mathbf{I}]_2^{M+1}\mathbf{D}_{M+1}\Big)+\frac{2\beta}{b-a}\Big([\mathbf{I}]_2^N\mathbf{D}_{N+1}\otimes[\mathbf{I}]_2^{M+2}\Big) - \frac{4\alpha}{(b-a)^2}\Big([\mathbf{I}]_2^N\mathbf{D}_{N+1}^2\otimes[\mathbf{I}]_2^{M+1}\Big)\Big]\mathbf{U}_1 = \mathbf{F}_1, \tag{28}$$

which can be formed as

$$\mathbf{A}_1\mathbf{U}_1 = \mathbf{F}_1, \tag{29}$$

where $\mathbf{F}_1$ and $\mathbf{U}_1$ are the $(N-1)(M)$ vectors they take the following forms:

$$\mathbf{F}_1 = [f_{1,1},...,f_{1,M}\mid\cdots\mid f_{N-1,1},...,f_{N-1,M}]^T,$$
$$\mathbf{U}_1 = [U_{1,1},...,U_{1,M}\mid\cdots\mid U_{N-1,1},...,U_{N-1,M}]^T,$$

and $\mathbf{A}_1$ is a matrix of dimension $N(N-1)\times(M+1)^2$, that can be defined as

$$\mathbf{A}_1 = \Big[\frac{2}{T}\Big([\mathbf{I}]_2^N\otimes[\mathbf{I}]_2^{M+1}\mathbf{D}_{M+1}\Big) + \frac{2\beta}{b-a}\Big([\mathbf{I}]_2^N\mathbf{D}_{N+1}\otimes[\mathbf{I}]_2^{M+2}\Big) - \frac{4\alpha}{(b-a)^2}\Big([\mathbf{I}]_2^N\mathbf{D}_{N+1}^2\otimes[\mathbf{I}]_2^{M+1}\Big)\Big].$$

For discretization the initial condition, we substitute (26) in (2) getting the following

$$\Big(\Phi_{[a,b]}^{(N)}(x)\otimes\Phi_{[0,T]}^{(M)}(0)\Big)\mathbf{U} = u_0(x), \quad a\le x\le b,$$

Now, for $0 < i < N$, we collocate the above equation at the collocation points $\{(x_i,0)\}$. After collocating, the previous equation becomes:

$$\Big(e_{i+1}^{N+1}\otimes e_1^{M+1}\Big)\mathbf{U}_2 = u_0(x_i), \tag{30}$$

then in matrix form using identity matrix

$$\Big([\mathbf{I}]_1^{N+1}\otimes e_1^{M+1}\Big)\mathbf{U}_2 = \mathbf{U}_0, \tag{31}$$

which can be formed as

$$\mathbf{A}_2\mathbf{U}_2 = \mathbf{U}_0, \tag{32}$$

where $\mathbf{U}_0$ and $\mathbf{U}_2$ are the $(N+1)$ vectors, they can be described as the following forms:

$$\mathbf{U}_0 = [u_0(x_0),...,u_0(x_N)]^T,$$
$$\mathbf{U}_2 = [U_{0,0},...,U_{N,0}]^T,$$

and $\mathbf{A}_2$ is a matrix of dimension $(N+1)\times(N+1)^2$, that has the following form

$$\mathbf{A}_2 = \Big([\mathbf{I}]_1^{N+1}\otimes e_1^{M+1}\Big).$$

Finally, to discrete the boundary conditions, we substitute (26) in (3). First, we deal with the left boundary to find the reduced form, then doing the same with the right boundary. Equation (3) will be

$$\Big(\Phi_{[a,b]}^{(N)}(a)\otimes\Phi_{[0,T]}^{(M)}(t)\Big)\mathbf{U} = g_1(t), \tag{33}$$

Now, for $1 < j < M$, we collocate the above equation at the collocation points $\{(a,t_j)\}$ for the first boundary condition. After collocating, the previous equation becomes:

$$\Big(e_1^{N+1}\otimes e_{j+1}^{M+1}\Big)\mathbf{U}_3 = g_1(t_j), \tag{34}$$

then in matrix form using identity matrix

$$\Big(e_1^{N+1}\otimes[\mathbf{I}]_2^{M+1}\Big)\mathbf{U}_3 = \mathbf{G}_1, \tag{35}$$

which can be formed as

$$\mathbf{A}_3\mathbf{U}_3 = \mathbf{G}_1, \tag{36}$$

where $\mathbf{G}_1$ and $\mathbf{U}_3$ are the (M) vectors, they can be described as the following forms:

$$\mathbf{G}_1 = [g_1(t_1),...,g_1(t_M)]^T,$$
$$\mathbf{U}_3 = [U_{0,1},...,U_{0,M}]^T,$$

and $\mathbf{A}_3$ is a matrix of dimension $(M)\times(M+1)^2$, that has the following form

$$\mathbf{A}_3 = \Big(e_1^{N+1}\otimes[\mathbf{I}]_2^{M+1}\Big).$$

Similarly, we can write the equation of the second boundary condition as the following form

$$\Big(e_{N+1}^{N+1}\otimes[\mathbf{I}]_2^{M+1}\Big)\mathbf{U}_4 = \mathbf{G}_2, \tag{37}$$

which can be formed as

$$\mathbf{A}_4\mathbf{U}_4 = \mathbf{G}_2, \tag{38}$$

where $\mathbf{G}_2$ and $\mathbf{U}_4$ are the (M) vectors, they can be described as the following forms:

$$\mathbf{G}_2 = [g_2(t_1),...,g_2(t_M)]^T,$$
$$\mathbf{U}_4 = [U_{N,1},...,U_{N,M}]^T,$$

and $\mathbf{A}_4$ is a matrix of dimension $(M)\times(M+1)^2$, that has the following form

$$\mathbf{A}_4 = \left(e_{N+1}^{N+1}\otimes[\mathbf{I}]_2^{M+1}\right).$$

The resulting system of equations can be described, from collecting equations (29), (32), (36) and (38), as the following

$$\mathbf{AU} = \mathbf{F}, \tag{39}$$

where $\mathbf{A}$ is a matrix of dimension $(N+1)^2\times(M+1)^2$, that has the form $\mathbf{A} = [\mathbf{A}_1 \mid \mathbf{A}_2 \mid \mathbf{A}_3 \mid \mathbf{A}_4]$. For $\mathbf{U}$ and $\mathbf{F}$, each one is a vector with dimension $(M+1)^2$, and take the following form

$\mathbf{U} = [\mathbf{U}_1 \mid \mathbf{U}_2 \mid \mathbf{U}_3 \mid \mathbf{U}_4]^T$,

$\mathbf{F} = [\mathbf{F}_1 \mid \mathbf{U}_0 \mid \mathbf{G}_1 \mid \mathbf{G}_2]^T$.

After solving the linear system described in (39), we can find the approximated solution to our problem (1).

4 Numerical Examples

In order to test the utility of the proposed method, we apply the new scheme to the following examples whose exact solutions are provided in each case. For both examples, we take $N = M$ and to show the efficiency of the present method for our problem in comparison with the exact solution, we calculate for different values of N the maximum error defined by

$$\|E\|_\infty = \max_{\substack{1\le i\le N-1\\ 1\le j\le M-1}} \left|U_{i,j} - u(x_i,t_j)\right|.$$

All the computations are carried out in double precision arithmetic using Matlab 7.9.0 (R2009b). To obtain sufficient accurate calculations, variable arithmetic precision (vpa) is employed with digit being assigned to be 32. The code was executed on a second generation Intel Core i52410M, 2.3 Ghz Laptop.

Example 1.[1] Consider the problem (1)–(3) with the initial condition $u(x,0) = x^2, \quad 0 \le x \le 1$, and the boundary conditions are given as

$$\begin{cases} u(0,t) = 0, \\ u(1,t) = \exp(t), \end{cases} \qquad 0 \le t \le 1,$$

and the exact solution $u(x,t) = x^2\exp(t)$, in this case the forcing function will be $f(x,t) = (x^2 + 2\beta x - 2\alpha)\exp(t)$.

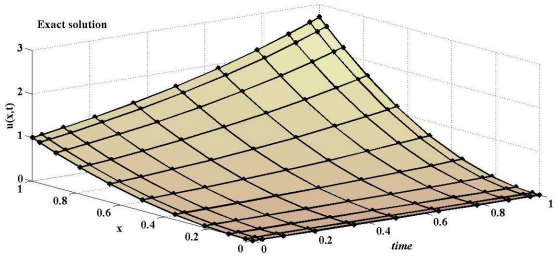
Example 2.[1] Consider the problem (1)–(3) with the initial condition $u(x,0) = \sin(x), \quad 0 \le x \le \pi$, and the boundary conditions are given as

$$\begin{cases} u(0,t) = 0, \\ u(\pi,t) = 0, \end{cases} \qquad 0 \le t \le 2,$$

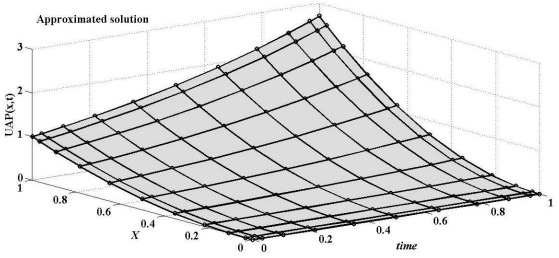
and the exact solution $u(x,t) = \sin(x)\exp(-t)$, in this case the forcing function will be $f(x,t) = \sin(x)\exp(-t)(\alpha - 1) + \beta\cos(x)\exp(-t)$.

Table 1: Max. $\|E\|_\infty$ errors with different values of N for Example 1.

N	$\alpha = 0.01, \beta = 1$	$\alpha = 0.001, \beta = 2$
4	8.60778E-05	6.71352E-05
5	3.21083E-06	1.64713E-06
6	9.85131E-08	6.13779E-08
7	2.48358E-09	8.80885E-10
8	5.24851E-11	1.45741E-11
9	8.31335E-13	8.13571E-13
10	3.39551E-12	3.62599E-12



(a) Exact solution



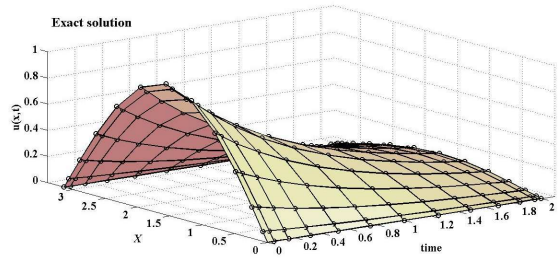
(b) Numerical solution

Fig. 1: Exact and Numerical solutions for $\alpha = 0.02$, $\beta = 2$ with $x \in [0,1]$ and $t \in [0,1]$ at $N = 10$ for Example 1.

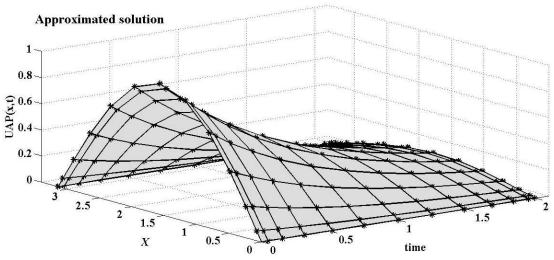
Table 2: Max. $\|E\|_\infty$ errors with different values of N for Example 2.

N	$\alpha = 0.01, \beta = 1$	$\alpha = 0.05, \beta = 2$
5	2.9530E-03	3.83772E-03
6	2.5459E-05	1.77285E-05
7	4.1972E-05	3.85403E-05
8	1.5390E-07	5.90046E-08
9	2.6529E-07	2.07852E-07
10	4.4326E-10	2.93808E-10
11	1.2614E-09	6.88776E-10
12	9.1676E-12	8.81836E-13

Example 3.[1] Consider the problem (1)–(3) with the initial condition $u(x,0) = 0, \quad 0 \le x \le 1$, and the

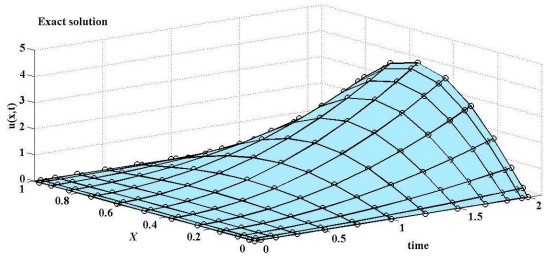


(a) Exact solution

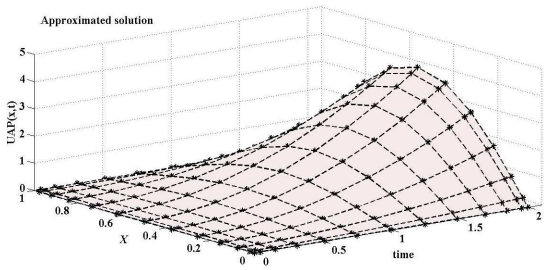


(b) Numerical solution

Fig. 2: Exact and Numerical solutions for $\alpha = 0.05, \beta = 2$ with $x \in [0,\pi]$ and $t \in [0,2]$ at $N = 12$ for Example 2.



(a) Exact solution



(b) Numerical solution

Fig. 3: Exact and Numerical solutions for $\alpha = 0.09, \beta = 2$ with $x \in [0,1]$ and $t \in [0,2]$ at $N = 12$ for Example 3.

boundary conditions are given as

$$\begin{cases} u(0,t) = 0, \\ u(\pi,t) = 0, \end{cases} \quad 0 \le t \le 2,$$

and the exact solution $u(x,t) = t^2 \sin(\pi x)$, in this case the forcing function will be $f(x,t) = \sin(\pi x)(2t + \alpha\pi^2 t^2) + \beta\pi t^2 \cos(\pi x)$.

Table 3: Max. $\|E\|_\infty$ errors with different values of N for Example 3.

N	$\alpha = 0.01, \beta = 1$	$\alpha = 0.09, \beta = 2$
5	1.9751E-02	1.15358E-02
6	1.0610E-04	1.20062E-04
7	2.2929E-04	8.19062E-05
8	7.8112E-07	5.90329E-07
9	1.4869E-06	3.50475E-07
10	3.1482E-09	1.71858E-09
11	6.0382E-09	1.30315E-09
12	2.1069E-11	1.94014E-11

5 Conclusion

In this work, we apply Legendre Pseudo–spectral method for one-dimensional advection–diffusion equation with Legendre–Gauss–Lobatto nodes. The differentiation matrices are used to represent the unknown functions. Several examples are introduced in this article show that the proposed numerical procedure is efficient and provides very accurate results even with using a small number of collocation points. The Pseudo–spectral scheme is a powerful approach for the numerical solution of parabolic advection–diffusion equation.

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